Cubic Matrix, Generalized Spin Algebra and Uncertainty Relation

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Abstract

We propose a generalization of spin algebra using three-index objects. There is a possibility that a triple commutation relation among three-index objects implies a kind of uncertainty relation among their expectation values.
1 Introduction

Matrix is an instrument indispensable to mathematics and physics. One of the main reason is as follows. Physical systems are often described by many variables, some of which are treated on an equal footing, e.g., coordinates of space. The systems have symmetries under some transformations for variables and such symmetry transformations, in many cases, form a group, e.g., rotations for coordinates of space form a rotational group. Use of matrices makes the analysis with many variables simple and systematic because an action for group elements can be represented by a matrix. In fact, the group theoretical analysis has been applied to a wide range of systems, e.g., the classification of elementary particles and the determination of interactions among them.[1] Considering its success, it is natural to ask the following questions:

1. Is there a generalization of matrix?

2. If there is a generalization, what advantages and applications for mathematics and physics?

For the first question, one might hit upon a many-index object such as $A_{m_1m_2...m_n}$ since a matrix is regarded as a two-index object. For the second one, a possible answer is that a new mechanics has been proposed based on many-index objects[2] and its basic structure has been studied from the algebraic point of view[3, 4]. This mechanics has a counterpart of the canonical structure in classical mechanics or Nambu mechanics[5], and is interpreted as its ‘quantum’ or ‘discretized’ version. It is also regarded as a generalization of Heisenberg’s matrix mechanics because a generalization of the Ritz rule is taken as a guiding principle, This mechanics has such interesting properties, but it is not clear whether it is applicable to a real physical system and what physical meanings many-index objects possess. There is also the possibility of examining an algebraic structure of many-index objects independent of its dynamics to provide information on their physical reality and implications. This is a main motivation of our work.

In this paper, we propose a generalization of spin algebra using three-index objects and point out a possibility that a triple commutation relation leads to a kind of uncertainty relation.
This paper is organized as follows. In the next section, we explain the definition of three-index object. We study a generalization of spin algebra in §3. In section 4, we discuss a connection between a triple commutation relation among three-index objects and an uncertainty among their expectation values. Section 5 is devoted to conclusions and discussion.

2 Cubic matrix

Here we state our definition of a cubic matrix and its related terminology. A cubic matrix is an object with three indices, $A_{lmn}$, which is a generalization of a usual matrix, such as $B_{mn}$. We refer to a cubic matrix whose elements possess cyclic symmetry, i.e., $A_{lmn} = A_{mnt} = A_{nml}$, as a cyclic cubic matrix. We define the hermiticity of a cubic matrix by $A_{l'm'n'} = A^*_{lmn}$ for odd permutations among indices and refer to a cubic matrix possessing hermiticity as a hermitian cubic matrix. Here, the asterisk indicates complex conjugation. A hermitian cubic matrix is a special type of cyclic cubic matrix, because it obeys the relations $A_{lmn} = A^*_{mnl} = A^*_{nml} = A^*_{nml} = A^*_{l'mn}$. We refer to the following form of a cubic matrix as a normal form or a normal cubic matrix:

$$A_{lmn} = \delta_{lm}a_{mn} + \delta_{mn}a_{nl} + \delta_{nl}a_{lm}. \quad (1)$$

A normal cubic matrix is also a special type of cyclic cubic matrix. The elements of a cubic matrix are treated as $c$-numbers throughout this paper.

We define the triple product among cubic matrices $A_{lmn}$, $B_{lmn}$ and $C_{lmn}$ by

$$(ABC)_{lmn} \equiv \sum_k A_{lmk}B_{lkn}C_{kmn}. \quad (2)$$

The resultant three-index object, $(ABC)_{lmn}$, does not always have cyclic symmetry, even if $A_{lmn}$, $B_{lmn}$ and $C_{lmn}$ are cyclic cubic matrices. Note that this product is, in general, neither commutative nor associative; that is,

\[\text{Awata, Li, Minic and Yoneya introduced many-index objects to realize the quantum version of Nambu bracket.} [6]\] They refer to three-index object as ‘cubic matrix’ and we use the naming in this paper although the definition of triple product is different each other.
(ABC)l_{mn} \neq (BAC)l_{mn} \text{ and } (AB(CDE))l_{mn} \neq (A(BCD)E)l_{mn} \neq ((ABC)DE)l_{mn}.

