We study the issue of simultaneous estimation of several phase shifts induced by commuting operators on a quantum state. We derive the optimal positive operator-valued measure corresponding to the multiple-phase estimation. In particular, we discuss the explicit case of the optimal detection of double phase for a system of identical qutrits and generalise these results to optimal multiple phase detection for $d$-dimensional quantum states.

I. INTRODUCTION

The issue of phase estimation has important applications in quantum computation and quantum information theory. For example, it was shown that the existing quantum algorithms can be described in a unified way as quantum interference processes among different computational paths where the result of the computation is encoded in a phase shift [1]. The design of optimal phase measurement procedures is also crucial in various tasks of atomic physics, such as for example methods for precision spectroscopy [2], and quantum interferometric experiments in quantum optics.

The problem of the optimal estimation of the value of a phase shift experienced by a quantum state has been extensively studied [3], and in particular a method to derive the optimal measurement procedure in the phase covariant case was reported in Ref. [4]. More recently, a general formulation of the phase estimation problem was considered in Ref. [5], where a method to derive the optimal positive-operator valued measurement (POVM) [6] for a generally degenerate phase shift operator was developed. In this work we introduce the problem of simultaneous estimation of several phase shifts undergone by a quantum physical system. More specifically, we address the problem of estimating the values of $M$ independent phase-shifts $\phi_j \,(j = 1, M)$, pertaining to the unitary transformation

$$\rho_{\phi_j} = e^{-i \sum_{j=1}^{M} \phi_j \hat{R}_j} \rho_0 e^{i \sum_{j=1}^{M} \phi_j \hat{R}_j} \quad (1)$$

where $\hat{R}_j$ represent $M$ commuting self-adjoint operators, which are in general degenerate on the Hilbert space $\mathcal{H}$ of the considered quantum system and each of them has a discrete spectrum $S_j$ ($S_j$ can be for example $\mathbb{Z}$, $\mathbb{N}$, or $\mathbb{Z}_q$, $q > 0$) [7]. In Eq. (1) $\rho_0$ is a generic initial pure state $|\psi_0\rangle\langle\psi_0|$ describing a quantum system with arbitrary dimension. We want to point out that the scenario of simultaneous estimation of several phases may be useful to improve the efficiency of quantum information processing tasks where several variables are encoded into phases in the same quantum states.

The paper is organised as follows. In Sect. II we derive a general treatment of the multiple phase estimation problem, extending to the multi-phase case the approach presented in [5] for the case of single phase estimation. In Sect. III we derive the optimal POVM and the corresponding estimation fidelity for a system of $N$ identically prepared “equatorial” three-dimensional systems. We want to point out that the possibility of encoding information in the states of three dimensional systems has been the object of several recent studies, for example in the context of quantum cloning [8] and quantum cryptography [9]. In Sect. IV we extend these results derived for qutrits to the case of quantum systems with arbitrary finite dimension $d$. Finally, we summarise and comment the results presented in Sect. V.

II. OPTIMAL POVM FOR MULTIPLE-PHASE ESTIMATION

In this section we derive the optimal POVM corresponding to the simultaneous estimation of several phase shifts experienced by a pure state $|\psi_0\rangle$, belonging to the Hilbert space $\mathcal{H}$, that undergoes the unitary transformation (1). Following the approach of Ref. [5], we treat the estimation problem in the general framework of quantum estimation theory [6]. According to this framework, we first define a cost function $C(\{\hat{\phi}_j\}, \{\phi_j\})$ which depends on the set of the $M$ estimated values $\{\hat{\phi}_j\}$, that are the results of the estimation procedure, and on the set of the $M$ true values...
so that the cost function becomes an even function of $M$ the operator $\hat{A}$.

We can then write the Hilbert space of the system as $H$, degenerate, and we will denote by $\{|\phi_j\rangle\}$ states that are orthogonal to $|\phi_j\rangle$.

