Classical paths in systems of fermions

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We implement in systems of fermions the formalism of pseudoclassical paths that we recently developed for systems of bosons and show that quantum states of fermionic fields can be described, in the Heisenberg picture, as linear combinations of randomly distributed paths that do not interfere between themselves and obey classical Dirac equations. Every physical observable is assigned a time-dependent value on each path in a way that respects the anticommutative algebra between quantum operators and we observe that these values on paths do not necessarily satisfy the usual algebraic relations between classical observables. We use these pseudoclassical paths to define the dynamics of quantum fluctuations in systems of fermions and show that, as we found for systems of bosons, the dynamics of fluctuations of a wide class of observables that we call "collective" observables can be approximately described in terms of classical stochastic concepts. Finally, we apply this formalism to describe the dynamics of local fluctuations of globally conserved fermion numbers.

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I. INTRODUCTION

Quantum fluctuations of physical observables can be manifest when they couple to another external system. For example, the Lamb shift in the absorption/emission lines of atoms is a direct consequence of their interaction with vacuum fluctuations of the electromagnetic fields. Similarly, quantum fluctuations in the energy-momentum tensor can affect the spacetime curvature and could result in dissipative effects in quantum cosmology and quantum gravity [1, 2].

The fluctuations and their dynamics can be described using the general formalism of
time dependent Green functions to obtain the correlation of physical observables at different times. Some aspects of quantum fluctuations of commuting physical observables, like photon shot noise in optical systems or the Casimir force between plates, can be simulated as classical stochastic fluctuations with well defined probability distribution functions. However, it is not clear in this context to what extent it is possible to describe additional aspects of the dynamics of quantum fluctuations using classical stochastic concepts.

In bosonic systems classicality is commonly associated with highly populated coherent states that can result, for example, from high temperatures or strong particle production in cosmology. In this context issues concerning classicality and decoherence of quantum fluctuations during and after inflation, and their time evolution were addressed. Fermionic statistics, on the contrary, prevents any state to be highly populated and, moreover, fermionic coherent states cannot be identified with classical trajectories, so it is generally conceded that there is no issue of classicality in quantum fluctuations of fermionic fields.

In a recent paper we have developed a new formalism of pseudoclassical incoherent paths to describe the dynamics of quantum fluctuations. New aspects of classical behaviour, in particular classical dynamics and decoherence, emerge in this formalism associated to fluctuations of collective observables which depend on a large number of degrees of freedom, even in scarcely populated states like the vacuum (a conclusion anticipated in by different means). The notion of classicality that we propose is quite different than the case which is often discussed in the literature: it is associated to collective observables rather than to a certain class of quantum states, namely coherent states. In we elaborated these ideas for a system of weakly interacting bosons. In this paper we show that this notion of classicality is not exclusive to systems of bosons, as it also emerges associated to fluctuations of collective observables in systems with a large number of fermions.

In the new formalism quantum states can be represented, in the Heisenberg picture, as linear combinations of randomly distributed pseudoclassical paths (PCP’s) that do not interfere between themselves. We use each set of random PCP’s to define the dynamics of quantum fluctuations in that state of the system. We remark that any single PCP cannot be identified with any quantum state. Instead, a whole set of PCP’s with their corresponding probabilities to randomly happen at any time is the only formal object that can describe the dynamics of quantum fluctuations in one state of the system. On each
path of a set every physical observable is assigned a time-dependent value in a way that is consistent with the commutation/anticommutation relations between quantum operators. In particular, the generalized canonical coordinates and their conjugate momentum operators get time-dependent values which obey classical equations of motion and depict a collection of harmonic oscillators with constrained initial conditions. On the other hand, we notice that these pseudoclassical time-dependent values on paths do not necessarily respect the usual algebraic relations that classical observables fulfill, as a consequence of the non-commutative relations between quantum operators.

We select collective observables which depend on a large number of bosonic or fermionic independent degrees of freedom because we realize that their time-dependent values on paths do approximately regain the usual algebraic relations between classical observables and, therefore, we can use the formalism of PCP’s to describe the dynamics of quantum fluctuations of collective observables in terms of unconstrained classical stochastic processes.

The formalism of PCP’s, for bosons or fermions, shows that we can give a description of quantum mechanical states in terms of non-interfering paths if we trade it for a non-trivial definition of the algebraic relations between the values on paths of physical observables. This formalism can help to understand the process of decoherence and the onset of classicality in quantum systems because it can be applied to closed systems without explicit reference to observers, measurement, or an environment. In some cases it may also become a useful tool for performing calculations, in particular numerical simulations.

This paper is organized in five sections. In section II we review the fundamental concepts of the formalism of PCP’s and implement them in a system with a single fermion mode. In section III we extend the formalism to a system with two non-interacting fermions, and then we generalize it to systems with any finite number of free fermions. In section IV we define collective observables and describe their dynamics in the context of a regularized QFT. Section V contains a summary of results. This paper deals basically with linear fermionic fields. A discussion of the general case of interacting fields is postponed to a forthcoming work.
II. PSEUDOCLASSICAL PATHS OF A SINGLE FERMIONIC MODE

In this section we present a detailed description of a system with a single fermionic mode in the new formalism of PCP’s. We will start building the set of random paths which describes the dynamics of quantum fluctuations in the vacuum state and, afterwards, will show how to build the sets of paths that describe states other than the vacuum.

The hamiltonian of the system is

\[ H = \kappa \left( a^\dagger a - \frac{1}{2} \right), \]  

(1)

where the operators \( a^\dagger, a \), which create and annihilate the excitations of the mode, obey anticommutation relations \( \{ a^\dagger, a \} = 1 \) and \( \{ a^\dagger, a^\dagger \} = \{ a, a \} = 0 \). The last two relations imply, in particular, that \( (a^\dagger)^2 = a^2 = 0 \) and, therefore, the orthonormal basis of eigenstates of the hamiltonian, which linearly span the Hilbert space of the states of the system, contains only two independent vectors: \( \{ |0\rangle, |1\rangle \} \). The action of the operators \( a^\dagger, a \) on this basis is given by the equations

\[ a^\dagger |0\rangle = 0, \quad a |1\rangle = |0\rangle \quad \text{and} \quad a^\dagger |0\rangle = |1\rangle, \quad a^\dagger |1\rangle = 0. \]

We will fix for simplicity \( \kappa = 1 \) and, therefore, \( H |0\rangle = -\frac{1}{2} |0\rangle \) and \( H |1\rangle = +\frac{1}{2} |1\rangle \).

The operator \( a^\dagger \) is the hermitic conjugate of the operator \( a \), so that the couple of operators

\[ \xi = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad \bar{\xi} = \frac{i}{\sqrt{2}}(a - a^\dagger), \]

(2)

are hermitic and obey anticommutation relations: \( \{ \xi, \xi \} = \{ \bar{\xi}, \bar{\xi} \} = 1 \) and \( \{ \bar{\xi}, \xi \} = 0 \).

The action of these operators in the basis of eigenstates of the hamiltonian is summarized in the equations \( \xi |0\rangle = \frac{1}{\sqrt{2}} |1\rangle, \xi |1\rangle = \frac{1}{\sqrt{2}} |0\rangle \) and \( \bar{\xi} |0\rangle = -\frac{1}{\sqrt{2}} |1\rangle, \bar{\xi} |1\rangle = \frac{1}{\sqrt{2}} |0\rangle \), and, therefore, they can be identified with Pauli matrices, \( \xi = \frac{1}{\sqrt{2}} \sigma_1 \) and \( \bar{\xi} = -\frac{1}{\sqrt{2}} \sigma_2 \). Notice that \( \xi^2 = \bar{\xi}^2 = \frac{1}{2} \).