The triple-commutator is defined by

\[ [A, B, C]l_{mn} \equiv (ABC + BCA + CAB - BAC - ACB - CBA)l_{mn} \]  \hspace{1cm} (3)

The triple-anticommutator is defined by

\[ \{A, B, C\}l_{mn} \equiv (ABC + BCA + CAB + BAC + ACB + CBA)l_{mn} \]  \hspace{1cm} (4)

If \( A_{l_{mn}}, B_{l_{mn}} \) and \( C_{l_{mn}} \) are hermitian cubic matrices, \( i[A, B, C]l_{mn} \) and \( \{A, B, C\}l_{mn} \) are also hermitian cubic matrices.

Further we define the product between cubic matrices \( A_{l_{mn}} \) and \( B_{l_{mn}} \) by

\[ (AB)l_m \equiv \sum_k A_{lmk}B_{klm}. \]  \hspace{1cm} (5)

If \( A_{l_{mn}} \) and \( B_{l_{mn}} \) are hermitian cubic matrices, the two-index object, \( (AB)l_m \), possess a hermiticity, i.e., \( (AB)l_m = (AB)^*l_m \). If \( A_{l_{mn}} \) and \( B_{l_{mn}} \) are cyclic cubic matrices, they commute with respect to this product, i.e., \( (AB)l_m = (BA)l_m \).

### 3 Generalized spin algebra

#### 3.1 Spin algebra

Here we review the spin algebra \( su(2) \). The algebra is defined by

\[ [J^a, J^b] = i\hbar \varepsilon^{abc} J^c, \]  \hspace{1cm} (6)

where \( J^a \)'s are spin variables \( (a = 1, 2, 3) \), \( \hbar \) is the reduced Planck constant and \( \varepsilon^{abc} \) is the Levi-Civita symbol. Matrices on the adjoint representation are written by \( 3 \times 3 \) matrices such as

\[ (J^a)_{mn} = -i\hbar \varepsilon^{amn}, \]  \hspace{1cm} (7)

where each of the indices \( m \) and \( n \) runs from 1 to 3. Further matrices on the spinor representation are written by \( 2 \times 2 \) matrices such as

\[ (J^a)_{mn} = \frac{\hbar}{2} (\sigma^a)_{mn}, \]  \hspace{1cm} (8)
where \( \sigma \)'s are Pauli matrices and each of the indices \( m \) and \( n \) runs from 1 to 2.

In general, matrices on the spin \( j \) representation are given by \( N \times N \) matrices:

\[
\begin{align*}
(J^1)_{mn} &= \frac{\hbar}{2} \left( \sqrt{n(N-n)} \delta_{mn+1} + \sqrt{m(N-m)} \delta_{mn-1} \right), \\
(J^2)_{mn} &= \frac{\hbar}{2i} \left( \sqrt{n(N-n)} \delta_{mn+1} - \sqrt{m(N-m)} \delta_{mn-1} \right), \\
(J^3)_{mn} &= \frac{\hbar}{2} \left( 2m - N - 1 \right) \delta_{mn},
\end{align*}
\]

(9)

where each of the indices \( m \) and \( n \) runs from 1 to \( N = 2j + 1 \). The Casimir operator \( \vec{J}^2 \) is given by

\[
(\vec{J}^2)_{mn} \equiv (J^1)_{mn}^2 + (J^2)_{mn}^2 + (J^3)_{mn}^2 = \hbar^2 j(j+1) \delta_{mn}.
\]

(10)
The spinor representation matrices (8) are obtained from (9) with \( j = \frac{1}{2} \), and the adjoint representation matrices (7) are obtained from (9) with \( j = 1 \) after making a suitable unitary transformation.

### 3.2 Generalization

Let us generalize the spin algebra (6) using hermitian cubic matrices. We consider the following \( 4 \times 4 \times 4 \) matrices as a counterpart of (7),

\[
\begin{align*}
(J^a)_{lmn} &= -i\hbar_C \varepsilon^{almn}, \\
(K^a)_{lmn} &= \hbar_C |\varepsilon^{almn}|,
\end{align*}
\]

(11)

where each of the indices \( a, l, m \) and \( n \) runs from 1 to 4 and \( \hbar_C \) is a new physical constant. The \( (J^a)_{lmn} \) and \( (K^a)_{lmn} \) satisfy the following algebra,

\[
\begin{align*}
[J^a, J^b, J^c] &= -i\hbar_C \varepsilon^{abcd} K^d, \\
[J^a, J^b, K^c] &= -i\hbar_C \varepsilon^{abcd} J^d, \\
[J^a, K^b, K^c] &= i\hbar_C \varepsilon^{abcd} K^d, \\
[K^a, K^b, K^c] &= i\hbar_C \varepsilon^{abcd} J^d,
\end{align*}
\]

