Transcending the Limits of Turing Computability∗

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Abstract

Hypercomputation or super-Turing computation is a “computation” that transcends the
limit imposed by Turing’s model of computability. The field still faces some basic questions,
technical (can we mathematically and/or physically build a hypercomputer?), cognitive (can
hypercomputers realize the AI dream?), philosophical (is thinking more than computing?).
The aim of this paper is to address the question: can we mathematically build a hypercom-
puter? We will discuss the solutions of the Infinite Merchant Problem, a decision problem
equivalent to the Halting Problem, based on results obtained in [9, 2]. The accent will be on
the new computational technique and results rather than formal proofs.

1 Introduction

Hypercomputation or super-Turing computation is a “computation” that transcends the limit
imposed by Turing’s model of computability; for a recent perspective see the special issue of
the journal Minds and Machines (12, 4, 2002). Currently there are various proposals to break
Turing’s Barrier by showing that certain classes of computing procedures have super-Turing power
(see [10, 30, 6, 7, 29, 11]). A specific class of computing procedures, [14, 9, 21, 2] make essential
use of some physical theory, relativity theory in [14], quantum theory [9, 21]; they reflect an
attitude advocated by Landauer [22, 23] (information is inevitably physical) and Deutsch [12, 13]
(the reason why we find it possible to construct, say, electronic calculators, and indeed why we
can perform mental arithmetic . . . is that the laws of physics “happen” to permit the existence of
physical models for the operations of arithmetic).

The aim of the present paper is to revisit the solutions offered in [9, 2]. We will focus on the
novelty of the approach and we will discuss its power and its limits. No proofs will be offered.

2 The Classical Merchant Problem

Recall that in the classical version of the Merchant Problem we have 10 stacks of coins, each stack
containing 100 coins, and we know that at most one stack contains only false coins, weighting
1.01 g; true coins weight 1 g. The problem is to find the stack with false coins (if any) by only one
weighting. The classical solution reduces the problem to the weighting of a special combination of
coins: one coin from the first stack, two coins from the second stack, . . . , ten coins from the tenth

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stack. If the false coins are present in the \( N \)-th stack, then the weight of the combination will be \( 55 + \frac{N}{100} \text{g} \); otherwise the weight is just 55 g. The Merchant Problem quoted above was widely spread in allies armies during the Second World War, cf. [20]. Probably the elegant solution described above was the very first solution of a computational problem bearing typical features of quantum computing, see an extended discussion in [9].

3 The Infinite Merchant Problem

In what follows we are going to consider the following generalization of the problem, the Infinite Merchant Problem: we assume that we have countable many stacks, given in some computable way, all of them, except at most one, containing true coins only. True coins weight 1 and false coins weight \( 1 + 2^{-j}, j > 0 \). Again we are allowed to take a coin from each stack and we want to determine whether all coins are true or there is a stack of false coins.

Next we will show that the Infinite Merchant Problem is classically undecidable by reducing it to the Halting Problem, i.e. the problem to decide whether an arbitrary Turing machine (TM) halts on an arbitrary input. Assume that a TM operates on positive integers and suppose, for the sake of contradiction, that there exists a TM HALT which can decide whether a TM \( T \) (given by its code \( \#T \), a positive integer) eventually stops on input \( x \):

\[
\text{HALT}(\#T, x) = \begin{cases} 
1, & \text{if } T(x) \text{ stops}, \\
0, & \text{otherwise}.
\end{cases}
\]

We construct a TM \( Q \)

\[
Q(x) = \begin{cases} 
1, & \text{if } \text{HALT}(x, x) = 0, \\
\text{loops forever}, & \text{otherwise},
\end{cases}
\]

and deduce the contradiction:

\[
\text{HALT}(\#Q, \#Q) = 1 \text{ iff } \text{HALT}(\#Q, \#Q) = 0.
\]

We next describe the reduction. Assume that we have a classical solution of the Infinite Merchant Problem and we are given a TM \( T \) and an input \( x \) for \( T \). We construct a computable sequence \( q_1, q_2, \ldots q_i \ldots \) as follows: if the computation of \( T(x) \) did not stop till the \( i \)-th step, then we put \( q_i = 1 \); if the computation halted at step \( i_0 \), then we put \( q_{i_0} = 1 + 2^{-j} \) and \( q_r = 1 \), for all \( r > i_0 \). The sequence \( q_i \) satisfies all conditions of the Infinite Merchant Problem and \( T(x) \) halts if and only if there is a false coin, i.e., \( q_{i_0} = 1 + 2^{-j} \), for some \( j \). This shows that the Infinite Merchant Problem is undecidable as the Halting Problem is undecidable.