The most general linear operator that can be defined on the Hilbert space of the single fermionic mode is a polynom of the type, \( O = \alpha_1 \cdot 1 + \alpha_2 \cdot \xi + \alpha_3 \cdot \bar{\xi} + \alpha_4 \cdot i \xi \bar{\xi} \), where the coefficients \( \alpha_j, j = 1, 2, 3, 4 \) are, in general, complex numbers. This linear set of operators is the Grassmann algebra generated by the anticommuting hermitic operators \( \xi, \bar{\xi} \). The operator \( \xi \) (or independently, \( \bar{\xi} \)) forms a complete representation of commuting observables in the Hilbert space. An operator \( O \) is hermitic if and only if the coefficients \( \alpha_j, j = 1, 2, 3, 4 \) are all real. In particular, the hamiltonian is

\[ H = i \bar{\xi} \xi \]  

(3)
Our first aim is now to write the action of a generic operator $O$ on the vacuum state in terms of the action on the same state of some linear combination of the identity operator $1$ and the generator $\xi$. In general, it can be immediately checked that $O|0\rangle = \left((\alpha_1 - \frac{i}{2}\alpha_4)\cdot 1 + (\alpha_2 - i\alpha_3)\xi\right)|0\rangle \equiv \mathcal{P}_o(1,\xi)|0\rangle$. It is also immediate to check that the operator $\mathcal{P}_o(1,\xi)$ defined through this equation is unique as the identity $\beta_1 \cdot 1|0\rangle = \beta_2 \cdot \xi|0\rangle$ only holds if $\beta_1 = \beta_2 = 0$. From the identities,

$$\tilde{\xi}|0\rangle = -i\xi|0\rangle, \quad H|0\rangle = -\frac{1}{2}|0\rangle,$$  \hfill (4)

we obtain, for example, that $\mathcal{P}_\xi(1,\xi) = -i\xi$ and $\mathcal{P}_h(1,\xi) = -\frac{1}{2} \cdot 1$.

The next stage in our programme is to identify the two eigenstates of the operator $\xi$: $|q_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|q_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, and their corresponding eigenvalues, $\lambda_+ = \frac{1}{\sqrt{2}}$ and $\lambda_- = -\frac{1}{\sqrt{2}}$. Then, we expand the vacuum state in the new basis: $|0\rangle = \frac{1}{\sqrt{2}}(|q_+\rangle + |q_-\rangle) \equiv \sum_{q=q_\pm}\psi(q)|q\rangle$, with $\psi(q_{\pm}) = \frac{1}{\sqrt{2}}$. The wavefunction of the vacuum in the basis of eigenstates of $\xi$, $\psi(q)$, is equally valued on each of the two disconnected points $|q_+\rangle, |q_-\rangle$ of the configuration space and so the probability $|\psi(q_{\pm})|^2 = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$ of each of them to happen randomly.

Moreover, using the relation $\xi|0\rangle = \frac{1}{\sqrt{2}}(\xi|q_+\rangle + \xi|q_-\rangle) = \frac{1}{\sqrt{2}}(\lambda_+|q_+\rangle + \lambda_-|q_-\rangle)$ we can assign to the operator $\xi$ the classical value

$$\xi_{cl}(q_+) = \lambda_+ = \frac{\langle q_+|\xi|0\rangle}{\langle q_+|0\rangle} = \frac{1}{\sqrt{2}}$$  \hfill (5)

on the first point, and

$$\xi_{cl}(q_-) = \lambda_- = \frac{\langle q_-|\xi|0\rangle}{\langle q_-|0\rangle} = -\frac{1}{\sqrt{2}}$$  \hfill (6)

on the second point of the configuration space. Thus, we have defined on configuration space a random variable $\xi_{cl}(q)$.

Furthermore, following the identity $O|0\rangle = \mathcal{P}_o(1,\xi)|0\rangle = \frac{1}{\sqrt{2}}(\mathcal{P}_o(1,\xi_{cl}(q_+))|q_+\rangle + \mathcal{P}_o(1,\xi_{cl}(q_-))|q_-\rangle)$, the generic operator $O$ should be assigned the random variable

$$O_{cl}(q_{\pm}) = \mathcal{P}_o(1,\xi_{cl}(q_{\pm})) = \frac{\langle q_{\pm}|\mathcal{P}_o(1,\xi)|0\rangle}{\langle q_{\pm}|0\rangle},$$  \hfill (7)

that gives the operator a pseudoclassical value at each point $|q_{\pm}\rangle$ of the vacuum configuration space. For example, looking at (4) we assign to the operator $\tilde{\xi}$ the random variable $\tilde{\xi}_{cl}(q)$ which takes the value

$$\tilde{\xi}_{cl}(q_+)= -i\xi_{cl}(q_+) = \frac{-i}{\sqrt{2}}$$  \hfill (8)
on the first point of the configuration space and
\[ \tilde{\xi}_c(q_-) = -i \xi_c(q_-) = \frac{i}{\sqrt{2}}. \] (9)
on the second point. Also according to (4), the free hamiltonian (3) is assigned on each of these points the constant value
\[ h_c(q_{\pm}) = -\frac{1}{2} = \langle 0|H|0 \rangle. \] (10)

Let then go a step further to describe the time dependence of the random variables we have just defined. In the Heisenberg picture, time dependence of the operator \( O \) is described by the expression
\[ O(t) = e^{iHt}Oe^{-iHt}, \]
where \( H \) is the hamiltonian of the system. The observable \( O(t) \) can then be associated following the same algorithm that we have described above with a new random variable \( O_c(t, q_{\pm}) \). Thus, each physical observable is given a time-dependent c-value at each point \( |q \rangle \) of the configuration space, that is a path whose realization probability is \( |\psi(q)|^2 \).

In particular, the time evolution of the operators \( \xi \) and \( \tilde{\xi} \) is described by the expressions
\[ \xi(t) = e^{iHt}\xi e^{-iHt} = \cos(t)\xi - \sin(t)\tilde{\xi} \] (11)
\[ \tilde{\xi}(t) = e^{iHt}\tilde{\xi} e^{-iHt} = \sin(t)\xi + \cos(t)\tilde{\xi}, \] (12)
which solve the differential equations
\[ \frac{d\xi(t)}{dt} = -\tilde{\xi}(t), \quad \frac{d\tilde{\xi}(t)}{dt} = \xi(t). \] (13)

On the vacuum state \( |0 \rangle \) these equations imply (see (4))
\[ \xi(t)|0 \rangle = e^{it}\xi|0 \rangle, \quad \tilde{\xi}(t)|0 \rangle = -ie^{it}\xi|0 \rangle, \] (14)
which, according to our formalism, mean that in the ground state \( |0 \rangle \) the time-dependent classical values of this pair of operators on each of the two paths \( |q_{\pm} \rangle \) are \( \xi_c(t, q_{\pm}) = e^{it}\xi_c(q_{\pm}) \) and \( \tilde{\xi}_c(t, q_{\pm}) = -ie^{it}\xi_c(q_{\pm}) \). They obey the classical equations of an harmonic oscillation
\[ \frac{d\xi_c(t; q_{\pm})}{dt} = -\tilde{\xi}_c(t; q_{\pm}), \quad \frac{d\tilde{\xi}_c(t; q_{\pm})}{dt} = \xi_c(t; q_{\pm}). \] (15)
with initial conditions fixed by (5 and 8) on the first path, and (6 and 9) on the second path. The operator \( i\tilde{\xi}\xi \) commutes with the hamiltonian so it does not change with time and, therefore, neither its classical value (10) on each path does.
The formalism of PCP’s can be extended to describe any other state in the Hilbert space of the single fermionic mode. Consider, for example, the normalized state $|\Psi\rangle = \cos(\theta)|0\rangle + \sin(\theta)e^{i\phi}|1\rangle$ and let expand it in the basis $\{|q_{\pm}\rangle\}$ of eigenstates of the generator $\xi$:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( (\cos(\theta) + \sin(\theta)e^{i\phi})|q_{+}\rangle + (\cos(\theta) - \sin(\theta)e^{i\phi})|q_{-}\rangle \right). \quad (16)$$