(12)

where the indices \( l, m \) and \( n \) are omit. There exists a subalgebra in (12) whose elements consist of \( J^a, J^b, J^c \) and \( K^d \) (or \( K^a, K^b, K^c \) and \( J^d \)). Here \( a, b, c \) and \( d \) are different numbers among them. For example, \( G^a = (J^1, J^2, J^3, K^4) \) satisfy the algebra:

\[
[G^a, G^b, G^c] = -i\hbar_C \varepsilon^{abcd} G^d.
\]

(13)
We refer to the above algebra (13) as a ‘cubic spin algebra’. The $G^a$’s satisfy the so-called ‘fundamental identity’:

$$[[G^a, G^b, G^c], G^d, G^e] = [[G^a, G^d, G^e], G^b, G^c]$$

$$+ [G^a, [G^b, G^d, G^e], G^c] + [G^a, G^b, [G^c, G^d, G^e]].$$  \(14\)

As a counterpart of spinor representation matrices (8), we define four kinds of hermitian $3 \times 3 \times 3$ matrices:

$$(S^1)_{lmn} \equiv \frac{\hbar c}{\sqrt{2}} \varepsilon_{lmn}, \quad (S^2)_{lmn} \equiv \frac{\hbar c}{i\sqrt{2}} \varepsilon_{lmn},$$

$$(S^3)_{lmn} \equiv \frac{\hbar c}{\sqrt{2}} (\delta_{lm} \varepsilon_{mn} + \delta_{mn} \varepsilon_{nl} + \delta_{nl} \varepsilon_{lm}),$$

$$(S^4)_{lmn} \equiv \frac{\hbar c}{\sqrt{2}} (\delta_{lm} \varepsilon_{mn} + \delta_{mn} \varepsilon_{nl} + \delta_{nl} \varepsilon_{lm}).$$  \(15\)

where each of the indices $l, m$ and $n$ runs from 1 to 3, $\varepsilon_{mn} \equiv (\delta_{m1} - \delta_{m2})\delta_{n3}$ and $\varepsilon_{mn} \equiv \delta_{m1}\delta_{n2} + \delta_{m2}\delta_{n1}$. It is shown that the variables $(S^a)_{lmn}$ yield the cubic spin algebra (13).  \(^3\)

For $(N + 1) \times (N + 1) \times (N + 1)$ matrices ($N \geq 3$) which satisfy the cubic spin algebra (13), we can find the example:

$$(G^1)_{lmn} = \frac{\hbar c}{81^{1/4}} \left( (\delta_{lm} - \delta_{lm} + \delta_{lm} - \delta_{lm} + \delta_{lm} + N - 1) 
\cdot (1 - \delta_{lN} + 1)(1 - \delta_{mN} + 1) \delta_{nN} + 
(\delta_{mn} - \delta_{mn} + \delta_{mn} - \delta_{mn} + \delta_{mn} + N - 1)(1 - \delta_{mN} + 1)(1 - \delta_{mN} + 1) \delta_{N} + 
(\delta_{nl} - \delta_{nl} + \delta_{nl} - \delta_{nl} + \delta_{nl} + N - 1)(1 - \delta_{N} + 1)(1 - \delta_{N} + 1) \delta_{mN} + 1
\right),$$

$$(G^2)_{lmn} = \frac{\hbar c}{81^{1/4} i} \left( (\delta_{lm} - \delta_{lm} + \delta_{lm} - \delta_{lm} + \delta_{lm} + N - 1) 
\cdot (1 - \delta_{lN} + 1)(1 - \delta_{mN} + 1) \delta_{nN} + 
(\delta_{mn} - \delta_{mn} + \delta_{mn} - \delta_{mn} + \delta_{mn} + N - 1)(1 - \delta_{mN} + 1)(1 - \delta_{mN} + 1) \delta_{N} + 
(\delta_{nl} - \delta_{nl} + \delta_{nl} - \delta_{nl} + \delta_{nl} + N - 1)(1 - \delta_{N} + 1)(1 - \delta_{N} + 1) \delta_{mN} + 1
\right),$$

$$(G^3)_{lmn} = \frac{\hbar c}{21^{1/4}} (\delta_{lm} g_{mn} + \delta_{mn} g_{nl} + \delta_{nl} g_{lm}),$$

$$(G^4)_{lmn} = \frac{\hbar c}{81^{1/4}} (\delta_{lm} g_{mn} + \delta_{mn} g_{nl} + \delta_{nl} g_{lm}).$$  \(16\)

