Linear Stability Analysis of Differentially Rotating Polytropes
— New results for the \( m = 2 \) \( f \)-mode dynamical instability —

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ABSTRACT

We have studied the \( f \)-mode oscillations of differentially rotating polytropes by making use of the linear stability analysis. We found that the critical values of \( T/|W| \) where the dynamical instability against the \( m = 2 \) \( f \)-mode oscillations sets in decrease down to \( T/|W| \approx 0.20 \) as the degree of differential rotation becomes higher. Here \( m \) is an azimuthal mode number and \( T \) and \( W \) are the rotational energy and the gravitational potential energy, respectively. This tendency is almost independent of the compressibility of the polytropes. These are the first exact results of the linear stability analysis for the occurrence of the dynamical instability against the \( m = 2 \) \( f \)-modes.

Subject headings: gravitational waves—stars: neutron—stars: rotation—stars: oscillation

1. Introduction

It is well known that rapidly rotating stars suffer from dynamical instability against nonaxisymmetric oscillations. In particular, the growth rate for the \( m = 2 \) or bar-type \( f \)-mode instability has been considered to be largest. The classical results about this instability were first studied by Riemann (1860). Riemann found that uniformly rotating incompressible fluids become dynamically unstable when the ratio of the rotational energy, \( T \), to the absolute value of the gravitational potential energy, \( W \), exceeds 0.27, i.e. \( \beta \equiv T/|W| \geq 0.27 \) (see also Bryan 1889).

Around 1970, Ostriker and coworkers developed a new numerical method to solve equilibrium configurations of rapidly rotating compressible stars (Ostriker & Mark 1968; Ostriker & Bodenheimer 1968; Mark 1968; Jackson 1970; Bodenheimer & Ostriker 1970, 1973; Bodenheimer 1971). Ostriker and coworkers also developed a new technique to explore stabilities of rotating equilibrium configurations by extending the tensor-virial technique of Chandrasekhar (Chandrasekhar 1969; Tassoul & Ostriker 1968; Ostriker & Tassoul 1969; Ostriker & Bodenheimer 1973). Their main result about the \( m = 2 \) dynamical instabilities of compressible stars was that rapidly rotating configurations become unstable when the condition \( \beta \geq 0.27 \) is satisfied irrespective of the compressibilities and the rotation laws as far as they investigated. Although there was no proof of this ”universal” nature of the critical value \( \beta_{\text{critical}} \approx 0.27 \), most people have believed that the critical value for the \( m = 2 \) \( f \)-mode dynamical instability is 0.27. This ”belief” has lasted rather long, although there appeared a paper that shows the tensor virial technique for stability analysis for differentially rotating stars gives neither the necessary condition nor the sufficient condition for the stability (Bardeen et al. 1977).

However, the situation has been changing because there appear many recent results of hydrodynamical simulations which show that configurations with lower values of \( \beta \) suffer from dynamical instabilities (see e.g. Tohline & Hachisu 1990; Pickett et al. 1996; Toman et al. 1998; Brown 2000; Centrella et al. 2001; for simulations of linearized equations see e.g. Liu 2002). Rough values of the critical configurations against nonaxisym-
metric perturbations are $\beta_{\text{critical}} \sim 0.16$ ($m = 2$) for self-gravitating toroidal polytropes with the polytropic index $N = 1.5$ (Tohline & Hachisu 1990), $\beta_{\text{critical}} \sim 0.2$ ($m = 2$) for $N = 1.5$ polytropic star-disk like configurations (Pickett et al. 1996), $\beta_{\text{critical}} \sim 0.24 \sim 0.28$ ($m = 2$) for $N = 5/4, 6/4, 10/4$ polytropes with the rotation laws that the distribution of the specific angular momentum is that of uniformly rotating polytropes with the same polytropic indices (Toman et al. 1998), $\beta_{\text{critical}} \sim 0.24$ ($m = 2$) for $N = 1.5$ polytropes with the angular velocity of the exponential decay with the distance from the rotation axis (Brown 2000), and $\beta_{\text{critical}} \sim 0.14$ ($m = 1$) for toroidal-like $N = 3.33$ polytropes (Centrella et al. 2001). Liu found $\beta_{\text{critical}} \sim 0.25$ ($m = 2$) for the angular momentum distribution which could be realized after accretion induced collapse of uniformly rotating massive white dwarfs, although his simulations were carried out by using the linearized equations.