On each of the two points $|q_{\pm}\rangle$ in the configuration space the random variable $\xi_{cl}(q_{\pm})$ takes the value $\frac{\langle q_{\pm}|\xi|\Psi\rangle}{\langle q_{\pm}|\Psi\rangle} = \pm \frac{\sqrt{2}}{\sqrt{2}}$, identical to those specified by eq. $(5)$ and $(6)$. In order to obtain in the new quantum state $|\Psi\rangle$ the random variable which corresponds to the generic operator $O$ we must find in the linear subspace spanned by the identity operator 1 and the operator $O$ on the vacuum state:

$$O|\Psi\rangle = \mathcal{P}_o(1, \xi)|\Psi\rangle. \quad (17)$$

This can be done through the following three steps: 1) we write the state $|\Psi\rangle$ as a linear combination of the operators 1 and $\xi$ acting on the vacuum: $|\Psi\rangle = (\cos(\theta) \cdot 1 + \sqrt{2}\sin(\theta)e^{i\phi}\xi)|0\rangle$; 2) now, use the rules that we have stated before in this section in order to obtain the action of the operator $O$ on the state $|\Psi\rangle$ in terms of the action of some new linear combination of the operators 1 and $\xi$ on the vacuum state: $O|\Psi\rangle = O(\cos(\theta) + \sqrt{2}\sin(\theta)e^{i\phi}\xi)|0\rangle = G(1, \xi)|0\rangle$; 3) note that the relation introduced in step 1) can be inverted into the relation $|0\rangle = \frac{1}{\cos^{2}(\theta) - \sin^{2}(\theta)e^{2i\phi}} (\cos(\theta) - \sqrt{2}\sin(\theta)e^{i\phi}\xi)|\Psi\rangle$, which is then introduced in the expression that we obtained from step 2) in order to get, as desired, the action of operator $O$ on state $|\Psi\rangle$ in terms of the action of some linear combination $\mathcal{P}_o(1, \xi)$ of the operators 1 and $\xi$ on such quantum state. The three described steps of this algorithm are well and uniquely defined, so the operator $\mathcal{P}_o(1, \xi)$ is unique for each operator $O$.

In particular, for the operator $\tilde{\xi}$ we obtain the identity

$$\tilde{\xi}|\Psi\rangle = \frac{i}{\cos^{2}(\theta) - \sin^{2}(\theta)e^{2i\phi}} (\sqrt{2}\sin(\theta)\cos(\theta)e^{i\phi} - (\cos^{2}(\theta) + \sin^{2}(\theta)e^{2i\phi})\xi)|\Psi\rangle, \quad (18)$$

which means that this operator is assigned, when the system is in the quantum state $|\Psi\rangle$, the random variable $\tilde{\xi}(q_{\pm}) = \frac{\langle q_{\pm}|\tilde{\xi}|\Psi\rangle}{\langle q_{\pm}|\Psi\rangle}$ whose value at each point of the configuration space is:

$$\tilde{\xi}_{cl}(q_{\pm}) = \frac{i}{\cos^{2}(\theta) - \sin^{2}(\theta)e^{2i\phi}} (\sqrt{2}\sin(\theta)\cos(\theta)e^{i\phi} - (\cos^{2}(\theta) + \sin^{2}(\theta)e^{2i\phi})\xi_{cl}(q_{\pm})). \quad (19)$$
The dynamical equations (13), when applied on the quantum state $|\Psi\rangle$,

$$\frac{d\xi(t)|\Psi\rangle}{dt} = -\tilde{\xi}(t)|\Psi\rangle,$$
$$\frac{d\tilde{\xi}(t)|\Psi\rangle}{dt} = \xi(t)|\Psi\rangle, \quad (20)$$

enforce that classical values of the operators $\xi$ and $\tilde{\xi}$ on each pseudoclassical path of the random set that describes the state $|\Psi\rangle$ still evolve according to classical equations of motion (14). Initial conditions are now fixed on each path by (5) and (6), respectively, for $\xi_{cl}(t=0)$ and (19) for $\tilde{\xi}_{cl}(t=0)$, instead of (8), (9). In general, the state of the system $|\Psi\rangle$ fixes only the initial conditions for its set of PCP’s, while the dynamics of these paths subsequently follows classical equations.

The action of the hamiltonian (3) on the quantum state $|\Psi\rangle$ can be expressed, using (18), as

$$H|\psi\rangle = -i\xi\tilde{\xi}|\Psi\rangle = \frac{1}{\cos^2(\theta) - \sin^2(\theta)e^{2i\phi}}(\sqrt{2}\sin(\theta)\cos(\theta)e^{i\phi}\xi - (\cos^2(\theta) + \sin^2(\theta)e^{2i\phi})/2)|\Psi\rangle, \quad (21)$$

which implies that the two paths of the set that describes quantum fluctuations in the generic state $|\Psi\rangle$ do not necessarily have the same energy, but

$$h_{cl}(q_\pm) = \frac{1}{\cos^2(\theta) - \sin^2(\theta)e^{2i\phi}}(\sqrt{2}\sin(\theta)\cos(\theta)e^{i\phi}\xi_{cl}(q_\pm) - (\cos^2(\theta) + \sin^2(\theta)e^{2i\phi})/2). \quad (22)$$

The two pseudoclassical paths either are no longer equally probable as can be seen from equation (16): when the system is in the quantum state $|\Psi\rangle$ the probability of the point $|q_+\rangle$ to happen randomly is $|\psi(q_+)|^2 = \frac{1}{2}[\cos(\theta) + \sin(\theta)e^{i\phi}]^2$, while the probability of the point $|q_-\rangle$ is $|\psi(q_-)|^2 = \frac{1}{2}[\cos(\theta) - \sin(\theta)e^{i\phi}]^2$.

Notice that although expressions (19) and (22) are ill-defined at the event $|q_-\rangle$ when $e^{i\phi} = \pm 1$ and $\sin(\theta) = \pm \cos(\theta)$, the probability of such event to happen randomly is zero, while the other event $|q_+\rangle$, then, should have an absolute probability equal to 1 to happen. If we consider this case as a limiting case when $\theta \rightarrow +\frac{\pi}{4}, +\frac{5\pi}{4}$ we can see that event $|q_-\rangle$ becomes less and less probable, but the corresponding values for the variables $\tilde{\xi}_{cl}(q_-)$ and $h_{cl}(q_-)$ grow towards infinity. In the case of the latter variable the probability of the event decreases faster than the divergence of the variable, so that this event contributes and infinitesimal amount to the average or expected value of the classical value $h_{cl}$ over the set of paths. On the contrary, the divergence in the expression for the former variable grows fast enough to contribute a finite amount of half the value of its total average $<\tilde{\xi}>$ over
paths, even though the event has a tiny probability to happen. The situation is similar for the event \( |q_+\rangle \) when \( e^{i\phi} = \pm 1 \) and \( \sin(\theta) = \mp \cos(\theta) \).

Let us add that although \( \xi_d(t, q_\pm) \) and \( \tilde{\xi}_d(t, q_\pm) \) obey classical equation of an harmonic oscillation, the energy of the PCP, as defined by equation (22), is not necessarily equal to \( \frac{1}{2}((\xi_d)^2 + (\tilde{\xi}_d)^2) \).

We will finish this section with two results that were already noticed in [8] but we remind them here for their relevance in the discussion of the next sections. The first result:

\[
\langle \Psi | O | \Psi \rangle = \langle \Psi | \mathcal{P}_o^\psi(1, \xi) | \Psi \rangle = O_{cl}(q_+)\langle \psi(q_+) |^2 + O_{cl}(q_-)\langle \psi(q_-) |^2 = \langle O_{cl} >_{PCP}, \tag{23}
\]

means not only that the average of pseudoclassical values of the observable \( O \) over the whole set or ensemble of random PCP’s that describes the state \( |\Psi\rangle \) is equal to the average value of the operator \( O \) in this state, but it also means that PCP’s do not interfere between themselves.

