Global constants in (2+1)–dimensional gravity

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Abstract. The extended conformal algebra so(2, 3) of global, quantum, constants of motion in 2+1 dimensional gravity with topology $\mathbb{R} \times T^2$ and negative cosmological constant is reviewed. It is shown that the 10 global constants form a complete set by expressing them in terms of two commuting spinors and the Dirac $\gamma$ matrices. The spinor components are the globally constant holonomy parameters, and their respective spinor norms are their quantum commutators.

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1. Introduction

Starting around 1989 there followed a period of intense activity in the field of (2+1)-dimensional gravity, and a number of different approaches were developed. Some of these are the reduced phase space quantization with ADM variables [1, 2, 3], quantization of the space of classical solutions of the first-order Chern-Simons theory [4, 5, 6], and quantization of the holonomy algebra [7].

In a pioneering article [1] Vincent Moncrief reduced (à la ADM) the Einstein equations for pure 2+1 gravity (with or without a cosmological constant $\Lambda$, and for spacetimes with compact Cauchy surface), to a time dependent, finite dimensional, Hamiltonian system on the cotangent bundle of the Teichmüller space of the Cauchy surface. At the same time Regge and I [7] were studying the first order, holonomy formalism of (2+1)-dimensional gravity, following Witten [4] and Achúcarro and Townsend [5]. In 1990 both Moncrief [8] and Carlip [6] started studying Cauchy surfaces diffeomorphic to the torus, since although it is, in principle, possible to determine the evolution of the Teichmüller parameters and their conjugate momenta (i.e., the reduced ADM variables), and therefore solve Hamilton’s equations implicitly, this procedure can be carried out in practice only for Cauchy surfaces diffeomorphic to the torus, since the spherical case is essentially trivial and the higher genus case particularly unyielding. At about the same time Regge, Zertuche and I [9] were studying the case of non-zero cosmological constant, and Carlip had determined [6] the relationship, for the case of the torus $T^2$, and with $\Lambda = 0$, between the reduced ADM variables and the holonomy parameters, as a time-dependent canonical transformation. Meanwhile Moncrief had found [8] a set of six global constants of the motion, for the case of zero cosmological constant, using a construction due to Martin [10]. Linear combinations of these global constants satisfy the Lie algebra of the Poincaré group ISO(1,2). These constants were exactly the traces of the SO(1,2) holonomies of the connections that Regge and I were studying.

A few years later Carlip and I [11] had extended the analysis of the relationship (both classical and quantum) between ADM variables and holonomy parameters, for the torus $T^2$, to $\Lambda \neq 0$, and it was natural to apply this to Moncrief’s global constants, since, although quantization in terms of the ADM variables was studied in [12], it involves non-trivial operator ordering ambiguities and the square root of the Laplace–Beltrami operator [13].

Firstly, Moncrief and I extended the construction of global constants to non-zero (negative) cosmological constant. This was straightforward [14], and the six new global constants were found to indeed reduce to those of [8] in the limit $\Lambda \to 0$, and to satisfy the Lie algebra of the anti-de Sitter group SO(2,2). Using the classical relationship found in [11] the constants are particularly simple in terms of the holonomy parameters, and easily quantized [14].

In a related classical construction [15] for zero cosmological constant in the unreduced, ADM, Hamiltonian formalism it was shown that on inclusion of the globally
constant part of the Hamiltonian (the full ADM Hamiltonian is a function of time times a global constant) three new global constants are derived and the algebra extends to that of the conformal algebra so(2,3), whose corresponding group is the conformal group of 3-dimensional Minkowski space. The same was true for $\Lambda \neq 0$ and three new quantum constants were obtained, which, together with the Hamiltonian, form a null SO(2,2) vector [14].

Here I show that there exactly no more global constants by expressing them in terms of two two-component commuting constant spinors (whose components are the globally constant holonomy parameters).

The plan of the paper is as follows. In section 2 the alternative classical and quantum descriptions, and the relationship between them, are discussed. In section 3 six global constants are constructed, and expressed either in ADM variables, or the holonomy parameters. Quantization is straightforward in terms of the holonomy parameters. In section 4 the extended algebra is calculated. In section 5 it is shown that the set of constants is complete by explicit use of a spinor representation, and in section 6 the role of the modular group is briefly discussed.