\(^3\)This choice is not unique, but it belongs to $(S^a)_{lmn}$ with $\varepsilon_{mn} = \epsilon_m \varepsilon_{mn}^1$ and $\varepsilon_{mn} = \epsilon_m \varepsilon_{mn}^2$. Here $\epsilon_m$ takes 1 or -1.
where each of the indices $l$, $m$ and $n$ runs from 1 to $N = 2j + 1$, and $g^3_{mn}$ and $g^4_{mn}$ are defined by

$$g^3_{mn} \equiv \epsilon_m (1 - \delta_{mN+1}) \delta_{nN+1}$$

and

$$g^4_{mn} \equiv \epsilon_m (\delta_{mn-1} - \delta_{mn+1} + \delta_{mn-N+1} - \delta_{mn+N-1})$$

$\cdot (1 - \delta_{mN+1})(1 - \delta_{nN+1}),$

respectively. Here $\epsilon_m$ takes 1 or $-1$.

As a comment, we can find $N \times N$ matrices $(M^a)_{mn}$ which satisfy the algebra $[M^a, M^b, M^c]_{mn} = -i\hbar^2 \varepsilon^{abcd} (M^d)_{mn}$. Here the triple commutator $[M^a, M^b, M^c]_{mn}$ is defined by

$$[M^a, M^b, M^c]_{mn} \equiv (M^a M^b M^c + M^b M^c M^a + M^c M^a M^b)$$

$$- M^b M^a M^c - M^a M^c M^b - M^c M^b M^a)_{mn}$$

$$= ([M^a, M^b] M^c)_{mn} + ([M^b, M^c] M^a)_{mn}$$

$$+ ([M^c, M^a] M^b)_{mn}$$

with a usual definition of the triple product among matrices:

$$(M^a M^b M^c)_{mn} \equiv \sum_{k,l} (M^a)_{mk} (M^b)_{kl} (M^c)_{ln}. \quad (20)$$

An example is as follows,

$$(M^a)_{mn} = \frac{1}{(j(j+1))^1/4} (J^a)_{mn}, \quad (M^4)_{mn} = -(j(j+1))^{1/4} \hbar \delta_{mn}, \quad (21)$$

where $(J^a)_{mn}$ is given in (9).

4 Uncertainty relation

4.1 Uncertainty relation in quantum mechanics

The uncertainty relation in quantum mechanics is expressed by

$$\delta x \delta p \geq \frac{\hbar}{2} \quad (22)$$
where \( \delta x \) and \( \delta p \) represent an uncertainty of position and its canonical momentum, respectively. This relation and a generalization are nicely formulated.\(^7\) For any observable \( A \), there is a correspondence to a hermitian operator \( \hat{A} \). The expectation value of \( A \) is defined by

\[
\langle A \rangle \equiv \int \psi^*(\vec{r})A\psi(\vec{r})d^3r,
\]

where \( \psi(\vec{r}) \) is a wave function and describes a state of the system. The uncertainty of the value of measurement for \( A \) is defined by

\[
\delta A \equiv \sqrt{\langle (A - \langle A \rangle)^2 \rangle} = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}.
\]

(24)

Here \( \delta A \) is a standard deviation which represents the magnitude of fluctuation about the mean value. For any observables \( A \) and \( B \), the following uncertainty relation holds;

\[
\delta A \delta B \geq \frac{1}{2} |\langle [A, B] \rangle|.
\]

(25)

Let us derive the above relation in the matrix formalism for a later convenience. The following relationship exists between the matrix \( A_{mn} \) in Heisenberg’s matrix mechanics and the hermitian operator \( \hat{A} \),

\[
A_{mn} = \int \phi_m^*(\vec{r})\hat{A}\phi_n(\vec{r})d^3r,
\]

(26)

where \( \phi_n(\vec{r}) \)s constitute a complete set of orthonormal functions.\(^5\) From (23) and (26), the \( \langle A \rangle \) is written down as

\[
\langle A \rangle = \sum_{m,n} a_m^* a_n A_{mn}
\]

(27)

for the wave function \( \psi(\vec{r}) = \sum_n a_n \phi_n(\vec{r}) \). In the same way, the expectation value of \( \hat{A}\hat{B} \) is written by

\[
\langle AB \rangle = \sum_{m,n} \sum_k a_m^* a_n A_{mk} B_{kn} \equiv \langle \hat{A}, \hat{B} \rangle.
\]