The second result on PCP’s that we want to bring to attention is related to the second momentum of the distribution of pseudoclassical values of the generic operator \( O \) over the ensemble of random paths:

\[
< (O^\dagger O)_{cl} >_{PCP} = \langle \Psi | O^\dagger O | \Psi \rangle = \langle \Psi | (\mathcal{P}_o^\psi(1, \xi))^\dagger \mathcal{P}_o^\psi(1, \xi) | \Psi \rangle = \langle O_{cl}^* O_{cl} >_{PCP}, \tag{24}
\]

It means that the average on PCP’s of pseudoclassical values for the operator \( O^\dagger O \) is equal to the second momentum of the distribution of \( O_{cl} \) over the paths. Although this equation holds in average for any operator \( O \) it does not necessarily hold on each one of the paths as can be readily seen, for instance, from equation (19): while \( (\tilde{\xi}^\dagger \xi)_{cl} = (\tilde{\xi}^2)_{cl} = \frac{1}{2} \), we find after some algebra that \( (\tilde{\xi}_d)^* (\tilde{\xi}_d) = \frac{(1+ \frac{1}{2} \sin^2(2\theta)(1+\cos(2\phi)) \mp 2 \sin(2\theta)(\cos^2(\theta)\cos(\phi) \mp \sin^2(\theta)\cos(3\phi)))}{2(1- \frac{1}{2} \sin^2(2\theta)(1+\cos(2\phi)))} \), where the \( \mp \) sign in the numerator stands for the value of the random variable at \( |q_+\rangle \) and \( |q_-\rangle \), respectively. Only after averaging these two possible values with their corresponding probabilities we recover the identity (24). This is the reason why we call these paths pseudoclassical paths: although they obey classical equations of motion, the values on paths of physical observables which do not commute with the representation of commuting observables we have chosen (in this example, the operator \( \xi \)) do not fulfill the usual algebraic relations between classical observables. Or stated in other words, \((O^\dagger O)_{cl} - (O_{cl})^* (O_{cl})\) is not necessarily zero on each one of the paths, neither even on any of them. This is a consequence of the non-commutative relations between operators in the algebra of quantum observables.
We will return to discuss this aspect of the formalism at the end of next section when we will introduce collective observables and show that classical algebraic relations are naturally recovered for this kind of operators.

III. PSEUDCLASSICLAL PATHS IN A SYSTEM OF MANY FERMIONS

In this section we discuss how to extend the formalism of pseudoclassical paths to systems that contain more than one fermion mode, for example a fermionic quantum field. The basic ideas are the same that we have already presented in the previous section, but there are now some subtleties related to the anticommutation relations between operators associated to different fermionic modes that we would like to notice. We will start, for the sake of simplicity, with a system with two free fermions whose Hamiltonian is:

\[ H = \kappa_a ^a a - \frac{1}{2} + \kappa_b ^b b - \frac{1}{2} = H_a + H_b. \]  

The operators \( a^\dagger, a \) create and annihilate excitations of a first fermion mode, and \( b^\dagger, b \) create and annihilate excitations of a second fermion mode. They obey anticommutation relations \( \{ a^\dagger, a \} = \{ b^\dagger, b \} = 1 \), and the anticommutator between any other pair of these operators is equal to zero. In particular, those relations imply that \( (a^\dagger)^2 = a^2 = (b^\dagger)^2 = b^2 = 0 \) and, moreover, the operations that create or annihilate excitations of the first fermionic mode anticommute, instead of commuting, with the operations that create or annihilate excitations of the second fermionic mode.

The basis of eigenstates of the Hamiltonian (25) can be obtained applying the creation operators \( a^\dagger, b^\dagger \) on the vacuum state \( |0; 0\rangle \). The anticommutation relations between the operators guarantee that the Fock space they generate contains only antisymmetric vectors:

\[ |0; 0\rangle \quad |1; 0\rangle \equiv a^\dagger |0; 0\rangle \quad |0; 1\rangle \equiv b^\dagger |0; 0\rangle \quad |1; 1\rangle \equiv a^\dagger b^\dagger |0; 0\rangle. \]  

Their corresponding eigenvalues can be read of the equalities

\[ H |0; 0\rangle = -\frac{\kappa_a + \kappa_b}{2} |0; 0\rangle \quad H |1; 0\rangle = +\frac{\kappa_a - \kappa_b}{2} |1; 0\rangle \quad H |0; 1\rangle = -\frac{\kappa_a - \kappa_b}{2} |0; 1\rangle \quad H |1; 1\rangle = +\frac{\kappa_a + \kappa_b}{2} |1; 1\rangle. \]  

The operators \( a^\dagger, b^\dagger \) are the hermitic conjugates of \( a \) and \( b \), respectively, so we can define two couples of hermitic linear combinations of them: the hermitic operators \( \xi, \tilde{\xi} \) introduced
in (2), together with the couple
\[ \zeta = \frac{1}{\sqrt{2}} (b + b^\dagger), \quad \bar{\zeta} = \frac{i}{\sqrt{2}} (b - b^\dagger). \tag{29} \]
These four operators anticommute each other:
\[ \{\xi, \bar{\xi}\} = \{\xi, \zeta\} = \{\bar{\xi}, \zeta\} = \{\bar{\xi}, \bar{\zeta}\} = \{\zeta, \bar{\zeta}\} = 0, \tag{30} \]
and each one is proportional to the identity operator when it is multiplied by itself, \( \xi^2 = \bar{\xi}^2 = \zeta^2 = \bar{\zeta}^2 = \frac{1}{2} \). They generate the Grassmann algebra of linear operators defined on the Hilbert space spanned by (26).

The two components \( H_a \) and \( H_b \) of the hamiltonian (25) can be expressed in terms of the new operators as \( H_a = \kappa_a (i\bar{\xi}\xi) \) and \( H_b = \kappa_b (i\bar{\zeta}\zeta) \), which commute each other \([H_a, H_b] = 0\). Therefore, \( e^{iHt} = e^{i(H_a+H_b)t} = e^{iH_ant} e^{iH_b t} \), and
\[ \xi(t) = e^{iHt}\xi e^{-iHt} = e^{iH_ant}\xi e^{-iH_ant}, \tag{31} \]
while
\[ \zeta(t) = e^{iHt}\zeta e^{-iHt} = e^{iH_b t}\zeta e^{-iH_b t}. \tag{32} \]
Similar expressions are obtained for \( \bar{\xi}(t) \) and \( \bar{\zeta}(t) \) and they imply
\[ \frac{d\xi(t)}{dt} = -\kappa_a \bar{\xi}(t), \quad \frac{d\bar{\xi}(t)}{dt} = \kappa_a \xi(t) \tag{33} \]
\[ \frac{d\zeta(t)}{dt} = -\kappa_b \bar{\zeta}(t), \quad \frac{d\bar{\zeta}(t)}{dt} = \kappa_b \zeta(t). \tag{34} \]
In order to build the set of PCP’s which describes a generic quantum state \( |\Psi\rangle \) of this system we need to choose a complete representation of commuting observables. The operators \( \xi \) and \( \zeta \) do not commute each other and, therefore, are not eligible as such a representation. Instead we use the pair of hermitic operators \( \xi \) and \( i\bar{\xi}\zeta \) that, according to (30), do commute each other.

So we expand the state \( |\Psi\rangle = \sum_q \psi(q)|q\rangle \) in the basis of common eigenstates \( |q\rangle \) to the complete representation of commuting observables that we have chosen. We know from the discussion of the previous section that each of these eigenstates is actually promoted to one PCP, whose probability to happen at any time is \( |\Psi(q)|^2 \). Each observable of our representation is naturally given as its classical value at each of these eigenstates its own
eigenvalue on it and then, exploiting identities of the kind \( O(t)\langle \Psi \rangle = \mathcal{P}_o^0(1, \xi, i\tilde{\zeta}; t)\langle \Psi \rangle \), any other physical observable \( O \) is also assigned a time-dependent value on each of them:

\[
O_{cl}(t; q) = \mathcal{P}_o^0(t; 1, \xi_{cl}(q), (i\tilde{\zeta})_{cl}(q)).
\] (35)

We can see from this last expression that whatever the quantum state \( \langle \Psi \rangle \) is, equations (33) and (34) enforce when they are applied on such state (see eq. (20)) that the two couples of observables \( \xi, \tilde{\xi} \) and \( \zeta, \tilde{\zeta} \) will be assigned following this formalism time-dependent values that obey classical equations of two decoupled harmonic oscillators,

\[
\begin{align*}
\frac{d\xi_{cl}(t, q)}{dt} &= -\kappa_a \tilde{\xi}_{cl}(t, q), & \frac{d\tilde{\xi}_{cl}(t, q)}{dt} &= \kappa_a \xi_{cl}(t, q) \tag{36} \\
\frac{d\zeta_{cl}(t, q)}{dt} &= -\kappa_b \tilde{\zeta}_{cl}(t, q), & \frac{d\tilde{\zeta}_{cl}(t, q)}{dt} &= \kappa_b \zeta_{cl}(t, q), \tag{37}
\end{align*}
\]
on each PCP, generically labeled as \( \langle q \rangle \), of the random set which describes that state \( \langle \Psi \rangle \).