2. Hamiltonian Dynamics

2.1. Reduced ADM Dynamics

It is known [14, 11] that (2+1)-dimensional gravity, with topology $R \times T^2$, the York time coordinate condition, and with or without a cosmological constant $\Lambda$, has (at least) two equivalent descriptions. In ADM quantization the reduced Einstein action can be written [1, 2]

$$I_1 = \int d\tau \left( p^\alpha dq^\alpha - H(q, p, \tau) \right), \quad \alpha = 1, 2$$

where the constant mean curvature $\tau$ labels the slices $T^2$, and $q_1, q_2, p^1, p^2$ are the reduced ADM canonical coordinates and momenta. In equation (1) the reduced, or effective, ADM Hamiltonian $H$ is just the spatial volume

$$H = \int_{T^2} d^2x \sqrt{g} = \frac{1}{\sqrt{\tau^2 - 4\Lambda}} \tilde{H}, \quad \tilde{H} = \sqrt{p_1^2 + e^{-2q_1} p_2^2}.$$  

and is not constant, since volume is not conserved, but is, instead, a function of time times a global constant $\tilde{H}$. From (1) the basic Poisson brackets are

$$\{ q_i, p^j \} = \delta_i^j$$

As discussed in [12, 13] quantization is achieved by replacing equation (3) with the commutators

$$[\hat{q}_i, \hat{p}^j] = i\hbar \delta_i^j,$$

representing the momenta as derivatives,

$$p^j = \frac{\hbar}{i} \frac{\partial}{\partial q_j},$$
and imposing the Schrödinger equation
\[ i\hbar \frac{\partial \psi(q, \tau)}{\partial \tau} = \hat{H}\psi(q, \tau), \] (6)
where the Hamiltonian \( \hat{H} \) is obtained from (2) by some suitable operator ordering. The obvious choice is that of equation (2), for which the Hamiltonian becomes
\[ \hat{H} = \frac{\hbar}{\sqrt{\tau^2 - 4\Lambda}} \Delta_0^{1/2}, \] (7)
where \( \Delta_0 \) is the ordinary scalar Laplacian for the constant negative curvature moduli space. Other orderings would correspond to operators \( \Delta_n \), the eight Maass Laplacians [16], which differ from \( \Delta_0 \) by terms of order \( \hbar \).

2.2. The holonomy representation

In the fully reduced holonomy representation [7, 9] the constraints are solved exactly, the Hamiltonian is zero, and (2+1)–dimensional gravity is described, for the torus, by the real, global, time-independent traces \( R_{1}^{\pm}, R_{2}^{\pm}, R_{12}^{\pm} \) of SL(2, R) holonomies which satisfy the Poisson bracket algebra (cyclical in the three \( \pm \) traces)
\[ \{ R_{1}^{\pm}, R_{2}^{\pm} \} = \mp \frac{1}{4\alpha} (R_{12}^{\pm} - R_{1}^{\pm} R_{2}^{\pm}). \] (8)
In equation (8) the two \( \pm \) copies refer to the decomposition of the spinor group of SO(2, 2) (the anti-de Sitter group) as SL(2, R) \( \otimes \) SL(2, R) [7], and the subscripts 1, 2 refer to two intersecting paths \( \gamma_1, \gamma_2 \) on \( T^2 \) with intersection number +1. The third holonomy, \( R_{12}^{\pm} \) corresponds to the path \( \gamma_1 \cdot \gamma_2 \), which has intersection number −1 with \( \gamma_1 \) and +1 with \( \gamma_2 \), and the constant \( \alpha \) is related to the cosmological constant through \( \Lambda = -1/\alpha^2 \). The holonomies of (8) can be parametrized as
\[ R_{1}^{\pm} = \cosh \frac{r_{1}^{\pm}}{2}, \]
\[ R_{2}^{\pm} = \cosh \frac{r_{2}^{\pm}}{2}, \]
\[ R_{12}^{\pm} = \cosh (r_{1}^{\pm} + r_{2}^{\pm})/2 \] (9)
where \( r_{1,2}^{\pm} \) are also real, global, time-independent (but undetermined) parameters which, from equation (8) satisfy
\[ \{ r_{1}^{\pm}, r_{2}^{\pm} \} = \mp \frac{1}{\alpha}, \quad \{ r_{1,2}^{\pm}, r_{1,2}^{\mp} \} = 0. \] (10)
With this parametrisation, the Chern-Simons action [4, 5] is [11]
\[ I_2 = \int \alpha (r_{1}^{-} dr_{2}^{-} - r_{1}^{+} dr_{2}^{+}) \] (11)
and quantization is achieved by replacing equations (10) with the commutators
\[ [\hat{r}_{1}^{\pm}, \hat{r}_{2}^{\pm}] = \mp i\hbar/\alpha, \quad [\hat{r}_{1,2}^{\pm}, \hat{r}_{1,2}^{\mp}] = 0. \] (12)
* Direct quantization of the algebra (8) gives an algebra related to the Lie algebra of the quantum group SU(2)q [9, 17], where \( q = \exp (4i\theta) \), tan \( \theta = -\hbar/8\alpha \). A (scaled) representation of the operators (9) leads to the commutators \( [\hat{r}_{1}^{\pm}, \hat{r}_{2}^{\pm}] = \pm 8i\theta \), which differ from (12) by terms of order \( \hbar^3 \).
2.3. Relationship between the ADM and holonomy descriptions