These results seem to imply that the critical value of $\beta$ for the occurrence of the $m = 2$ bar-type dynamical instability cannot be considered to be universal any more. In order to see the unstable nature of rotating stars, we need to investigate sequences of rotating configurations systematically. For that purpose, the best method would be the linear stability analysis rather than hydrodynamical simulations.

Except for non-self-gravitating disks, linear stability analyses for rapidly rotating self-gravitating stars have been done only for limited kinds of compressibilities and rotation laws, and for limited types of oscillations. Luyten (1990, 1991) found a new dynamical instability for differentially rotating polytropes. This instability seems to belong to the shear instability which is also found in the theory of accretion disks (see e.g. Papaloizou & Pringle 1984; Narayan & Goodman 1989). This instability appears for differentially and very slowly rotating stars and disappears for differentially but very rapidly rotating stars. However, Luyten’s analysis was applied only to very limited kinds of rotation laws and the displacements were restricted only in the plane parallel to the equator. Very recently, Shibata et al. (2002, 2003) have investigated equilibrium sequences of differentially rotating polytropes by using linear stability analysis, the numerical scheme of which is the same as that employed in this paper, and shown that the dynamical instability of a new type can set in even when $\beta$ is $\sim 0.05$ if the degree of differential rotation is very high.

However, even the classical $m = 2$ $f$-mode instability has not been fully studied by the linear stability analysis. Thus, in this paper, we will study systematically the dynamical stability of rapidly and differentially rotating polytropic stars by using the linear stability analysis. It should be noted that the linear stability analysis of differentially rotating polytropes in this paper will give exact values of $\beta_{\text{critical}}$ for the onset of $m = 2$ $f$-mode dynamical instabilities for the first time.

2. Numerical Method

The method to investigate stability by solving linearized basic equations is the same as that used in Karino et al. (2000, 2001). Therefore, we will summarize the essence of the method briefly.

2.1. Equilibrium states

Axisymmetric equilibrium configurations of rotating Newtonian stars are obtained by solving the following equations in the spherical polar coordinates ($r, \theta, \phi$):

\[ \frac{1}{\rho} \nabla p = -\nabla \phi + r \sin \theta \Omega^2 \hat{e}_R, \tag{1} \]

\[ \Delta \phi = 4\pi G \rho, \tag{2} \]

where $p$, $\rho$, $\phi$, $\Omega$, $\hat{e}_R$, and $G$ are the pressure, the density, the gravitational potential, the angular velocity, a unit vector in the $R$ direction of the cylindrical coordinates $(R, \varphi, z)$, and the gravitational constant, respectively. As for the equation of state, we assume the polytropic relation as follows:

\[ p = K \rho^{\frac{1}{N}} \tag{3} \]

where $K$ and $N$ are the polytropic constant and the polytropic index, respectively. The rotation law is assumed as follows:

\[ \Omega = \frac{\Omega_c A^2}{(r \sin \theta)^2 + A^2}, \tag{4} \]

where $\Omega_c$ is the central angular velocity. The quantity $A$ is a parameter which represents the degree of differential rotation. The rotation becomes more differential as $A$ becomes smaller. We
call this rotation law as the \( j \)-constant rotation law.