As an example we will explicitly construct here the set of PCP’s that describes the vacuum state \( \langle 0; 0 \rangle \) of this system. The generalization to other states is straightforward following the three-steps algorithm discussed in the previous section and we will only outline how to apply it to the two-fermions system.

The first stage then is to find the way to express the action on the vacuum state \( \langle 0; 0 \rangle \) of any operator \( O \) in the Grassmann algebra generated by \( \xi, \tilde{\xi}, \zeta \) and \( \tilde{\zeta} \) in terms of the action on the same state of some linear combination of the set of operators generated by the complete representation of commuting observables, \( \xi \) and \( i\tilde{\zeta} \):

\[
O\langle 0; 0 \rangle = (\alpha_1 \cdot 1 + \alpha_2 \cdot \xi + \alpha_3 \cdot (i\tilde{\zeta}) + \alpha_4 \cdot \xi(i\tilde{\zeta}))\langle 0; 0 \rangle \equiv \mathcal{P}_o(1, \xi, i\tilde{\zeta})\langle 0; 0 \rangle. \tag{38}
\]
The operator \( \mathcal{P}_o(1, \xi, i\tilde{\zeta}) \) defined by this expression is unique because the action of the identity operator 1 on the vacuum does not excite any mode, the action of the operator \( \xi \) excites the mode \( \langle 1; 0 \rangle \), the action of the operator \( (i\tilde{\zeta}) \) excites \( \langle 1; 1 \rangle \) and the action of \( (i\xi\tilde{\zeta}) \) excites \( \langle 0; 1 \rangle \). Therefore, the only linear combination of these four operators which can give the number zero when acting on the vacuum is the trivial combination.

For example, we already know that \( \tilde{\xi}\langle 0; 0 \rangle = -i\xi\langle 0; 0 \rangle \) because of the identity \( a\langle 0; 0 \rangle = 0 \). A little more tricky is the way we describe the action of the operator \( \zeta \) on the vacuum. We first note that \( \zeta = 2\tilde{\xi}(\tilde{\xi}\zeta) = -2(\xi\zeta)\tilde{\xi} \). Therefore, \( \zeta\langle 0; 0 \rangle = -2(\xi\zeta)\tilde{\xi}\langle 0; 0 \rangle \), and substituting in this expression the previous result we get \( \zeta\langle 0; 0 \rangle = 2(i\xi\zeta)\tilde{\xi}\langle 0; 0 \rangle = 2\xi(i\tilde{\xi})\langle 0; 0 \rangle \). Now,
using the identity $b|0;0\rangle = 0$, we get $\tilde{\zeta}|0;0\rangle = -i\zeta|0;0\rangle = -2i\xi(i\tilde{\zeta})|0;0\rangle$. Using the anticommutation relations between the generators of the Grassmann algebra we can write down similar expression for any other operator $O$.

The next stage is to obtain the basis of common eigenvectors to the complete representation of commuting observables we have chosen, $\xi$ and $(i\tilde{\zeta})$:

$$|q_+;q_+\rangle \equiv \frac{1}{2} \left((|0;0\rangle + |1;0\rangle) + (|0;1\rangle + |1;1\rangle)\right)$$

$$|q_+;q_-\rangle \equiv \frac{1}{2} \left((|0;0\rangle + |1;0\rangle) - (|0;1\rangle + |1;1\rangle)\right)$$

$$|q_-;q_+\rangle \equiv \frac{1}{2} \left((|0;0\rangle - |1;0\rangle) - (|0;1\rangle - |1;1\rangle)\right)$$

$$|q_-;q_-\rangle \equiv \frac{1}{2} \left((|0;0\rangle - |1;0\rangle) + (|0;1\rangle - |1;1\rangle)\right),$$

and their corresponding eigenvalues,

$$\xi|q_+;q_+\rangle = \frac{1}{\sqrt{2}}|q_+;q_+\rangle \quad \text{and} \quad i\tilde{\zeta}|q_+;q_+\rangle = +\frac{1}{2}|q_+;q_+\rangle$$

$$\xi|q_+;q_-\rangle = \frac{1}{\sqrt{2}}|q_+;q_-\rangle \quad \text{and} \quad i\tilde{\zeta}|q_+;q_-\rangle = -\frac{1}{2}|q_+;q_-\rangle$$

$$\xi|q_-;q_+\rangle = -\frac{1}{\sqrt{2}}|q_-;q_+\rangle \quad \text{and} \quad i\tilde{\zeta}|q_-;q_+\rangle = +\frac{1}{2}|q_-;q_+\rangle$$

$$\xi|q_-;q_-\rangle = -\frac{1}{\sqrt{2}}|q_-;q_-\rangle \quad \text{and} \quad i\tilde{\zeta}|q_-;q_-\rangle = -\frac{1}{2}|q_-;q_-\rangle.$$
Of course, \((i\xi\tilde{\zeta})_{cl}(q_{\pm};q_{\pm}) = \xi_{cl}(q_{\pm};q_{\pm}) \cdot (i\xi\tilde{\zeta})_{cl}(q_{\pm};q_{\pm})\), because the two operators of our representation commute each other. Then, \(\zeta_{cl}(q_{\pm};q_{\pm}) = 2(i\xi\tilde{\zeta})_{cl}(q_{\pm};q_{\pm}) \cdot \xi_{cl}(q_{\pm};q_{\pm})\) and \(\tilde{\zeta}_{cl}(q_{\pm};q_{\pm}) = -i\xi_{cl}(q_{\pm};q_{\pm})\). The dependence on time of these random variables can now be obtained from the equations of motion \((36), (37)\).

Once we have stated the rules to define the random variables on configuration space associated to each operator we can pick \(\xi_{cl}\) and \(\zeta_{cl}\) as the two independent variables, noticing then that \((i\xi\tilde{\zeta})_{cl} = \frac{1}{2}\xi_{cl}/\xi_{cl}\). In the vacuum the four equally probable random events in the configuration space correspond to the four possibilities: \(\xi_{cl} = \pm \frac{1}{\sqrt{2}}\), \(\zeta_{cl} = \pm \frac{1}{\sqrt{2}}\).