In fact the two problems are equivalent. Indeed, assume that we could classically solve the Halting Problem. To every sequence \( (q_i) \) satisfying the conditions of the Infinite Merchant Problem we associate the TM \( T \) such that \( T(i) = 1 \) if \( q_i = 1 \), and \( T(i) = 0 \) otherwise. The TM \( T' \) defined by \( T'(0) = \min\{i \mid T(i) = 0\} \) halts at 0 if and only if there an \( i_0 \) such that \( q_{i_0} = 1 + 2^{-j} \), i.e., \( T'(0) \) halts if and only if there are false coins in the system. Hence, a classical solution of the Halting Problem will produce a classical solution for the Infinite Merchant Problem.

The above discussion shows that undecidability is determined by the impossibility to decide in a finite of time the answer to an infinite number of questions, “does the first stack contain a false coin?”, “does the second stack contain a false coin?”, etc. This might be caused either by the fact that the time of the computation grows indefinitely or by the fact that the space of computation grows indefinitely or both. The classical theories of computability and complexity (see, for example, [5]) do not give any indication in this respect. In the following section we will show that time can be made finite provided we use a specific probabilistic strategy.

2
In this section we present, in a slightly different way, the probabilistic solution proposed in [9]. We will adopt the following strategy. We are given a probability $\theta = 2^{-n}$ and we assume that we work with a “device” described below\(^1\) with sensitivity given by a real $\varepsilon = 2^{-m}$. Then, we compute classically a time $T = T_{\theta,\varepsilon}$ and run the “device” on a random input for the time $T$. If we get a click, then the system has false coins; if we don not get a click, then we conclude that with probability greater than $1 - \theta$ all coins are true. An essential part of the method is the requirement that the time limit $T$ is classically computable.

The “device” (with sensitivity $\varepsilon$) will distinguish the values of the iterated quadratic form $\langle Q'(x), x \rangle = \sum_{i=1}^{\infty} q_i^2 |x_i|^2$, by observing the difference between averaging over trajectories of two discrete random walks with two non-perturbed and perturbed sequences $t_i, t_i$ of “stops”. The non-perturbed sequence corresponds to equal steps $\delta_m = 1, t_i = \sum_{m=0}^{+\infty} \delta_m$, and the perturbed corresponds to the varying steps $\Delta_m, 0 < \Delta_m < \delta_m, t_i = \sum_{m=0}^{t} \Delta_m$. We work with the intersections of $l_2$ with the discrete Sobolev class $l_2^2$ of square-summable sequences with the square norm

$$|x|^2 = \sum_{m=1}^{\infty} |x_m - x_{m-1}|^2,$$

(1)

and the discrete Sobolev class $\tilde{l}_2^2$ of weighted-summable sequences with the square norm

$$\|x\|_1^2 = \sum_{m=1}^{\infty} \frac{1 - \Delta_m}{\Delta_m} |x_m - x_{m-1}|^2.$$

(2)

By natural extension from cylindrical sets we can define the Wiener measures $\tilde{W}$ and $W$ on the spaces of trajectories of the perturbed and non-perturbed random walks respectively and use the absolute continuity $\tilde{W}$ with respect to $W$: that is for every $W$-measurable set $\Omega$,

$$\tilde{W}(\Omega) = \frac{1}{\prod_{l=1}^{\infty} \sqrt{\Delta_l}} \int_{\Omega} e^{-\sum_{m=1}^{\infty} \frac{|x_m - x_{m-1}|^2}{\Delta_m}} dW.$$

Assume that the “device” revealing the exponential growth of the quadratic form of the iterations $\langle Q'(x), x \rangle$ clicks if

$$\langle Q'(x), x \rangle \geq \|x\|^2 + \varepsilon \|x\|^2.$$

Thus the “device” sensitivity is defined in terms of the Sobolev norm.