The above two descriptions appear quite different in concept and in structure, but classically can be related because of the availability of explicit classical solutions for the torus case. The conformal spatial metric $g^{-1/2}g_{ij}$ is in fact parametrized by the parameters $q^1, q^2$

$$g^{-1/2}g_{ij} = \left( e^{-q^1} + \frac{(q^2)^2 e^{q^1}}{q^2 e^{q^1}}, \frac{q^2 e^{q^1}}{e^{q^1}} \right)$$  \hspace{1cm} (13)

Now, for a suitable triad $e^a = e^a_\mu dx^\mu$ whose spatial components $e^a_i$ are related to the two metric through

$$g_{ij} = e^a_i e^b_j \eta_{ab}, \quad \eta_{ab} = diag(-1, 1, 1), \quad a, b, c = 0, 1, 2,$$  \hspace{1cm} (14)

and its corresponding spin connection

$$\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{bc}, \quad \epsilon^{012} = -\epsilon_{012} = 1$$  \hspace{1cm} (15)

the traces of holonomies $R^\pm_i$ (equations (9)) are just the Wilson loops

$$R^\pm_i = Tr \left( \exp \Delta^\pm_i \right), \quad i = 1, 2, 12$$  \hspace{1cm} (16)

where

$$\Delta^\pm_i = \int_{\gamma_i} \lambda^\pm(a)$$  \hspace{1cm} (17)

and the $\lambda^\pm a$ are "shifted connections" defined by

$$\lambda^\pm a = \omega^a \pm \sqrt{-\Lambda} e^a.$$  \hspace{1cm} (18)

These integrated shifted connections (17) and therefore the traces (16) can be calculated directly from the classical solutions (13), and their relation to the parametrisation (9) is

$$(r^\pm_{1,2})^2 = \Delta^\pm_{1,2} \Delta^\pm_{1,2} \eta_{ab},$$  \hspace{1cm} (19)

From equations (13)–(18) we have the following [11]

$$m = \left( r_1^- e^{it/\alpha} + r_1^+ e^{-it/\alpha} \right) \left( r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha} \right)^{-1},$$

$$\pi = -\frac{i\alpha}{2\sin \frac{\pi}{\alpha}} \left( r_2^+ e^{it/\alpha} + r_2^- e^{-it/\alpha} \right)^2,$$  \hspace{1cm} (20)

where, in equation (20) $m = m_1 + im_2$ are the complex moduli, and $\pi = \pi^1 + i\pi^2$ their complex momenta, related to the ADM coordinates and momenta through

$$m_1 = q^2, m_2 = e^{-q^1}, \pi^1 = p_2, \pi^2 = -p_1 e^{q^1}$$  \hspace{1cm} (21)

and the mean curvature $\tau$ is related to $t$ in equations (20)–(21) by $\tau = -\frac{2}{\alpha} \cot \frac{2t}{\alpha}$, and is monotonic in the range $t \in (0, \pi\alpha/2)$. Note that since the $r_1^\pm, r_2^\pm$ are arbitrary the moduli and momenta of equation (20) (and therefore, from equation (21), also the ADM variables) can have arbitrary initial data at some initial time $t_0$. 


That the two descriptions are related classically through a time-dependent canonical transformation can be seen, from equations (20)–(21), from the comparison of the classical actions (1) and (11)

\[ I_1 = I_2 + \int d(\pi^1 m_1 + \pi^2 m_2) = I_2 + \int d(p_2 q^2 - p_1) \]  

Using (20) and (21) the ADM Hamiltonian (2) becomes

\[ H = \frac{\alpha}{2\sqrt{\tau^2 - 4\Lambda}}(r_1^+ r_2^- - r_1^+ r_2^-), \quad \bar{H} = \frac{\alpha}{2}(r_1^- r_2^+ - r_1^+ r_2^-). \]  

and quantum mechanically, there would be no ordering ambiguity, as can be seen from equation (12). It should be noted, however, that no representation is known for the fundamental quantized holonomy parameters \( r_{1,2}^\pm \) which guarantees positivity of the reduced Hamiltonian (23), though some progress has been made using the representation of [18].

Comparing the two quantum theories is much more subtle, and I will say very little. The ADM representation looks like a standard Schrödinger picture quantum theory, with time-dependent states \( \psi(q, \tau) \) whose evolution is determined by a Hamiltonian operator, equation (7). The holonomy representation resembles a Heisenberg picture quantum theory, characterized by time-independent states \( \psi(r_{1,2}^\pm) \) (though clearly some choice of polarization would be necessary) and time-dependent operators (20)–(21). A polarization and unitary transformation between the two representations was constructed in [18].

3. Constants of the motion

It is known [10] that the traces of SO(1,2) holonomies, for \( \Lambda = 0 \), of connections integrated along arbitrary closed loops, are absolutely conserved quantities (i.e., they are gauge invariant and invariant under non-singular deformations of the loops within the vacuum spacetimes).