One equilibrium state can be characterized by specifying three parameters as follows: the polytropic index \( N \), the parameter \( A \) and the parameter describing the amount of rotation which can be represented by the following axis ratio:

\[
q \equiv r_p/r_e ,
\]

where \( r_p \) and \( r_e \) are the polar and equatorial radii, respectively. These equations are solved by applying the proper boundary conditions and using the SFNR method of Eriguchi and Müller (1985). Equilibrium models in this paper have been solved by using mesh numbers \( (r \times \theta) = (42 \times 19) \).

2.2. Perturbed States

As mentioned before, the linearized equations are solved by the scheme developed by Karino et al. (2000, 2001). The perturbed quantities are expanded as follows:

\[
\delta f(r, \theta, \varphi, t) = \sum_m \exp(-i(\sigma t - m\varphi)) f_m(r, \theta),
\]

where \( \delta f \) means the Euler perturbation of the corresponding quantity. Here, \( m \) is the azimuthal mode number and \( \sigma \) is the eigenfrequency of the oscillation mode. As for the density and the pressure perturbations, we assume the adiabatic oscillation mode. As for the density and the \( \delta \) mode number and \( \sigma \) the obtained eigenvalues can be considered to be accurate enough. We have checked the eigenvalues by changing the mesh number. For \( N = 1.0 \) polytropes with extremely rapid rotation \( (q = 0.2) \), the accuracy of the obtained eigenfrequency can be estimated by \( \delta \sigma(m, n) \equiv |(\sigma_m - \sigma_n)/\sigma_m| \), where \( \sigma_k \) denotes the eigenvalue when the mesh number in the \( r \)-direction is \( k \). For \( A = 1.0 \) models, \( \delta \sigma(42, 30) = 1.8 \% \), and \( \delta \sigma(42, 36) = 0.6 \% \).

3. Numerical Results

By fixing the values of \( A \) and \( N \) and by changing the value of \( q \), we can obtain eigenvalues of \( m = 2 \) \( f \)-mode oscillations along an equilibrium sequence. After that by changing the values of \( A \) and/or \( N \) and following the same procedure, we can obtain eigenstates for many equilibrium sequences. In this paper, we have examined sequences with \( N = 0.0, 1.0 \) and \( 1.5 \), and mainly \( 1.0 \leq A^{-1} \). It may be noted that for models with large values of \( A \), mass begins to shed from the equatorial surface before the dynamical instability sets in. Hence it is not necessary to consider larger values of \( A \) more than the value for which mass shedding occurs.

In Figure 1, the real part of eigenfrequencies of the \( N = 1.0 \) sequences with \( A = 1.0 \) is plotted against the value of \( \beta \). In this plane, the \( m = 2 \) \( f \)-mode has two branches for lower values of \( \beta \). The upper branch corresponds to the co-rotating mode and the lower one corresponds to the counter-rotating mode. These two branches merge into two complex conjugate branches for higher values of \( \beta \). The positive imaginary part is shown in Figure 2. The model for which the two branches merge corresponds to the critical configuration from which dynamically unstable configurations follow. For this sequence, i.e. \( N = 1.0 \) and \( A = 1.0 \) sequence, the critical value where the dynamical instability for \( m = 2 \) \( f \)-mode sets in is \( \beta \simeq 0.266 \).

The eigenvalues of configurations of \( N = 1 \) polytropic sequences with several values of \( A \) are shown against the value of \( \beta \): real parts in Figure 3 and imaginary parts in Figure 4. It is clear from Figures 3 and 4 that the values of \( \beta \) where two real modes change to two complex conjugate modes become smaller as the degree of differential rotation becomes higher. It means that the critical point of dynamical instability will appear for smaller values of \( \beta \) as the degree of differential rotation becomes significant.