The whole formalism can be repeated for any quantum state \(|\Psi\rangle\) other than the vacuum. First, we need to expand the state \(|\Psi\rangle\) in the basis \((39)-(42)\): \(|\Psi\rangle = \psi(+/+)|q_{+};q_{+}\rangle + \psi(+-)|q_{+};q_{-}\rangle + \psi(-+)|q_{-};q_{+}\rangle + \psi(--)|q_{-};q_{-}\rangle\). Each PCP correspond to one of the points \(|q_{\pm}^{a};q_{\pm}^{b}\rangle\) in the configuration space and its probability to happen randomly is equal to the modulus squared of the amplitude of the wavefunction, \(|\langle q|\Psi\rangle|^{2}\). The four events are no longer necessarily equally probable, but the classical values on each of them of the random variables assigned to the operators that form our complete representation of commuting observables are still given by their corresponding eigenvalues \((43)-(46)\). Then, the action of the operator \(O(t)\) on the state \(|\Psi\rangle\) is written as the action on the vacuum of some operator \(\mathcal{G}(t;1,\xi,i\tilde{\zeta})\): \(O(t)|\Psi\rangle = (\chi_{0,0}(t) \cdot 1 + \chi_{1,0}(t) \cdot \xi + \chi_{1,1}(t) \cdot (i\tilde{\zeta}) + \chi_{0,1}(t) \cdot \xi(i\tilde{\zeta}))|0;0\rangle \equiv \mathcal{G}(t;1,\xi,i\tilde{\zeta})|0;0\rangle\). This step is always possible and uniquely defined because the action on the vacuum of each one of the four operators \(1,\xi,(i\tilde{\zeta})\) and \((i\xi\tilde{\zeta})\) excites a different mode in the basis \((26)\) of eigenstates of the hamiltonian \((25)\). Finally, the relation between the states \(|\Psi\rangle\) and \(|0;0\rangle\) is inverted \(|0;0\rangle = \mathcal{J}(1,\xi,i\tilde{\zeta})|\Psi\rangle\) and introduced in the previous expression to get the desired result: \(O(t)|\Psi\rangle = \mathcal{G}(t;1,\xi,i\tilde{\zeta})\mathcal{J}(1,\xi,i\tilde{\zeta})|\Psi\rangle = \mathcal{P}_{cl}^{\psi}(t;1,\xi,i\tilde{\zeta})|\Psi\rangle\). This last expression can be used to define the classical value of the operator \(O\) in \(|q\rangle\) when the quantum system is described in its state \(|\Psi\rangle\) as \(O_{cl}(t;q) = \mathcal{P}_{cl}^{\psi}(t;1,\xi_{cl}(q), (i\xi\tilde{\zeta})_{cl}(q))\).

We see that although each PCP corresponds to one of the eigenstates \(|q_{\pm};q_{\pm}\rangle\), the way how we actually define the classical values of physical observables explicitly depends on the quantum state of the system \(|\Psi\rangle\). Therefore, the PCP cannot be identified with the quantum state described by the eigenstate \(|q\rangle\). Instead the whole set of four random PCP’s, which according to \((33)-(34)\) should still obey classical equation of motion, does describe the dynamics of the system in its quantum state \(|\Psi\rangle\). The state of the system, therefore, enters this description by: first, fixing the different probabilities of each of the four points of the
configuration space; second, defining the initial conditions on the set of classical equations (33)-(34) for each of the PCP’s; third, defining the algebraic relations between the values of different physical observables on each PCP.

The formalism can now be straightforward generalized to a fermionic system containing many modes. We define a couple of hermitic operators \( \xi_l, \bar{\xi}_l \) associated to each one of these modes. The integer index \( l = 0, 1, 2, ... \) labels the different modes. The set of operators \( \xi_0, i\bar{\xi}_0\xi_1, i\bar{\xi}_1\xi_2, ... \) forms a complete representation of commuting observables and the Hilbert space of the states of the system can be linearly spanned in the basis of common eigenstates to all the operators in this representation. Each common eigenstate can be labeled by a sequence of + or - signs (for example, \(|+, +, -, +, -, -, ..., \rangle\)) each input corresponding to one of the fermionic modes. In total, \( 2^N \) different eigenstates, where \( N \) is the number of different modes. A + sign in the zeroth position corresponds to an eigenstate with eigenvalue \( \frac{1}{\sqrt{2}} \) for the operator \( \xi_0 \). Inversely, a - sign correspond to an state with eigenvalue \( \frac{1}{\sqrt{2}} \) for this operator. And similarly, a \( \pm \) sign in the \( l \)-th position corresponds to an state with eigenvalue \( \pm \frac{1}{2} \) for the operator \( i\bar{\xi}_{l-1}\xi_l, l \geq 1 \). In the quantum state \( |\Psi\rangle = \sum_q \psi(q)|q\rangle \) each of the \( 2^N \) random pseudoclassical paths \( |q\rangle \) which describe the system has a probability \( |\psi(q)|^2 \) to happen. In the vacuum all the paths are equally probable and, therefore, this probability is \( 1/2^N \).

Let us now turn back to equations (23) and (24), which describe the statistical behaviour of the random variables over the ensemble of PCP’s. They relate the first and second momentum of the random variable \( O_{cl} \) to the expected values of the operator \( O \) and \( O^\dagger O \), respectively. The proof of these equations that we gave in the previous section also proves the assertions for systems with many fermionic modes. We noticed then that (24) holds only on average over the whole ensemble of PCP’s but does not necessarily hold on each specific PCP and understood that, as a consequence of the non-commutative relations between quantum operators, the usual algebraic relations between classical observables are not necessarily fulfilled by the time-dependent values \( O_{cl}(t) \) defined on the non-interfering paths. There are, nevertheless, a certain class of operators, which we call ”collective” operators for reasons that will become clear in the next section, for which \( (O^\dagger O)_{cl} - (O_{cl})^*O_{cl} \sim 0 \) not only in the average sense of equation (24), but under the stronger requirement that

\[
\sigma_O \equiv \left[ <(O^\dagger O)_{cl} - (O_{cl})^*O_{cl}>_{PCP}^2 \right]^{1/2} \ll <(O_{cl})^*O_{cl}>_{PCP} .
\]
In this sense \((O^\dagger O)_\text{cl} \sim (O_{\text{cl}})^* O_{\text{cl}}\) on each PCP independently of the quantum state \(|\Psi\rangle\) that they describe and we can conclude that values on paths of collective observables approximately recover the usual algebraic relations between classical observables. This condition is naturally satisfied for some operators which depend on the dynamics of many modes because for these operators the random variable \((O^\dagger O)_\text{cl} - (O_{\text{cl}})^* O_{\text{cl}}\) can be expressed as a quadratic function of many other independent random variables associated to individual modes and, according to the central limit theorem, condition (48) is satisfied. Moreover, higher statistical moments of the collective random variable are highly suppressed.

IV. PCP’S IN FERMIONIC FIELD THEORIES

We are now ready to develop the formalism of PCP’s for the Dirac theory of free spinors. We consider a single fermionic field \(\psi(x)\) in Minkowski space-time, whose lagrangian density is

\[
\mathcal{L} = \bar{\psi} (i \partial_\mu \gamma^\mu - M) \psi,
\]

where \(\gamma^\mu\) are 4D Clifford matrices and \(M\) is the mass of the spinor field. The hamiltonian of the system is:

\[
H = \int d^3x \, \bar{\psi}(\vec{x}) (-i \gamma^j \partial_j + M) \psi(\vec{x}).
\]

In order to regularize the theory in the infrared limit we impose periodic boundary conditions on the large three-dimensional box \([0, X] \times [0, X] \times [0, X]\). The spinor field \(\psi(\vec{x})\) and its conjugate \(\bar{\psi}(\vec{x}) \equiv \psi^\dagger(\vec{x}) \gamma^0\) can then be expanded in a discrete series of Fourier modes:

\[
\psi(\vec{x}) = \sum_{n_1, n_2, n_3 = -N}^{+N} \frac{1}{X^{3/2}} \sum_s \left( a^s_{\vec{n}} u^s_{\vec{n}} e^{+2\pi i \vec{n} \cdot \vec{x}/X} + b^s_{\vec{n}} \bar{u}^s_{\vec{n}} e^{-2\pi i \vec{n} \cdot \vec{x}/X} \right),
\]

\[
\bar{\psi}(\vec{x}) = \sum_{n_1, n_2, n_3 = -N}^{+N} \frac{1}{X^{3/2}} \sum_s \left( a^s_{\vec{n}} \bar{u}^s_{\vec{n}} e^{-2\pi i \vec{n} \cdot \vec{x}/X} + b^s_{\vec{n}} \bar{u}^s_{\vec{n}} e^{+2\pi i \vec{n} \cdot \vec{x}/X} \right).
\]
the equation \([-i\gamma^j \partial_j + M] u^s_n e^{+2\pi i n \cdot \vec{x}/X} = \kappa_n \gamma^0 u^s_n e^{+2\pi i n \cdot \vec{x}/X}\), while \(v^{s=\pm}_n\) solve the equation \([-i\gamma^j \partial_j + M] v^{s=\pm}_n e^{-2\pi i n \cdot \vec{x}/X} = -\kappa_n \gamma^0 v^{s=\pm}_n e^{-2\pi i n \cdot \vec{x}/X}\), where \(\kappa_n = \left|\frac{2\pi n}{X}\right|^2 + M^2\). They obey orthogonality relations of the kind \(u^s_n \bar{u}^{s'}_n = v^{s}_n \bar{v}^{s'}_n = \delta^{ss'}\).