Here for \( \Lambda < 0 \) it is instead appropriate to use the integrated shifted connections (17) for SL(2,\( \mathbb{R} \)) \( \otimes \) SL(2,\( \mathbb{R} \)), the spinor decomposition of SO(2,2), the anti–de Sitter group. As in [8] these traces were computed for 3-different classes of loops corresponding to the paths \( \gamma_1, \gamma_2 \) and "twisting loops" \( \gamma_1 \cdot \gamma_2 \). In spatial coordinates \( x^1, x^2 \) these loops would correspond to having \( x^2 = \text{constant}, x^1 = \text{constant} \), whereas the "twisting loops" have \( x^1 = \frac{m}{n} x^2 \), for integer \( m, n \), respectively. The twisting loops do not give new independent conserved quantities but, instead, are functions of those coming from the loops \( \gamma_1 \) and \( \gamma_2 \).

3.1. ADM Variables

The following absolutely conserved quantities were obtained [14], as in [8, 10], from the traces of the SL(2,\( \mathbb{R} \)) holonomies

\[ C_1^\pm = C_1 \pm 2\sqrt{-\Lambda}C_4, \]
$C_2^\pm = C_2 \mp 2\sqrt{-\Lambda}C_5,$
$C_3^\pm = C_3 \pm \sqrt{-\Lambda}C_6,$  \hspace{1cm} (24)

where

$C_1 = \frac{1}{2} e^{-q^1 \tau} \left\{ \sqrt{1 - 4\Lambda \bar{H}} - p_1 \right\},$
$C_2 = \frac{1}{2} e^{q^1 \tau} \left\{ \sqrt{1 - 4\Lambda \bar{H}} + p_1 \right\},$
$C_3 = \frac{1}{2} e^{q^1 \tau} \left\{ q^2 \sqrt{1 - 4\Lambda \bar{H}} - p_1 - p_2 e^{-q^1} \right\},$
$C_4 = \frac{1}{2} \left\{ p_2 e^{-2q^1} + 2q^1 p_1 - p_2 (q^2)^2 \right\},$
$C_5 = \frac{1}{2} p_2,$
$C_6 = p_1 - q^2 p_2.$ \hspace{1cm} (25)

The above quantities $C_1 - C_6$ are quasi-linear in the momenta $p_1, p_2$, as is the ADM Hamiltonian itself (see equation (2)), and in the limit $\Lambda \to 0$ reduce to those defined in [8]. That they are conserved quantities can be verified directly through Hamilton’s equations (where $H$ is given by equation (2))

\[
\frac{dq^i}{d\tau} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q^i}, \quad i = 1, 2. \hspace{1cm} (26)
\]

The $C_1 - C_6$ are related as follows to the perhaps more familiar quantities $J_{ab}, P_c$ through

$P_0 = -\frac{1}{2} (C_1 + C_2),$  
$P_1 = \frac{1}{2} (C_1 - C_2),$  
$P_2 = C_3,$  
$J_{12} = C_5 - C_4,$  
$J_{02} = C_4 + C_5,$  
$J_{01} = -C_6$ \hspace{1cm} (27)

which, from equation (3), satisfy the Lie algebra $so(2, 2)$ of the anti–de Sitter group

\[
\{J_{ab}, J_{cd}\} = \eta_{ac}J_{bd} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc} + \eta_{bd}J_{ac},
\]

\[
\{P_a, P_b\} = \Lambda J_{ab},
\]

\[
\{J_{ab}, P_c\} = \eta_{ac}P_b - \eta_{bc}P_a
\]

with the identity

\[
P_a J_{bc} \epsilon^{abc} = C_4 C_2 - C_5 C_1 - C_3 C_6 = 0, \hspace{1cm} (29)
\]

The constant part $\bar{H}$ of the ADM Hamiltonian (2) is related to these quantities through

\[
\Lambda \bar{H}^2 = (C_3)^2 - C_1 C_2 = P_a P^a
\]

\[
\bar{H}^2 = (C_6)^2 + 4 C_4 C_5 = -J_{ab} J^{ab}
\]
Quantization of these constants and the Hamiltonian (equation (2)) in terms of these ADM variables has been discussed in [12].