The positivity condition for the operator $\chi_n$ is satisfied.

In the above equation $\chi$ is a positive operator satisfying the completeness constraints needed for the normalization of the POVM $\int d\mu(\bar{\phi}_j) = 1$, where $1$ denotes the identity operator. Actually, using Eq. (3) and the invariance of the trace under cyclic permutations it can be shown that $p(\bar{\phi}_j|\phi_j) = p(\bar{\phi}_j - \phi_j)$ if and only if $d\mu(\bar{\phi}_j)$ is covariant. Therefore the optimization of the phase estimation procedure can be performed by finding the positive operator $\chi$ that minimises the average cost for a given cost function $\bar{C}(\phi_j)$, and a generic initial state $\rho_0$.

We will now show explicitly how to derive the optimal POVM for a broad class of cost functions and initial states $\rho_0$. First of all we will choose the representation where all the operators $\hat{A}_j$ are diagonal. We have assumed that the operators $\hat{A}_j$ commute, so we can identify a common basis of eigenvectors.

The positivity of the operator $\chi$ implies the inequalities

$$|\chi_{\{n_j\}\{m_j\}}| \leq \sqrt{\chi_{\{n_j\}}\chi_{\{m_j\}}},$$

In order to accomplish this task, we first rewrite Eq. (2) more conveniently as

$$\bar{C} = \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \ldots \int_0^{2\pi} d\phi_M \rho_0(\{\phi_j\}) \int_0^{2\pi} d\tilde{\phi}_1 \int_0^{2\pi} d\tilde{\phi}_2 \ldots \int_0^{2\pi} d\tilde{\phi}_M C(|\phi_j\rangle, \{\phi_j\}) p(\{\phi_j\}|\phi_j\rangle) ,$$

where $p_0(\{\phi_j\})$ is the a priori probability density for the true values $\{\phi_j\}$ and $p(\{\phi_j\}|\phi_j\rangle)$ is the conditional probability of estimating the set of values $\{\phi_j\}$ given the true values $\{\phi_j\}$. The average cost is minimized by optimizing the POVM $d\mu(\{\phi_j\})$, which appears in the definition of the conditional probability as follows

$$p(\{\phi_j\}|\phi_j\rangle) d\phi_1 d\phi_2 \ldots d\phi_M = \text{Tr}[d\mu(\{\phi_j\}) e^{-i \sum_{j=1}^M \phi_j \hat{A}_j} \rho_0 e^{i \sum_{j=1}^M \phi_j \hat{A}_j}].$$

In this way the component $d\mu(\{\phi_j\})$ of the POVM acting on $\{\phi_j\}$ is covariant. Actually, using Eq. (3) and the invariance of the trace under cyclic permutations it can be shown that $p(\{\phi_j\}|\phi_j\rangle) = p(\{\phi_j - \phi_j\})$ if and only if $d\mu(\{\phi_j\})$ is covariant. Therefore the optimization of the phase estimation procedure can be performed by finding the positive operator $\chi$ that minimises the average cost for a given cost function $\bar{C}(\phi_j)$ and a generic initial state $\rho_0$.

We will now show explicitly how to derive the optimal POVM for a broad class of cost functions and initial states $\rho_0$. First of all we will choose the representation where all the operators $\hat{A}_j$ are diagonal. We have assumed that the operators $\hat{A}_j$ commute, so we can identify a common basis of eigenvectors. The operators $\hat{A}_j$ are generally degenerate, and we will denote by $\{|n_j\}_\nu$ a choice of (normalized) eigenvectors corresponding to eigenvalue $n_j$ for the operator $\hat{H}_j$, by $\Pi_{\{n_j\}}$ the projector onto the corresponding degenerate eigenspace and by $\nu$ a degeneracy index, whose maximum value corresponds to the dimension of the degenerate eigenspace.