Two cases may appear. If for some $T > 0$, $\langle Q^T(x), x \rangle \geq \|x\|^2 + \varepsilon \|x\|^2$, then the “device” has clicked and we know for sure that there exist false coins in the system. However, it is possible that at some time $T > 0$ the “device” hasn’t (yet?) clicked because $\langle Q'(x), x \rangle < \|x\|^2 + \varepsilon \|x\|^2$. This may happen because either all coins are true, i.e., $\langle Q'(x), x \rangle < \|x\|^2 + \varepsilon \|x\|^2$ for all $t > 0$, or because at time $T$ the growth of $\langle Q^T(x), x \rangle$ hasn’t yet reached the threshold $\|x\|^2 + \varepsilon \|x\|^2$. In the first case the “device” will never click, so at each stage $t$ the test-vector $x$ produces “true” information; we can call $x$ a “true” vector. In the second case, the test-vector $x$ is “lying” at time $T$ as we do have false coins in the system, but they were not detected at time $T$; we say that $x$ produces “false” information at time $T$.

If we assume that there exist false coins in the system, say at stack $j$, but the “device” does not click at the moment $T$, then the test-vector $x$ belongs to the indistinguishable set

$$\mathcal{F}_x,T = \{x \in l_2^2 \mid \left( (1 + \gamma)T - 1 \right) |x_j|^2 < \varepsilon \|x\|^2, \text{ for some } j \}.$$

\(^1\)As in [9] we use quotation marks when referring to our mathematical “device.”
In [9] it was proven that the Wiener measure of the indistinguishable set tends to zero as $T \to \infty$:

$$\tilde{W}(F_{\varepsilon,T}) \leq \left(\frac{\varepsilon}{(1 + \gamma)^T - 1 - \varepsilon \cdot \prod_{m=1}^{\infty} \Delta_m}\right)^{1/2}.$$ 

This fact is not enough to realize the scheme described in the beginning of this section: we need a more precise result, namely we have to prove that $\tilde{W}(F_{\varepsilon,T})$ converges computably to zero. And, indeed, this is true because:

$$\tilde{W}(F_{\varepsilon,T}) \leq \eta, \text{ provided } t > \log_{1+\gamma} \left(\frac{\varepsilon}{\eta^2 \prod_{m=1}^{\infty} \Delta_m} + 1 + \varepsilon\right).$$

Denote by $P(N)$ the a priori probability of absence of false coins in the system. Then, the a posteriori probability that the system contains only true coins, when the “device” did not click after running the experiment for the time $T$, is

$$P_{\text{non-click}}(N) > 1 - \frac{1 - P(N)}{P(N)} \cdot \frac{\sqrt{\varepsilon}}{\sqrt{(1 + \gamma)^T - 1 - \varepsilon \cdot \prod_{m=1}^{\infty} \Delta_m}}.$$ 

5 A Brownian Solution Based on Resonance Amplification

In [1] the idea to consider a single act of quantum computation as a scattering process was suggested.

We will first illustrate the method by describing a simple quantum scattering system realizing the quantum C\text{NOT} gate, i.e., a quantum gate satisfying exactly to the same truth-table as the classical controlled-NOT gate. The C\text{NOT} device has two input and output channels. Each channel can be only in two different states, say $|0\rangle, |1\rangle$. The in and out states of the control-channel are the same, $|I_{in}\rangle = |I_{out}\rangle$, but the in and out states of the current-channel may be different, $|J_{in}\rangle \neq |J_{out}\rangle$, depending upon the state $|J_{in}\rangle$ and the control-channel state. The classical controlled-NOT gate has the following truth-table:

<table>
<thead>
<tr>
<th>$I_{in}$</th>
<th>$J_{in}$</th>
<th>$I_{out}$</th>
<th>$J_{out}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

which describes the effect of the device on the above in states,

$$|I_{in} J_{in}\rangle \quad \longrightarrow \quad |I_{out} J_{out}\rangle$$

| $I_{in} J_{in}\rangle$ | $\longrightarrow$ | $I_{out} J_{out}\rangle$ |
|-----------------|-----------------|
| $|0 0\rangle$ | $\longrightarrow$ | $|0 0\rangle$ |
| $|0 1\rangle$ | $\longrightarrow$ | $|0 1\rangle$ |
| $|1 0\rangle$ | $\longrightarrow$ | $|1 1\rangle$ |
| $|1 1\rangle$ | $\longrightarrow$ | $|1 0\rangle$ |