3.2. Holonomy parameters

In terms of the time independent global parameters $r_{1,2}^\pm$ the $C_1^\pm - C_3^\pm$, equation (24) are, from equations (20)–(21), quite simple, and evidently globally conserved

$$C_1^\pm = (r_1^\mp)^2, \quad C_2^\pm = (r_2^\mp)^2, \quad C_3^\pm = r_1^\mp r_2^\mp,$$

and quantization is indeed straightforward in terms of these parameters. From equations (24), (25) and (27) the combinations

$$\hat{j}_a^\pm = \frac{1}{2} \epsilon_{abc} J_{bc}^\pm \pm \alpha P_a$$

are just

$$\hat{j}_0^\pm = \pm \frac{\alpha}{2} ((\hat{r}_1^\mp)^2 + (\hat{r}_2^\mp)^2),$$

$$\hat{j}_1^\pm = \pm \frac{\alpha}{2} ((\hat{r}_1^\mp)^2 - (\hat{r}_2^\mp)^2),$$

$$\hat{j}_2^\pm = \pm \frac{\alpha}{2} (\hat{r}_1^\mp \hat{r}_2^\mp + \hat{r}_2^\mp \hat{r}_1^\mp).$$

Note that the $j_a^+$ depend only on the $r^+$'s and the $j_a^-$ only on the $r^-$'s, and that the only ordering ambiguity is in $\hat{j}_2^\pm$, equation (35). With this symmetric ordering the combinations (32) satisfy, using equations (12), the two ($\pm$) Lie algebras of $so(1,2) \approx sl(2, \mathbb{R})$

$$[\hat{j}_a^\pm, \hat{j}_b^\pm] = 2i \hbar \epsilon_{abc} \hat{j}_c^\pm, \quad [\hat{j}_a^+, \hat{j}_b^-] = 0,$$

Now the generators $\hat{j}_a^\pm$ and $\hat{H}$ are not all independent. There are 3 Casimirs corresponding to the classical identities (29) and (30)

$$\hat{j} = \hat{j}_a^+ \hat{j}_a^- = \frac{3\hbar^2}{4}, \quad \hat{H}^2 - \frac{1}{2} \hat{j}_a^+ \hat{j}_a^- = \frac{\hbar^2}{2}.$$

Note that for the only ordering ambiguity is in $\hat{j}_2^\pm$ (equation (35)) any other ordering would only produce terms of $O(\hbar^2)$ on the right hand side of equation (36) and equation (37).

4. The Extended Algebra

In a related classical construction [15] for zero cosmological constant the constants $C_1 - C_6$ (25) were used as generators of isometries in the unreduced, ADM, Hamiltonian formalism. Consider the constant part $\hat{H}$ (equation (23)) of the ADM Hamiltonian. It can be checked, using equation (12), that $\hat{H}$ does not commute with all the $sl(2, \mathbb{R})$ generators, equations (33)–(35) (though it does commute with $\hat{j}_a^+ + \hat{j}_a^- = \epsilon_{abc} J_{bc}$). Since

$$[\hat{H}, \hat{r}_i^\pm] = i \frac{\hbar}{2} \hat{r}_i^\pm,$$
a new globally constant three-vector $\hat{v}_a$ is defined through

$$[\hat{H}, \hat{J}_a^\pm] = \pm i\hbar \hat{v}_a,$$

(39)

where

$$\hat{v}_0 = -\frac{\alpha}{2}(\hat{r}_1^+ \hat{r}_1^- + \hat{r}_2^+ \hat{r}_2^-),$$

$$\hat{v}_1 = \frac{\alpha}{2}(\hat{r}_1^+ \hat{r}_1^- - \hat{r}_2^+ \hat{r}_2^-),$$

$$\hat{v}_2 = -\frac{\alpha}{2}(\hat{r}_1^+ \hat{r}_2^- + \hat{r}_2^+ \hat{r}_1^-).$$

(40)

Note that from (12) there are also no ordering ambiguities in the $\hat{v}_a$.

The extended algebra of the ten $\hat{H}, \hat{J}_a^\pm, \hat{v}_a, a = 0, 1, 2$ (equations (23), (33)–(35), (40)), then closes as follows

$$[\hat{H}, \hat{J}_a^\pm] = \pm i\hbar \hat{v}_a$$

$$[\hat{J}_a^+, \hat{J}_b^\pm] = 2i\hbar \epsilon_{abc} \hat{J}^c \pm$$

$$[\hat{J}_a^+, \hat{J}_b^-] = 0$$

$$[\hat{H}, \hat{v}_a] = \frac{i\hbar}{2}(\hat{J}_a^+ - \hat{J}_a^-)$$

$$[\hat{v}_a, \hat{v}_b] = -\frac{i\hbar}{2}\epsilon_{abc}(\hat{J}_a^+ + \hat{J}_a^-)$$

$$[\hat{J}_a^+, \hat{v}_b] = i\hbar(\mp\eta_{ab}\hat{H} + \epsilon_{abc} \hat{v}_c)$$

(41)

with the 3 additional identities (making, with equation (37), a total of 6 identities)

$$\hat{v}_a^\alpha \hat{J}_a^\pm = \hat{J}_a^\pm \hat{v}_a^\alpha = \pm \frac{3i\hbar}{2} \hat{H}, \text{and}$$

$$\hat{v}_a \hat{v}_a - \hat{H}^2 = -\frac{\hbar^2}{2}.$$

(42)