Once we put (51) and (52) into (50) we obtain the following expression for the Hamiltonian of the system

\[
H = \sum_{n,s} \kappa_n \left( (a^s_{n} a^{s\dagger}_{n} - \frac{1}{2}) + (b^s_{n} b^{s\dagger}_{n} - \frac{1}{2}) \right),
\] (53)

which describes a collection of free fermionic modes.

We introduce for each mode, labeled by the pair of indexes \(s, n\), a couple of hermitic operators \(\xi^s_n\) and \(\tilde{\xi}^s_n\) as we did in the previous section. The task of choosing a complete representation of commuting observables is much simplified after having regularized the theory in the IR and UV: then we have a large but finite number of fermionic modes that we can order in a sequence. The set of hermitic operators \(\xi_0, i\tilde{\xi}_0, i\tilde{\xi}_1, \ldots\) forms a complete representation of commuting observables and we can expand any quantum state \(|\Psi\rangle\) of the system in the basis of common eigenstates \(|q\rangle\) to the operators in this representation.

Following the formalism that we have developed in the two previous sections we can give to each physical observable \(O(\vec{x}, t)\) a "classical" value \(O_{cl}(\vec{x}, t)\) at each of these common eigenstates \(|q\rangle\), which are so promoted to describe space-time dependent pseudoclassical field configurations (PCFC’s) whose probabilities to randomly happen are, again, given by \(|\langle q | \Psi \rangle|^2\). We remark again that although each PCFC corresponds to a certain eigenstate \(|q\rangle\), the way how we actually assign classical values to the physical observables, exploiting identities like (17), explicitly depends on the quantum state \(|\Psi\rangle\) of the system and, therefore, the path cannot be identified with the quantum state described by \(|q\rangle\).

The state \(|\Psi\rangle\) of the quantum field is then represented as a linear combination of randomly distributed PCFC’s. Each field operator, in particular \(\psi(\vec{x})\) and its conjugate \(\bar{\psi}(\vec{x})\), is assigned a time-dependent value on each of these paths. The operator equation

\[
[i\partial_\mu \gamma^\mu - M] \psi(t, \vec{x}) = 0,
\] (54)

and its hermitic conjugate

\[
\bar{\psi}(t, \vec{x}) [i\partial_\mu \gamma^\mu + M] = 0,
\] (55)
when are applied on the quantum state of the system

\[ [i\partial_\mu \gamma^\mu - M] \psi(t, \vec{x})|\Psi\rangle = 0, \quad \bar{\psi}(t, \vec{x}) [i\partial_\mu \gamma^\mu + M] |\Psi\rangle = 0, \] (56)

guarantee that on each of the random PCFC’s in the set that describes the quantum state
the classical values \( \psi(t, \vec{x})_{cl} \) and \( \bar{\psi}(t, \vec{x})_{cl} \) obey classical equations of motion:

\[ [i\partial_\mu \gamma^\mu - M] \psi(t, \vec{x})_{cl} = 0, \quad \bar{\psi}(t, \vec{x})_{cl} [i\partial_\mu \gamma^\mu + M] = 0. \] (57)

On the other hand, the initial conditions on each of the PCFC in the set that describes
the state, as well as their probabilities to randomly happen, are different for each quantum
state of the system, as we have discussed in the previous sections. Let us remark that
\( (\bar{\psi})_{cl} = (\psi^\dagger \gamma^0)_{cl} \) is determined from the action of the operator on the ket-state \( \bar{\psi}|\Psi\rangle \) and,
therefore, is not equal to \( (\psi_{cl})^\dagger \gamma^0 \) which results from its action on the bra-state \( \langle \Psi|\bar{\psi} \).

We can now consider the addition of a linear interaction term to the quantum field free
theory that we are discussing. If the coupling is weak enough we can work in perturbation
theory. Then, the Hilbert space of states can still be built as the Fock space of free quanta and
the algebra of the quantum operators is not altered. So, the whole formalism of PCFC’s can
be repeated. Now there is a new interacting term that appears in the hamiltonian of the sys-
tem and modify the dynamics of the quantum operators, \( O(t) = e^{i(H_0 + \lambda H_{int})t}Oe^{-i(H_0 + \lambda H_{int})t}, \)
in the Heisenberg picture. In consequence, it also modifies the time-dependent values of the
field operators on each PCFC. We can see from the operator equations (54) and (55) that
\( \psi(t, \vec{x})_{cl} \) and \( \bar{\psi}(t, \vec{x})_{cl} \) will still obey exactly classical equations of motion because the term
added to the equations is linear in the operators \( \psi(t, \vec{x}) \) and \( \bar{\psi}(t, \vec{x}) \).

As a simple example, we can add a Yukawa coupling term to the lagrangian (49). This
interaction adds a new term to the operator equation:

\[ [i\partial_\mu \gamma^\mu - M] \psi(t, \vec{x}) + g\phi(t, \vec{x})\psi(t, \vec{x}) = 0. \] (58)

This equation implies, when it is applied on the quantum state \( |\Psi\rangle \), that the dynamics of
the pseudoclassical values on its PCFC’s gets modified:

\[ [i\partial_\mu \gamma^\mu - M] \psi(t, \vec{x})_{cl} + g\phi(t, \vec{x})_{cl}\psi(t, \vec{x})_{cl} = 0 \]

and a similar equation can be obtained for \( (\bar{\psi})_{cl} \). In the new term we have made use of
the identity \( (\phi \psi)_{cl} = (\phi)_{cl}(\psi)_{cl} \), which can be easily proved because in a weakly interacting
theory the operators $\phi$ and $\psi$ act on different sectors of the Hilbert space, and have taken advantage of the linearity of (58) in the fermion field operator. The interaction term does not modify the initial conditions to be fixed on the PCFC’s that describe the quantum state, which depend only on the relations between operators when they act on the quantum state. Neither it does modify the random probability of each path. A discussion of, in general, non-linear interactions will be presented in a forthcoming work.

In the picture that we have presented quantum states of fermionic fields are represented as linear combination of randomly distributed paths that obey classical Dirac equations and do not interfere between themselves. The formalism is reminiscent of the consistent histories approach of Gell-Mann and Hartle [10], wherein the Feynman paths [11] which contribute to a certain process are grouped into equivalence classes or non-detailed histories that are approximately incoherent. On the contrary, in the formalism that we have developed the random paths do not interfere at all and obey ”deterministic” equations of motion at the price of redefining the algebraic relations between the values on paths of physical observables.

We can use this formalism of PCFC’s to describe quantum fluctuations of fermionic fields in terms of pseudoclassical stochastic processes. In particular, we can use it to describe local fluctuations of globally conserved numbers using classical concepts. We present here two examples that could be interesting: local fluctuations of the energy and fermion number in a finite volume $V$ contained in the whole box $[0, X]^3$ and parametrically larger than the UV cutoff that we have introduced.

The operator $\mathcal{H}_V = \int_V d^3x ~ \bar{\psi}(\vec{x}) (-i\gamma^j \partial_j + M) \psi(\vec{x})$ which describes the energy contained in the finite volume $V$ was already introduced in [12] and discussed in [8] in the context of a bosonic field theory. Similarly the fermion number contained in the same volume is described by the operator $\mathcal{B}_V = \int_V d^3x ~ \bar{\psi}(\vec{x}) \gamma^0 \psi(\vec{x})$.