We will now generalise the projection method developed in [5] and define $H_\parallel$ as the Hilbert space spanned by the (normalized) vectors $|\{n_j\}_\parallel \times \Pi_{\{n_j\}}|\psi_0\rangle \neq 0$ with the choice of the arbitrary phases such that $|\{n_j\}_\parallel|\psi_0\rangle > 0$. We can then write the Hilbert space of the system as $H = H_\parallel \otimes H_\perp$, where the component $H_\perp$ is spanned by states that are orthogonal to $|\psi_0\rangle$. Hence the POVM can be chosen of the block diagonal form on $H_\parallel \otimes H_\perp$, i.e. $d\mu(|\phi_j\rangle) = d\mu_\parallel(|\phi_j\rangle) \oplus d\mu_\perp(|\phi_j\rangle)$. In this way the component $d\mu_\parallel(\{\phi_j\})$ of the POVM acting on $H_\perp$ can be chosen arbitrarily because it does not contribute to the average cost. Therefore, the optimisation of the estimation procedure can be performed by optimising only the component $d\mu_\parallel(\{\phi_j\})$ of the POVM.

In order to optimise the POVM we can assume that $\Pi_{\{n_j\}}|\psi_0\rangle \neq 0$ for all values of $\{n_j\}$, since the resulting POVM will be optimal also for states having zero projection for some of these values. Due to the covariance property (4) and to the argument followed above, we can also write $\chi = \chi_\parallel \otimes \chi_\perp$. Thus, the problem reduces to finding the positive operator $\chi_\parallel$ that minimizes the cost $\bar{C}$ in Eq. (2).

In order to accomplish this task, we first rewrite Eq. (2) more conveniently as

$$\bar{C} = \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \ldots \int_0^{2\pi} d\phi_M C(|\phi_j\rangle, \{\phi_j\}) \text{Tr}[\chi e^{-i \sum_{j=1}^M \phi_j \hat{H}_j} \rho_0 e^{i \sum_{j=1}^M \phi_j \hat{H}_j}].$$

We will now express the operator $\chi_\parallel$ on the $|\{n_j\}_\parallel\rangle$ basis as follows

$$\chi_\parallel = \sum_{\{n_j\}_\parallel, \{m_j\}_\parallel} |\{n_j\}_\parallel\rangle\langle\{m_j\}_\parallel| \chi_{\{n_j\}_\parallel\{m_j\}_\parallel}.$$

The positivity condition for the operator $\chi$ implies the inequalities

$$|\chi_{\{n_j\}_\parallel\{m_j\}_\parallel}| \leq \sqrt{\chi_{\{n_j\}_\parallel}\chi_{\{m_j\}_\parallel}},$$
where the last equality comes from the POVM completeness relation $\int d\mu_{\parallel}(\phi) = 1$.

The cost functions we will consider are $2\pi$-periodic functions in the variables $\{\phi_j\}$, and therefore they can be written as

$$C(\{\phi_j\}) = -\sum_{l_1,l_2,\ldots,l_m=-\infty}^{\infty} c_{l_j} e^{i\sum_j l_j \phi_j},$$  

(8)

with the condition $c_{l_j} = c_{-l_j}$ due to the fact that the cost is a real and even function. By performing the integrals in Eq. (5) and exploiting the relation $\int_0^{2\pi} d\phi e^{i(n-m)\phi} = \delta_{n,m}/2\pi$ we arrive at the following form of the average cost

$$\bar{C} = -c_0 - \sum_{\{l_j\} \neq 0} c_{l_j} \sum_{\{m_j - n_j\} = \{l_j\}} \langle\psi_0|\{n_j\}\rangle\langle\{m_j\}|\psi_0\rangle \chi_{\{n_j\}\{m_j\}},$$  

(9)

where the expression $\{l_j\} \neq 0$ under the first summation symbol means that the sum contains all the values of the indexes $l_j$ apart from the case where they are all zero (this contribution corresponds to the term $c_0$ in Eq. (9)) and the expression $\{m_j - n_j\} = \{l_j\}$ under the second summation means that the equality $m_j - n_j = l_j$ must hold for all values of the index $j$.