(43)

The above 10-dimensional algebra (41) is isomorphic to the Lie algebra of so(2, 3), whose corresponding group is the conformal group of 3-dimensional Minkowski space [15]. The dilatation $D$ is to be identified with $-\hat{H}$, the translations with $\hat{P}_a^\pm$, and the conformal accelerations with $\hat{P}_a^\pm$, where $\hat{P}_a^\pm = \alpha \hat{P}_a \pm \hat{v}_a$. Note that, in contrast to the generators $\hat{j}_a^+$ and $\hat{j}_a^-$ (equations (33)–(35)) of the two commuting sl(2, $\mathbb{R}$) subalgebras (equation (36)), the group extension, that is, the Hamiltonian $\hat{H}$ (equation (23) and the vectors $\hat{v}_a$ (equation (40)) require both the $\pm$ global parameters, and therefore both the two sl(2, $\mathbb{R}$) algebras.

5. Spinor representation

The fact that the Hamiltonian $\hat{H}$ and the three new constants $\hat{v}_a$ define a null SO(2, 2) vector (see equation (43)) suggests the use of the commuting two–component constant spinors $\hat{r}^+, \hat{r}^-$ defined by

$$\hat{r}^\pm = \begin{pmatrix} \hat{r}_1^\pm \\ \hat{r}_2^\pm \end{pmatrix}$$

(44)
which satisfy, from (12) \([\hat{r}^+, \hat{r}^-] = 0\), and whose respective norms are just the commutators

\[
e^{AB} \hat{e}^+_B \hat{r}^+_A = \hat{r}^+_A \hat{e}^+_B = [\hat{r}^+_2, \hat{r}^+_1] = \pm \frac{i\hbar}{\alpha}, \quad A, B = 1, 2
\]

(45)

with \(\epsilon^{12} = -\epsilon^{21} = 1\) and \(\hat{r}^+_A = \epsilon_{AB} \hat{r}^+_B\).

In terms of these commuting two–component constant spinors \(\hat{r}^+, \hat{r}^-\) the ten constants \(\hat{H}, \hat{j}_a, \hat{v}_a\) are

\[
\begin{align*}
\hat{v}_0 &= -\frac{\alpha}{2} \hat{r}^+ T \mathbb{I} \hat{r}^- \\
\hat{v}_1 &= \frac{\alpha}{2} \hat{r}^+ T \sigma_3 \hat{r}^- \\
\hat{v}_2 &= \frac{\alpha}{2} \hat{r}^+ T \sigma_1 \hat{r}^- \\
\hat{H} &= -i \frac{\alpha}{2} \hat{r}^+ T \sigma_2 \hat{r}^- \\
\hat{j}_0^\pm &= \mp \frac{\alpha}{2} \hat{r}^+ T \mathbb{I} \hat{r}^\pm \\
\hat{j}_1^\pm &= \pm \frac{\alpha}{2} \hat{r}^+ T \sigma_3 \hat{r}^\pm \\
\hat{j}_2^\pm &= \pm \frac{\alpha}{2} \hat{r}^+ T \sigma_1 \hat{r}^\pm
\end{align*}
\]

(46) (47) (48) (49) (50) (51) (52)

where the \(\sigma_1, \sigma_2, \sigma_3\) are the usual Pauli matrices.

That there are precisely no more than these ten constants can be seen by writing these constants in terms of the Dirac \(\gamma\) matrices satisfying

\[
\{\gamma_a, \gamma_b\} = 2\eta_{ab}
\]

(53)

with \(\eta_{ab} = \text{diag}(-1, 1, 1, -1)\) the SO(2, 2) (anti–de Sitter) metric, and the four–component constant spinors

\[
\hat{r} = \begin{pmatrix} \hat{r}^+ \\ \hat{r}^- \end{pmatrix}
\]

(54)

with \(\hat{r}^+, \hat{r}^-\) given in equation (44).

Consider the sixteen linearly independent \(4 \times 4\) matrices \(\Gamma_i, i = 1,...,16\)

\[
\Gamma_i = \mathbb{I}, \gamma_a, \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \gamma_a\gamma_5, \gamma_a\gamma_b, a \neq b.
\]

(55)

There should therefore be at most sixteen global constants of the form

\[
\hat{r}^T \Gamma_i \hat{r}.
\]

(56)

Consider the representation [9]

\[
\gamma_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = i\sigma_2 \otimes \sigma_3 = i \left( \begin{array}{cc} \sigma_2 \\ -\sigma_2 \end{array} \right)
\]