(28)

\(^5\)Here we treat a case with a discrete spectrum for simplicity, but it is straightforward to extend a case with a continuous one.
Here \( \vec{B} \) stands for a complex vector whose component is \( \sum_n B_{kn} a_n \). Then the uncertainty relation (25) is shown by use of the Schwarz inequality \( |\vec{\mathcal{A}}|^2 |\vec{\mathcal{B}}|^2 \geq |(\vec{\mathcal{A}}, \vec{\mathcal{B}})|^2 \) and the relation \( \hat{A}\hat{B} = \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{2}\{\hat{A}, \hat{B}\} \). Hence the uncertainty relation is understood as a consequence of algebraic relation between physical variables. If the expectation value \( \langle [\hat{A}, \hat{B}] \rangle \) does not vanish, it is not possible to measure the values of \( A \) and \( B \) simultaneously as precisely as possible. The uncertainty relation (22) is derived from the commutation relation for \( \hat{x} \) and \( \hat{p} \), i.e., \( [\hat{x}, \hat{p}] = i\hbar \).

### 4.2 Generalized uncertainty relation

We have seen that the uncertainty relation \( \delta A \delta B \geq \frac{1}{2} |\langle C \rangle| \) is derived from the commutation relation such as \( [A, B]_{mn} = iC_{mn} \) in quantum mechanics. Then, we have a question whether there is an uncertainty relation originated from the triple commutation relation such as \( [A, B, C]_{lmn} = iD_{lmn} \). (The typical triple commutation relation is the cubic spin algebra discussed in the previous section.) In the following, we will find that an inequality such as \( \delta A \delta B \delta C \geq \frac{1}{6} |\langle D \rangle| \) is derived with certain types of definitions for expectation values of many-index objects. We define the expectation value for a cubic matrix \( A_{lmn} \) by

\[
\langle A \rangle_c \equiv \sum_{l,m,n} |a_la_ma_n| e^{i(\theta_{lm} + \theta_{mn} + \theta_{nl})} A_{lmn},
\]

where \( a_l \) is a complex number and \( \theta_{lm} \) is a real antisymmetric object, \( \theta_{lm} = -\theta_{ml} \). Then the expectation value of \( (ABC)_{lmn} \) is written by

\[
\langle ABC \rangle_c = \sum_{l,m,n} |a_la_ma_n| e^{i(\theta_{lm} + \theta_{mn} + \theta_{nl})} \sum_k A_{lmk} B_{lkn} C_{kmn}
\]

\[
\equiv \sum_{l,m,n} \sum_k A^{(k)}_{lm} B^{(k)}_{mn} C^{(k)}_{mn} = \sum_k \text{Tr}(A^{(k)}C^{(k)}B^{(k)}) \tag{30}
\]

where \( A^{(k)}_{lm} \equiv |a_la_m|^{1/2} e^{i\theta_{lm}} A_{lmk} \). Further we define the expectation value for a two-index object \( B_{lm} \) by

\[
\langle B \rangle_s \equiv \sum_{l,m} |a_la_m| e^{i(\theta_{lm} - \theta_{ml})} B_{lm} \tag{31}
\]
The expectation value of \((A^2)_{lm}\) is written by

\[
\langle A^2 \rangle_s = \sum_{l,m} |a_l a_m| e^{i(\theta_{lm} - \theta_{ml})} \sum_k A_{lk} A_{km} = \sum_{l,m} \sum_k A_{lm}^{(k)} A_{lm}^{(k)} = \sum_{l,m} \sum_k A_{lm}^{(k)} A_{ml}^{*(k)} \equiv |A^{(k)}|^2.
\]

(32)

In this way, two kinds of expectation values* are introduced without any guiding principle. It is an important subject to pursue their physical implication. By use of the inequality:

\[
(\sum_{k_1} |A^{(k_1)}|^2)(\sum_{k_2} |B^{(k_2)}|^2)(\sum_{k_3} |C^{(k_3)}|^2) \geq |\sum_k Tr(A^{(k)} C^{(k)} B^{(k)})|^2
\]

(33)

and the relation:

\[
\langle ABC \rangle_c = \frac{1}{6}([A, B, C]_c + \{A, B, C\}_c),
\]