The dependency of the critical points of dynamical instability on the degree of differential rotation and the compressibility can clearly be seen from Figure 5. In this figure, the values of \( \beta \) at the critical points for \( N = 0.0, 1.0 \) and \( 1.5 \) polytropes are plotted against the parameter \( A^{-1} \). The critical values of \( \beta \) for the \( m = 2 \) \( f \)-mode dynamical instability depend significantly on the values of \( A \) but depend only slightly on the values of \( N \) for the high degree of differential rotation. Note that, as it should be, the critical value of \( \beta_{\text{critical}} \) approaches
to $\beta_{\text{critical}} = 0.27$ when the rotation law tends to the rigid rotation ($A^{-1} \to 0$) and the unperturbed configuration becomes the Maclaurin spheroid.

In Figures 6-8, the eigenfunctions of the density perturbation for $N = 1$ polytropes with $A = 1.0$ but with different values of $q$ are plotted against the normalized distance from the center of the star at selected values of $\theta$. In slowly rotating regions, the eigenfunctions are almost similar to those of $f$-mode oscillations for the Maclaurin spheroid, i.e. monotonically increasing functions of the distance from the center (see e.g. Tassoul 1978). As the stellar rotation becomes faster, however, the eigenfunctions on and near the equator attain their maximum values on the way to the surface but not on the surface.

4. Discussions

By making use of the linear stability analysis, we have shown that, for $m = 2$ $f$-mode oscillations, higher degree of differential rotation makes the stars more unstable than previously thought. Although results of recent dynamical simulations have shown the same tendency, it should be noted that our present results have been obtained by the systematic analysis of the oscillations and that we can understand the occurrence of this type of instability from a more rigorous study.

Since the parameter regions for the existence of dynamically unstable configurations are widened, we need to reconsider some astrophysical situations under the light of the present results. There are several situations where high differential rotation would be realized. One example can be outcomes of collapsing gases or stars, i.e. star formation stages or core collapse stages of massive stars or accretion-induced collapse processes of stars. Other example may be outcomes of collision or merging of two stars, i.e. merging of two neutron stars in a binary neutron star system.

In the star formation process, if the angular momentum is not lost from the gaseous system, the rotation of the final young stellar objects would be extremely rapid and highly differential. Thus they might suffer from the dynamical instability discussed in this paper. Although the outcomes of the dynamical instability have not been fully understood yet, several authors have argued that dynamical instabilities during the star formation stages are related to binary formations (see e.g. Bonnell 1994; Matsumoto & Hanawa 1999), and others have suggested that they would result in new systems consisting of a central young stellar object and a disk around it (see e.g. Bate 1998).

In order to obtain a final answer to this problem, reliable nonlinear simulations and high-resolution infrared/radio observations about star forming regions are required.

Another example is related to newly-born neutron stars. There are several ways for neutron stars to be formed. One is the formation via core collapse of massive stars followed by Type II supernova explosions. In recent simulations, it is shown that the compact object formed from an iron core collapse is rotating rapidly and differentially with $\beta \sim 0.2$ (Zwerger & Müller 1997; Dimmelmeier et al. 2002). Another is a merging of two neutron stars in a binary neutron star system. In the merging process in a binary neutron star system, there appears a rapidly rotating massive neutron star with high degree of differential rotation (see e.g. Shibata and Uryu 2000). Neutron stars are also considered to be formed by accretion onto massive white dwarfs. In this accretion induced scenario, the outcome would be a rapidly and differentially rotating massive neutron star (see e.g. Liu & Lindblom 2000). Therefore, for these differentially rotating neutron stars, it would be very likely that the dynamical instability would set in easily and lead the systems to nonaxisymmetric configurations just depending on the degree of differential rotation and irrespective of the compressibility. Those deformed and rapidly rotating compact objects would emit a large amount of gravitational waves. Therefore, such young neutron stars could be important targets of ground-based laser interferometric detectors of gravitational waves (see e.g. Brown 2000; Shibata et al. 2002)\(^2\).