(57)
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\[ \gamma_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \sigma_3 \otimes \sigma_3 = \begin{pmatrix} \sigma_3 \\ -\sigma_3 \end{pmatrix} \]  
(58)

\[ \gamma_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{I} \otimes \sigma_1 = \begin{pmatrix} -\mathbb{I} \\ -\mathbb{I} \end{pmatrix} \]  
(59)

\[ \gamma_3 = \begin{pmatrix} -i & i \\ i & i \end{pmatrix} = -i\sigma_1 \otimes \sigma_3 = i \begin{pmatrix} -\sigma_1 \\ \sigma_1 \end{pmatrix} \]  
(60)

with

\[ \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \mathbb{I} \otimes \sigma_2 = i \begin{pmatrix} \mathbb{I} \\ -\mathbb{I} \end{pmatrix}. \]  
(61)

With this representation then of the sixteen possible global constants, equation (56), five are identically zero, and one a numerical constant, due to the commutators (12). The remaining ten are linear combinations of the ten global constants $\hat{H}, \hat{v}_i, \hat{j}_i^\pm, i = 0, 1, 2$, as I shall show.

There are eight block diagonal combinations, namely $\gamma_0, \gamma_1, \gamma_3, \mathbb{I}$ and

\[ \gamma_0\gamma_1 = -\sigma_1 \otimes \mathbb{I} = - \begin{pmatrix} \sigma_1 \\ \sigma_1 \end{pmatrix} \]  
(62)

\[ \gamma_0\gamma_3 = -i\sigma_3 \otimes \mathbb{I} = -i \begin{pmatrix} \sigma_3 \\ \sigma_3 \end{pmatrix} \]  
(63)

\[ \gamma_1\gamma_3 = \sigma_2 \otimes \mathbb{I} = i \begin{pmatrix} \sigma_2 \\ \sigma_2 \end{pmatrix} \]  
(64)

\[ \gamma_2\gamma_5 = -i\mathbb{I} \otimes \sigma_3 = i \begin{pmatrix} -\mathbb{I} \\ \mathbb{I} \end{pmatrix} \]  
(65)

and six of them combine to give the six constants $\hat{j}_a^\pm$ as

\[ \hat{j}_0^\pm = \pm \frac{\alpha}{4} \hat{r}^T (\mathbb{I} \pm i\gamma_2\gamma_5) \hat{r} \]  
(66)

\[ \hat{j}_1^\pm = \frac{\alpha}{4} \hat{r}^T (\gamma_1 \pm i\gamma_0\gamma_3) \hat{r} \]  
(67)

\[ \hat{j}_2^\pm = \frac{\pm \alpha}{4} \hat{r}^T (\gamma_0\gamma_1 \mp i\gamma_3) \hat{r} \]  
(68)

The two unused block–diagonal elements, namely $\gamma_0$ and $\gamma_1\gamma_3$ do not give other global constants, since

\[ \hat{r}^T \gamma_0 \hat{r} = i(\hat{r}^+ T\sigma_2 \hat{r}^+ - \hat{r}^- T\sigma_2 \hat{r}^-) = [\hat{r}^+_1, \hat{r}^+_2] - [\hat{r}^-_1, \hat{r}^-_2] = -\frac{2i\hbar}{\alpha} \]  
(69)
and 
\[ \hat{r}^T \gamma_1 \gamma_3 \hat{r} = \hat{r}^+ T \sigma_2 \hat{r}^+ + \hat{r}^- T \sigma_2 \hat{r}^- = -i(\hat{r}_1^+ \hat{r}_2^- + \hat{r}_1^- \hat{r}_2^+) = 0 \] (70)

The remaining eight possible combinations are all block off-diagonal, with four symmetric, and four anti-symmetric, so the available constants are all of the form 
\[ (\hat{r}^+ \hat{r}^-) \begin{pmatrix} c \\ d \end{pmatrix} (\hat{r}^+ \hat{r}^-) = \hat{r}^+ d \hat{r}^+ + \hat{r}^- c \hat{r}^- \] (71)

for some $2 \times 2$ matrices $c$ and $d$. For the four anti-symmetric matrices $\gamma_1 \gamma_2, \gamma_2 \gamma_3, \gamma_0 \gamma_5$ and $\gamma_5$ (i.e. with $c^T = -d$) the two terms in (71) cancel since $[\hat{r}^+, \hat{r}^-] = 0$. The four symmetric matrices $\gamma_0 \gamma_2, \gamma_2, \gamma_1 \gamma_5, \gamma_3 \gamma_5$ (i.e. with $c^T = d$) instead combine to give the four constants $\hat{H}$ and $\hat{v}_a$ respectively, as follows.

\[ \hat{H} = \frac{\alpha}{4} \hat{r}^T \gamma_0 \gamma_2 \hat{r} \] (72)
\[ \hat{v}_0 = \frac{\alpha}{4} \hat{r}^T \gamma_2 \hat{r} \] (73)
\[ \hat{v}_1 = i \frac{\alpha}{4} \hat{r}^T \gamma_1 \gamma_5 \hat{r} \] (74)
\[ \hat{v}_2 = -i \frac{\alpha}{4} \hat{r}^T \gamma_3 \gamma_5 \hat{r} \] (75)