Let us now consider the following inequality

$$\text{sign}(c_{l_j}) \sum_{\{m_j - n_j\} = \{l_j\}} \langle\psi_0|\{n_j\}\rangle\langle\{m_j\}|\psi_0\rangle \chi_{\{n_j\}\{m_j\}} \leq \sum_{\{m_j - n_j\} = \{l_j\}} |\langle\psi_0|\{n_j\}\rangle||\langle\{m_j\}|\psi_0\rangle|,$$  

(10)

where we remind that we chose $\langle\{n_j\}|\psi_0\rangle > 0$. The above relation becomes an equality if the conditions

$$\chi_{\{n_j\}\{m_j\}} = \text{sign}(c_{m_j - n_j})$$  

(11)

can be fulfilled. In this case the minimum cost takes the simple form

$$\bar{C} = -c_0 - \sum_{\{l_j\} \neq 0} c_{l_j} \sum_{\{m_j - n_j\} = \{l_j\}} |\langle\psi_0|\{n_j\}\rangle||\langle\{m_j\}|\psi_0\rangle|.$$  

(12)

Notice however that the positivity of $\chi$ is not generally guaranteed for any set of values of sign$(c_{l_j})$.

Let us now define a general class of cost functions, that extends the one considered by Holevo [4], with

$$c_{l_j} \geq 0, \forall \{l_j\} \neq 0.$$  

(13)

For this class the conditions (11) are trivially satisfied for all the sets of values $\{n_j\}$ and the optimal POVM takes the explicit form

$$d\mu_{\parallel}(\{\phi_j\}) = \frac{d\phi_1}{2\pi} \cdots \frac{d\phi_M}{2\pi} |e(\{\phi_j\})\rangle\langle e(\{\phi_j\})|,$$  

(14)

where the vectors $|e(\{\phi_j\})\rangle$ are defined as

$$|e(\{\phi_j\})\rangle = \sum_{\{n_j\}} e^{i\sum_j n_j \phi_j} |\{n_j\}\rangle.$$  

(15)

In the following sections we will illustrate more explicitly the results presented here by considering specific examples.

**III. DOUBLE PHASE ESTIMATION FOR QUTRITS**

As a simple application of the concepts presented above, let us consider the optimal double phase estimation for $N$ identical three-dimensional quantum systems (qutrits) all in the state

$$|\psi(\phi, \theta)\rangle = \frac{1}{\sqrt{3}} (|0\rangle + e^{i\phi}|1\rangle + e^{i\theta}|2\rangle),$$  

(16)

where $\{|0\rangle, |1\rangle, |2\rangle\}$ represents a basis for the qutrit. In the language of the previous section we identify $\phi_1 = \phi, \phi_2 = \theta$, $H_1 = |1\rangle\langle 1|, H_2 = |2\rangle\langle 2|$ and $|\psi_0\rangle = (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}$ for each qutrit. For the composite system of $N$ qutrits we
have \( H_1 = \sum_{k=1}^{N} |1\rangle \langle 1 |_k \) and \( H_2 = \sum_{k=1}^{N} |2\rangle \langle 2 |_k \), where \( |j\rangle \langle j | \) denotes the projection operator onto the state \( |j\rangle \) of the \( k\)-th qutrit. The operators \( H_1 \) and \( H_2 \) commute, and they are diagonalised in the basis \( |N-n_1-n_2,n_1,n_2\rangle \) of the states where \( N-n_1-n_2 \) qutrits are in the state \( |0\rangle \), \( n_1 \) in the state \( |1\rangle \) and \( n_2 \) in the state \( |2\rangle \). The symbol \( \nu \) represents the degeneracy index of the corresponding subspace, and in particular it ranges from 1 to \( N!/(N-n_1-n_2)!n_1!n_2! \).