Concerning the rotation law dependence of the critical values for the $m = 2$ $f$-mode dynamical instability, we have performed linear stability analysis for equilibrium models with the $v$-constant rotation law (see e.g. Karino et al. 2001). How-

\(^2\)When we consider these compact stars, the effect of general relativity becomes important. In relation to the dynamical instability, however, it is only argued that the critical values might become slightly smaller by the general relativistic effect (Shibata, Baumgarte, & Shapiro 2000).
ever, the effect of differential rotation seems very weak and the dynamical instability occurs only for models with very stiff equations of state and/or extremely small values of the parameter $A$. Additionally, the change of the critical values of $\beta$ is very little. The critical values for $N = 0.5$ polytropes are $T/W \sim 0.266$ for $A = 0.8$ and $T/W \sim 0.260$ for $A = 0.5$.

5. Summary

In this paper, we have performed linear stability analysis of equilibrium sequences of differentially rotating polytropes in Newtonian gravity and obtained critical values of $\beta$, i.e. $\beta_{\text{critical}}$, which determine the equilibrium states where the dynamical instability against $m = 2$ $f$-mode oscillations sets in. The rotation law which we have employed is the $j$-constant law (Eq. (4)). Our important finding is that the critical points of $m = 2$ $f$-mode dynamical instability depend strongly on the degree of differential rotation while they depend weakly on the compressibility. The critical values of $\beta_{\text{critical}}$ decreases as the degree of differential rotation becomes higher. Therefore, we should not use the "universal" value of $\beta_{\text{critical}} = 0.27$ for the stability analysis of rapidly rotating objects with high degree of differential rotation from now on.

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REFERENCES

Fig. 1.— Real part of the eigenvalue of the $m = 2$ $f$-mode in units of $\sqrt{4\pi G \rho_c}$ is plotted against the ratio $T/|W|$ for polytropes with $N = 1.0$ and $A = 1.0$. Here $\rho_c$ denotes the central density. In slowly rotating regions, the $f$-mode has two real branches, but when the stellar rotation exceeds a certain critical point, these two branches merge into one branch, in this real part of the eigenvalue plane, since eigenvalues are complex conjugate. This corresponds to the critical point where the $m = 2$ $f$-mode dynamical instability sets in.

Fig. 2.— Positive imaginary part of the eigenvalue for the same mode as Figure 1 is plotted against $T/|W|$. In slowly rotating regions, the imaginary part of the eigenfrequency vanishes and it implies that the configurations are dynamically stable. After the stellar rotation reaches a certain critical point, the imaginary part of the eigenvalues appears and grows very rapidly. This corresponds to the critical point where the $m = 2$ $f$-mode dynamical instability sets in.
Fig. 3.— Real part of the eigenvalues around the merging points is plotted against $T/|W|$ for polytropic stars with $N = 1.0$ and $A = 0.7, 0.8, 0.9, 1.0$. As the degree of differential rotation becomes higher, the values of $T/|W|$ at the critical points become lower.

Fig. 4.— Same as Figure 3 but for the imaginary part of eigenvalues.

Fig. 5.— Values of $T/|W|$ at the critical points where the $m = 2$ $f$-mode dynamical instability sets in are plotted against the degree of differential rotation, $A^{-1}$. The stellar models are polytropes with $N = 0.0, 1.0$ and 1.5. As the degree of differential rotation becomes higher, the critical value of $T/|W|$ tends to decrease.

Fig. 6.— Eigenfunctions of the density perturbation for the bar-mode oscillation of a differentially rotating polytropic star with $N = 1.0$, $A = 1.0$, and $q = 0.90$ are plotted against the normalized distance from the rotation axis. Here $R_s(\theta)$ is the radius to the surface along the the polar angle $\theta$. Three curves correspond to the eigenfunctions on the spokes with $\theta = \pi/6$ (curve 1), $\theta = \pi/3$ (curve 2), and $\theta = \pi/2$ (curve 3). The value of the eigenfunctions is normalized so that the maximum absolute value among all the perturbed quantities becomes unity.
Fig. 7.— Same as Figure 6 but for $q = 0.55$.

Fig. 8.— Same as Figure 6 but for $q = 0.30$. 