(34)

the following uncertainty relation is derived,

\[
\delta A \delta B \delta C \geq \frac{1}{6}|\langle [A, B, C]_c \rangle| = \frac{1}{6}|\langle D \rangle_c|,
\]

(35)

where the uncertainty \(\delta A\) is defined by

\[
\delta A \equiv \sqrt{\langle (A - \langle A\Delta \rangle_s \Delta)^2 \rangle_s}.
\]

(36)

Here \(\Delta_{lmn} = \delta_{lm} \delta_{mn}\) and hence \(\langle A \Delta \rangle_s = \sum_m |a_m|^2 A_{mmm}\). Note that there is the identity \([A, B, \Delta]_{lmn} = 0\) for arbitrary cyclic cubic matrices \(A_{lmn}\) and \(B_{lmn}\).

Finally we discuss a physical implication on the uncertainty relation (35). Let us assume that the 4-dimensional space-time coordinates are described by cubic matrices \((X^\mu)_{lmn}\) \((\mu = 0, 1, 2, 3)\) which satisfy the following relation,

\[
[X^1, X^2, X^3]_{lmn} = -i l_P^2 (X^0)_{lmn},
\]

(37)

*In case that \(\theta_{lm} = \frac{1}{2}(\beta_m - \beta_l)\), the expectation values \(\langle A \rangle_c\) and \(\langle B \rangle_s\) are reduced to \(\langle A \rangle_c \equiv \sum_{l,m,n} |a_l a_m a_n| A_{lmn}\) and \(\langle B \rangle_s \equiv \sum_{l,m} a_l^* a_m B_{lm}\) where \(a_m = |a_m| e^{i\beta_m}\).
where \( l_P \) is the Planck length defined by

\[
l_P \equiv \sqrt{\frac{2G\hbar}{c^3}}.
\]

Here \( G \) is the Newton constant and \( c \) is a speed of light. From the above argument, the following uncertainty relation can be derived,

\[
\delta X^1 \delta X^2 \delta X^3 \geq \frac{l_P^2}{6}|\langle X^0 \rangle|.
\]

(38)

Many people have discussed uncertainty relations concerning the measurement of space-time distances based on various kinds of thought experiments.[8, 9, 10, 11, 12, 13] Among them, the relation\[9, 10, 11, 12\] like

\[
(\delta r)^3 \sim \frac{l_P^2 r}{2} \sim \frac{l_P^2 c \delta t}{2}
\]

is deeply related to ours (38). Here \( \delta r \) and \( \delta t \) are an uncertainty of space distance \( r \) and a time period of observer, respectively. The Lorentz covariant form has been proposed in Ref.[12];

\[
|\varepsilon_{\mu\nu\rho \sigma} n^\mu \delta x^\nu_{i_1} \delta x^\rho_{i_2} \delta x^\sigma_{i_3}| \geq l_P^2 \delta x^\mu_{i_4} n_\mu,
\]

(39)

where \( \delta x^\mu_{i} \)'s are the four vectors to define a space-time volume and \( n_\mu \) is any four vector which represents the velocity vector of an observer. There is a possibility that the space-time uncertainty relation (39) has the origin from the algebraic relation such as

\[
[X^\mu,X^\nu,X^\rho] = -i l_P^2 \varepsilon^{\mu\nu\sigma} X_\sigma.
\]

(40)

5 Conclusions and discussion

We have studied a generalization of spin algebra using cubic matrices and pointed out a possibility that a triple commutation relation among cubic matrices implies a kind of uncertainty relation among their expectation values. As a physical implication, we suggest that the space-time uncertainty relation can be connected to the triple commutation relation such as

\[
[X^1,X^2,X^3]_{lmn} = -i l_P^2 (X^0)_{lmn}.
\]

The physical meanings of cubic matrices have not been completely understood and there still exist several questions. The \((J^a)_{mn}\)'s are representation matrices that operate a representation space called spin space. We have a question whether \((G^a)_{lmn}\) also act as generators or not and what kind of representation space exists. In quantum mechanics, the matrix element \(A_{mn}\) is interpreted as a probability amplitude between the state described by \(\phi_m\) and that described by \(\phi_n\). However the meaning of the cubic matrix element
$A_{lmn}$ has not been known yet. Further the derivation of uncertainty relation (35) seems tricky because the definition of expectation values looks ad hoc. We need to find the meaning of $\alpha_l$ and $\theta_{lm}$. The physical implication on expectation values could be a key to find a way out. It is also an interesting subject to find the relationship between the space-time uncertainty relations derived from string/M theories [13] and ours (38).

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References

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