Since the state of the \( N \) qutrits is symmetric under any permutation performed on the qutrits, the states \( |\{n_j\}\rangle \) defined in the previous section by the projection method correspond in this case to the symmetric normalised states of the \( N \) qutrits, which we will simply denote as \( |N-n_1-n_2,n_1,n_2\rangle \) (such a state is an equally weighted superposition of \( N!/(N-n_1-n_2)!n_1!n_2! \) components corresponding to all the possible permutations of states with \( N-n_1-n_2 \) qutrits in the state \( |0\rangle \), \( n_1 \) in the state \( |1\rangle \) and \( n_2 \) in the state \( |2\rangle \)).

In this case the optimal POVM (14) for the cost functions of the generalised Holevo form (13) takes the form

\[
d\mu(\phi, \theta) \equiv \frac{d\phi d\theta}{2\pi} |e(\phi, \theta)| |e(\phi, \theta)| ,
\]

where \( |e(\phi, \theta)| = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} e^{i(n_1\phi+n_2\theta)} |N-n_1-n_2,n_1,n_2\rangle \).

Let us now compute the fidelity of the optimal double phase estimation procedure. As a cost function we can choose for example \( 1 - F \), where \( F \) is the fidelity of the estimated state \( |\psi(\phi, \theta)\rangle \) with respect to the true state \( |\psi(\phi, \theta)\rangle \). This cost belongs to the class (13), and therefore the corresponding optimal POVM is the one written above. This choice of the cost function is particularly interesting because the fidelity is the figure of merit usually adopted to describe other processes in quantum information theory, such as for instance cloning transformations, and therefore it allows a direct comparison of the efficiency of optimal phase estimation with other procedures.

By the covariance of the procedure we can write the fidelity as

\[
F(\phi, \psi) = |\langle \psi_0 | \psi(\phi, \theta) \rangle|^2 = \frac{1}{9} [3 + 2 \cos \phi + 2 \cos \psi + 2 \cos(\phi - \psi)] ,
\]

where

\[
|\psi_0\rangle = \frac{1}{\sqrt{3^N}} \sum_{j=0}^{N-j} \sum_{k=0}^{N-j} \sqrt{M(N,j,k)} |N-j-k,j,k\rangle .
\]

In the above equation we have defined \( M(N,j,k) = \frac{N!}{(N-j-k)!j!k!} \).

The average fidelity \( \bar{F} \) of the procedure is then given by

\[
\bar{F} \equiv \int F(\phi, \psi) Tr[\rho_0 d\mu(\phi, \psi)]
\]

\[
= \frac{1}{9} \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \frac{2\pi}{2\pi} [3 + 2 \cos \phi + 2 \cos \psi + 2 \cos(\phi - \psi)] |\langle \psi_0 | \psi(\phi, \theta) \rangle|^2 .
\]

By performing the integration in Eq. (20) we have

\[
\bar{F} = \frac{1}{3} + \frac{1}{3^N} \sum_{j,p=0}^{N-j} \sum_{k=0}^{N-j-p} \sum_{q=0}^{N-j} \sqrt{M(N,j,k)M(N,p,q)}
\]

\[
\times [\delta_{k,q}(\delta_{j,p+1} + \delta_{j+1,p}) + \delta_{j,p}(\delta_{k,q+1} + \delta_{k+1,q}) + \delta_{j+1,p}\delta_{k,q+1} + \delta_{j,p+1}\delta_{k+1,q}]
\]

\[
= \frac{1}{3} + \frac{2}{3^N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-j-1} \sum_{q=0}^{N-j} M(N,j,k) \frac{N-j-k}{j+1} .
\]

We want to point out that the fidelity (21) corresponding to the optimal double phase estimation for qutrits is always smaller than the one for equatorial qubits, given in Ref. [10], where a single phase is estimated.