If we want to find the ”classical” expressions for these two operators in the set of PCFC’s that describe a generic quantum field theoretic state $|\Psi\rangle$ we would need to write the action on this state of each of the operators in terms of the action on the same state of some function of the operators in a complete representation of commuting observables. We have done this excersise in previous sections and, therefore, we will present here a different approach that will shortcut the way towards this aim and will give us a different perspective on PCFC’s.
We know from equation (24) that
\[
< (H_V(t))_{cl} >_{PCFC} = \int_V d^3 x \quad < (\bar{\psi}(t, \vec{x}) \left( -i \gamma^j \partial_j + M \right) \psi(t, \vec{x}))_{cl} >_{PCFC}
\]
\[
= \int_V d^3 x \quad < (\psi_{cl}(t, \vec{x}))^{\dagger} \gamma^0 \left( -i \gamma^j \partial_j + M \right) \psi_{cl}(t, \vec{x}) >_{PCFC},
\]
and similarly
\[
< (B_V(t))_{cl} >_{PCFC} = \int_V d^3 x \quad < (\bar{\psi}(t, \vec{x}) \gamma^0 \psi(t, \vec{x}))_{cl} >_{PCFC}
\]
\[
= \int_V d^3 x \quad < (\psi_{cl}(t, \vec{x}))^{\dagger} \psi_{cl}(t, \vec{x}) >_{PCFC}.
\]
Although these equations hold only on average over the whole ensemble of PCFC’s and not necessarily on each of them, we gave in (48) a condition that ensures that
\[
< (H_V(t))_{cl} >_{PCFC} \simeq \int_V d^3 x \quad (\psi_{cl}(t, \vec{x}))^{\dagger} \gamma^0 \left( -i \gamma^j \partial_j + M \right) \psi_{cl}(t, \vec{x}),
\]
and
\[
< (B_V(t))_{cl} >_{PCFC} \simeq \int_V d^3 x \quad (\psi_{cl}(t, \vec{x}))^{\dagger} \psi_{cl}(t, \vec{x}),
\]
on each PCFC, if the random variables \[ [(H_V)_{cl} - f_V d^3 x \quad (\psi_{cl}(\vec{x}))^{\dagger} \gamma^0 \left( -i \gamma^j \partial_j + M \right) \psi_{cl}(\vec{x})] \]
and \[ [(B_V)_{cl} - f_V d^3 x \quad (\psi_{cl}(\vec{x}))^{\dagger} \psi_{cl}(\vec{x})] \]
are quadratic in many independent random variables. We called this kind of observables collective observables and will show below that these two operators are in fact ”collective” operators, but for now let us just assume it to get to some conclusions:

a) We can describe the dynamics on paths of ”collective” observables like (60) and (61) in terms of classical stochastic concepts, from the classical Dirac equations (57) or (59) that \( \psi_{cl}(t, \vec{x}) \) obeys. Only the initial conditions and the actual distribution of paths are constrained by the quantum state of the system \(| \Psi \rangle \).

b) Then, the time correlation function of any operator can be defined on each PCFC as usual in statistical mechanics [13], \( f_o(t_1 - t_2) = (O_{cl}(t_1))^{\ast} O_{cl}(t_2) - (O_{cl}(t_2))^{\ast} O_{cl}(t_1) \), and the lifetime of the fluctuation is defined as the inverse width of its Fourier transform. We proved in [8] that the average over the ensemble of PCFC’s coincides, as it should do, with the usual definition of the time scale of quantum fluctuations.

We will conclude this section giving a formal proof that justifies (60) and (61). Let start introducing the Fourier expansions (51) and (52) in the expressions for the operators \( H_V \) and \( B_V \). Their classical values on paths can then be expressed as [15]:
\[
(H_V)_{cl} = \sum_{\vec{n}, \vec{m}} \sum_{s,s'} \kappa_{\vec{n}, \vec{m}} \left[ (a^s_{\vec{n}} u^s_{\vec{m}} + b^s_{-\vec{n}} v^s_{-\vec{m}}) (a^{s'}_{\vec{m}} \gamma^0 u^{s'}_{\vec{m}} - b^{s'}_{-\vec{m}} v^{s'}_{-\vec{m}}) \right]_{cl} F(\vec{n} - \vec{m}, X), \quad (62)
\]
\( (B_V)_{cl} = \sum_{\vec{n}, \vec{m}} \sum_{s,s'} \left[ (a_{\vec{n}}^{\dagger} \bar{u}^s_{\vec{n}} + b_{\vec{n} \rightarrow \vec{m}} \bar{v}^s_{\vec{n}})(a_{\vec{m}}^{\dagger} \gamma^0 u^s_{\vec{m}} + b_{\vec{m} \rightarrow \vec{n}}^{\dagger} \gamma^0 v^s_{\vec{m}}) \right] F(\vec{n} - \vec{m}, X). \) (63)

On the other hand the approximate expressions (60),(61) can be expressed using the same Fourier expansions as:

\( (H_V)_{cl}^{approx} = \sum_{\vec{n}, \vec{m}} \sum_{s,s'} \kappa_{\vec{m}} \left[ ([a_{\vec{n}}^{\dagger}]_{cl} \bar{u}^s_{\vec{n}} + [b_{\vec{n} \rightarrow \vec{m}}^{\dagger}]_{cl} \bar{v}^s_{\vec{n}})([a_{\vec{m}}^{\dagger}]_{cl} \gamma^0 u^s_{\vec{m}} - [b_{\vec{m} \rightarrow \vec{n}}^{\dagger}]_{cl} \gamma^0 v^s_{\vec{m}}) \right] F(\vec{n} - \vec{m}, X), \) (64)

\( (B_V)_{cl}^{approx} = \sum_{\vec{n}, \vec{m}} \sum_{s,s'} \left[ ([a_{\vec{n}}^{\dagger}]_{cl} \bar{u}^s_{\vec{n}} + [b_{\vec{n} \rightarrow \vec{m}}^{\dagger}]_{cl} \bar{v}^s_{\vec{n}})([a_{\vec{m}}^{\dagger}]_{cl} \gamma^0 u^s_{\vec{m}} + [b_{\vec{m} \rightarrow \vec{n}}^{\dagger}]_{cl} \gamma^0 v^s_{\vec{m}}) \right] F(\vec{n} - \vec{m}, X). \) (65)

We need to prove that \( (H_V)_{cl} - (H_V)_{cl}^{approx} \) and \( (B_V)_{cl} - (B_V)_{cl}^{approx} \) are bilinear in the random variables associated to the single modes. Comparing both expressions we can see that it is enough to prove that expressions of the kind \([a_{\vec{n}}^{\dagger}]_{cl} a_{\vec{m}}^{\dagger} - [a_{\vec{n}}^{\dagger}]_{cl}[a_{\vec{m}}^{\dagger}]_{cl}\), which involve only two different fermion modes are bilinear. We know from (38) that they are.

V. SUMMARY

We have extended to systems of fermions the formalism of pseudoclassical paths that we recently developed for systems of weakly interacting bosons and have shown that fermionic quantum states can also be represented in the Heisenberg picture as linear combinations of randomly distributed paths which do not interfere between themselves. Every physical observable was assigned a time-dependent value on each path in a way that respects the anticommutation relations between fermionic operators and, in consequence, these values on paths do not necessarily respect the usual algebraic relations between classical observables. Each path depicts a collection of pseudoclassical harmonic oscillators with constrained initial conditions.

We used these paths to define the dynamics of quantum fluctuations in systems of fermions without reference to an environment or any additional external system. Then, we selected ”collective” observables which depend on many fermionic degrees of freedom and found that, as we found in [8] for collective operators which depend on many bosonic modes, their values on PCP’s do approximately fulfill the usual algebraic relations between classical observables and, therefore, the dynamics of these collective fluctuations can be described in terms of unconstrained classical stochastic processes.

In this setup, we showed that quantum fluctuations of fermionic fields obey classical Dirac
equations and described the dynamics of local fluctuations of globally conserved fermion numbers.

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[14] we use units in which $\hbar = 1, c = 1$.
[15] We have included all the dependence on the geometry of the volume $V$ inside the spatial
integrals $F(\vec{k}, X) = \int_V d^3x \frac{1}{X} \exp \left(-2\pi i \frac{\vec{k} \cdot \vec{x}}{X} \right)$, that can be analytically computed for simple geometries. If $V = [0, X]^3$ is the whole box, $F(\vec{k}, X) = 0$ except for $F(\vec{k} = 0, X) = 1$. 