6. The Quantum Modular Group

So far I have not mentioned the role of the "large diffeomorphisms" - diffeomorphisms that cannot be continuously deformed to the identity. For the torus $T^2$ this group—also known as the modular group—acts on the torus modulus and momentum and holonomy parameters as

\[ S : m \rightarrow -m^{-1}, \quad \pi \rightarrow m^2 \pi, \quad r_1^\pm \rightarrow r_2^\pm, \quad r_2^\pm \rightarrow -r_1^\pm, \]
\[ T : m \rightarrow m + 1, \quad \pi \rightarrow \pi, \quad r_1^\pm \rightarrow r_1^\pm + r^\pm_2, \quad r_2^\pm \rightarrow r_2^\pm, \] (76)

and generates the entire group of large diffeomorphisms of $R \times T^2$. It can be seen from equation (76) that this group acts properly discontinuously on the ADM variables, (the configuration or Teichmüller space variables, see equation (21)) and in fact splits this space into fundamental regions that are interchanged by the action of the group.

But on the holonomy parameters the group is not so well behaved. Even so, with the ordering of equation (20) (the only ambiguity), the quantum action of the modular group is the same as the classical one, with no $O(\hbar)$ corrections, and is generated by the SO(2, 2) discrete anti-de Sitter subgroup of the conformal group SO(2, 3) by conjugation with the operators $U_T = U_T^+ U_T^-$, and $U_S = U_S^+ U_S^-$ [18, 19] where

\[ U_T^\pm = \exp \frac{i}{2\hbar} (j_0^\pm + j_1^\pm) = \exp \mp \frac{i\alpha}{2\hbar} C_2^\mp = \exp \mp \frac{i\alpha}{2\hbar} (r_2^\pm)^2, \]
\[ U_S^\pm = \exp \frac{i\pi}{2\hbar} j_0^\pm = \exp \mp \frac{i\pi\alpha}{4\hbar} (C_1^\mp + C_2^\mp) = \exp \mp \frac{i\pi\alpha}{4\hbar} ((r_1^\pm)^2 + (r_2^\pm)^2). \] (77)
Moreover, it is easy to check that the commutators (12) and the ADM Hamiltonian (23) are invariant under the transformations (76). The initial constants \( \hat{j}_a^\pm \) transform into linear combinations of themselves, as do the group extensions \( \hat{v}_a \), in such a way that the quantum algebra (equation (36)) and the identities (equation (37)) are invariant under the transformations (76).

**7. Conclusion**

It is shown that there are exactly ten, and no more, global constants satisfying the Lie algebra of \( \text{so}(2,3) \), whose corresponding group is the conformal group of 3-dimensional Minkowski space. This is achieved by using two commuting two–component constant spinors whose norms are their respective commutators. With the six central elements (equations (37), (42) and (43)) this leaves precisely four arbitrary global variables corresponding to the required two degrees of freedom. These ten constants are easily expressed and quantized in terms of the holonomy parameters.

It would be desirable to try to implement the operator analogues of \( C_1 - C_6 \), or alternatively \( \hat{j}_a^\pm \) on a partially reduced quantization, working on Teichmüller space for the torus (i.e. the 2-dimensional hyperbolic space with global coordinates \( q^1, q^2 \) and Riemannian metric \( (dq^1)^2 + e^{2q^1}(dq^2)^2 \)) instead of moduli space, but unfortunately there is no known ordering of the operator analogues of expressions (25) which reproduces the \( \text{so}(2,2) \) algebra as would be expected from classical considerations.

It would also be desirable to be able to express the three-vector \( \hat{v}_a \) (equation (40)) in terms of the ADM variables. This is possible in principle, from its definition (39) but unfortunately the \( \hat{v}_a \) and \( \hat{H} \) (equation (23)) require both the \( \pm \) global parameters. But, as can be seen from equation (31) that would imply that the \( \hat{v}_a \) should be expressed as a square roots of quadratic combinations of the \( C_1 - C_6 \). This is no surprise, since \( \hat{H} \) is, from (2), or from (30) itself a square root of ADM variables, and is related to the \( \hat{v}_a \) through equation (43).

For completeness it should be noted that although only the case of negative cosmological constant was discussed there would seem to be no obstruction to the discussion for \( \Lambda \) positive or zero (see the discussion in [11]). For example, for \( \Lambda > 0 \), the parameters \( r^\pm_1, r^\pm_2 \) would be unchanged but the “shifted connections” (18), and in consequence, the holonomies (9) and the corresponding de Sitter or \( \text{sl}(2,\mathbb{C}) \) generators \( j_a^\pm, a = 0, 1, 2 \), equation (32) would be complex conjugates of each other rather than real and independent as here, for \( \Lambda < 0 \).

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