We want also to stress that, as in the case of qubits, there is a relation between optimal double phase estimation and optimal cloning for states of the form (16). Actually, the fidelity of the optimal double phase estimation (21) for a single qutrit \( (N = 1) \) coincides with the cloning fidelity for the optimal \( 1 \rightarrow M \) cloning transformations, that take a single input equatorial qutrit and produce \( M \) output copies [11], in the limit of an infinite number of output copies, i.e. \( M \rightarrow \infty \). Moreover, this result is consistent also with the relation between optimal state estimation [12] and optimal cloning [13] for input qutrits whose state is completely unknown (not restricted to the form (16)).
We want to point out that other figure of merits could be considered in order to evaluate the efficiency of the phase estimation procedure. For example, a mean periodic “variance” \( V(\phi, \theta) = 2(\sin^2 \phi/2 + \sin^2 \theta/2) \) could be considered as a cost function. In this case the cost function still belongs to the class (13), and therefore, by explicitly calculating the average variance in a similar way as for the average fidelity, we arrive at the form

\[
\bar{V} = \int V(\phi, \psi) \text{Tr}[\rho_0 d\mu(\phi, \psi)] = 2 - \frac{2}{3N} \sum_{j=0}^{N-1} \sum_{k=0}^{j-1} M(N, j, k) \sqrt{\frac{N - j - k}{j + 1}}. \tag{22}
\]

### IV. MULTIPLE PHASE ESTIMATION FOR SYSTEMS WITH ARBITRARY DIMENSION

In this section we generalise the results derived above for qutrits to the case of multiple phase estimation for systems with arbitrary finite dimension \( d \) (qudits). We will consider the optimal multiple phase estimation for \( N \) identical \( d \)-dimensional quantum systems all in the state

\[
|\psi(\{\phi_j\})\rangle = \frac{1}{\sqrt{d}} (|0\rangle + e^{i\phi_1}|1\rangle + e^{i\phi_2}|2\rangle + \ldots + e^{i\phi_{d-1}}|d-1\rangle), \tag{23}
\]

where \( \{|0\rangle, |1\rangle, |2\rangle, \ldots |d-1\rangle\} \) represents a basis for each system.

In the language of section II, we are considering the estimation problem for \( d - 1 \) phases corresponding to the operators \( \hat{H}_j = |j\rangle\langle j|, j = 1, \ldots, d - 1 \), and \( |\psi_0\rangle = (|0\rangle + |1\rangle + |2\rangle + \ldots + |d-1\rangle)/\sqrt{d} \) for each system. For the composite system of \( N \) qudits we have \( \hat{H}_j = \sum_{k=1}^{N} |j\rangle\langle k| \), where as in the previous section \( |j\rangle\langle k| \) denotes the projection operator onto the state \( |j\rangle \) of the \( k \)-th qudit. The operators \( \hat{H}_j \) commute, and they are diagonalised in the basis \( |n_0, n_1, n_2, \ldots, n_{d-1}\rangle \), where \( n_0 \) qudits are in the state \( |0\rangle \), \( n_1 \) in the state \( |1\rangle \) and so on, with \( \sum_{j=0}^{d-1} n_j = N \). As in the case of qutrits, \( \nu \) represents the degeneracy index of the corresponding subspace, and in particular it ranges from 1 to the multinomial \( N!/(N - n_1 - n_2 - \ldots - n_{d-1})!n_1!n_2!\ldots n_{d-1}! \). Analogously to the case of qudits, the POVM is optimised by choosing the symmetric normalised states of the \( N \) qudits, which we will simply denote as \( |n_0, n_1, n_2, \ldots, n_{d-1}\rangle \).

The optimal POVM for the cost functions of the generalised Holevo form (13) is given by (14), with \( M = d - 1 \) and

\[
|e(\{\phi_j\})\rangle = \sum_{\{n_j\}} e^{i\sum_{j=1}^{d-1} n_j \phi_j} |n_0, n_1, n_2, \ldots, n_{d-1}\rangle. \tag{24}
\]

In the above equation the sum over \( \{n_j\} \) means that the variables \( n_j \) take all the possible non negative values compatible with the constraint \( \sum_{j=0}^{d-1} n_j = N \).

Let us now compute the fidelity of the optimal multiple phase estimation procedure derived above. As in the case of qutrits we choose a cost function of the form \( 1 - F \), where \( F \) is the fidelity of the estimated state \( |e(\{\phi_j\})\rangle \) with respect to the true state \( |\psi(\{\phi_j\})\rangle \). This cost belongs to the class (13), and therefore the corresponding optimal POVM is the one mentioned above. By the covariance of the procedure we can write the fidelity as

\[
F(\{\phi_j\}) = \langle \psi_0 | \psi(\{\phi_j\}) \rangle^2 = \frac{1}{d^2} \left[ d + 2 \sum_{j=1}^{d-1} \cos \phi_j + 2 \sum_{j>k} \cos(\phi_j - \phi_k) \right] \tag{25}
\]

where

\[
|\psi_0\rangle = \frac{1}{\sqrt{d^N \nu}} \sum_{\{n_j\}} \sqrt{\frac{N!}{n_0!n_1!n_2!\ldots n_{d-1}!}} |n_0, n_1, n_2, \ldots, n_{d-1}\rangle. \tag{26}
\]

The average fidelity \( \bar{F} \) of the procedure is now given by

\[
\bar{F} = \frac{1}{d^2} \int_0^{2\pi} \frac{d\phi_0}{2\pi} \ldots \int_0^{2\pi} \frac{d\phi_{d-1}}{2\pi} F(\{\phi_j\}) \langle \psi_0 | \psi(\{\phi_j\}) \rangle^2. \tag{27}
\]

By performing the integrations in Eq. (27) we have
\[
\bar{F} = \frac{1}{d} + \frac{d - 1}{d^{N+1}} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-n_1-1} \ldots \sum_{n_{d-1}=0}^{N-n_1-n_2-\ldots-n_{d-2}} \frac{N!}{(N-n_1-n_2-\ldots-n_{d-1})!n_1!n_2!\ldots n_{d-1}!} \\
\sqrt{\frac{N-n_1-n_2-\ldots-n_{d-1}}{d+1}}. 
\] 
\]

(28)

Notice that the fidelity decreases as a function of the dimension \(d\). For example, in the case of multiple phase estimation on the state of a single qudit it takes the simple form

\[
\bar{F}_1 = \frac{2d - 1}{d^2}. 
\]

(29)

We also want to point out that the above fidelity is larger than the fidelity of estimation of a single qudit in a completely unknown pure state (not restricted to be of the form (23)). Actually, the fidelity of such a universal procedure, which we will call \(F_{\text{univ,1}}\), is given by [12]

\[
F_{\text{univ,1}} = \frac{2}{d+2}. 
\]

(30)

V. CONCLUSIONS

In this paper we have addressed the problem of simultaneous estimation of several phase shifts induced by a unitary transformation acting on a quantum system. We have derived in a general way the optimal estimation procedure for an arbitrary number of phase shifts and for a wide class of cost functions. We have then specialised the results obtained to the case of “equatorial” qutrits and then generalised them to the case of quantum system with arbitrary finite dimension. An interesting result that was emphasised in this work is the connection between optimal double phase estimation and optimal double phase covariant cloning for qutrits. We expect also that such a connection is valid in arbitrary finite dimension.

Before closing the paper, we want to point out that the scenario of simultaneous estimation of several phases considered in this paper may be exploited to design schemes where several variables are encoded into phases in the same quantum states and in this way the efficiency of quantum information processing tasks may be improved.

ACKNOWLEDGEMENTS

This work has been supported in part by the EC programs ATESIT (Contract No. IST-2000-29681) and QUPRODIS (Contract No. IST-2002-38877).

[7] The case of non-commuting operators is highly non trivial and is not considered in this paper.