The quantum C\text{NOT} gate operates not only on the “classical” states $|0\rangle$ and $|1\rangle$, $(C\text{NOT}|ij\rangle = |ik\rangle$, where $i, j \in \{0, 1\}$ $k = i \oplus j \mod 2)$, but also on all their linear combinations,

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \quad \longrightarrow \quad \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|11\rangle + \alpha_{11}|10\rangle.$$

This quantum transformation can be presented via the unitary matrix
with respect to the canonical basis \((e_1, e_2, e_3, e_4) = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)\). More importantly, \(C_{\text{NOT}}\) is universal\(^2\) and truly quantum as it cannot be written as a tensor product of two binary operators \(U = U_1 U_2, U_1 \neq I\).

We claim that the matrix \(U\) in (3) can be realized as a scattering matrix of a special quantum dot. First, here is the motivation. Consider Figure 1 in which two isolated quantum wires are placed in proximity and there is a window region in which the two wires are coupled. An electron moving in the window region oscillates between the two quantum wires and the probability of the electron exiting into a specific quantum wire depends on the length of the window. This “switching phenomenon” was discovered by del Alamo and Eugster [3] and intensely discussed in literature, see for instance [15, 4]. We can arrange the setup in such a way that, under normal conditions, the electron exits from the same wire it enters, but switches to the other wire when an extra potential is applied, a realization of the \(C_{\text{NOT}}\) gate.

![Figure 1: \(C_{\text{NOT}}\) as quantum dot](image)

We continue with the mathematical model and assume that the quantum dot is inserted in an one-dimensional quantum wire \(R = (-\infty, \infty) = R_+ \cup R_-\) between the wires \(R_+, R_-\) and a single electron may be redirected according to the state of the dot. We assume also that the inner Hamiltonian of the quantum dot is presented by a finite diagonal matrix which is either \(A_1 = \text{diag} \{\alpha_2^1, \alpha_3^2, \ldots, \alpha_N^2\}\) or \(A_2 = \text{diag} \{\alpha_2^2, \alpha_3^2, \ldots, \alpha_N^2\}\), with positive diagonal elements, \(\alpha_2^1 < \alpha_2^2 < \alpha_3^2 < \cdots < \alpha_N^2\). We assume that the quantum dot is inserted in an one-dimensional quantum wire \(-\infty < x < \infty\) at the origin and a proper boundary condition is satisfied (see (9)) for connecting it with the Schrödinger operator on the wire defined in the space of square-integrable vector-functions \(L_2(R_+, \mathcal{E})\)

\[
l = -\frac{d^2}{dx^2}.
\]

One could assign to the above quantum system a product space \(H_e \times H_d\) constituted respectively by the states of the electron and the states of the dot, and consider an evolution of the system generated by the total Hamiltonian \(H_e + H_d + H_{\text{int}}\) with a proper interaction term. This would lead to a quite sophisticated problem of quantum mechanics, similar to three-body problems, see for instance [26]. We assume now that the state of the dot is selected independently and thus reduce the above problem to the corresponding one-body problem for an electron scattered in the quantum wire depending on the state of the dot. The corresponding device should be called rather quantum relay rather than quantum gate; however, it may be transformed into a quantum gate if the state of the dot is obtained as a quantum state with finite life-time. Practically the

\(^2\)Every classical computable function can be computed by a small universal set of gates like \{OR, NOT\} or \{NAND\}. A set of quantum gates \(S\) is called universal if any unitary operation can be approximated with an arbitrary accuracy by a quantum circuit involving gates in \(S\); see more in [16, 17, 19, 8].
model suggested below is acceptable if the life-time of the state of the dot is long enough during the scattering experiment. The corresponding general "zero-range" quantum Hamiltonian (solvable model) is described as a self-adjoint extension $A_\beta$ of the orthogonal sum $l \oplus A$ restricted to $l_0 \oplus A_0$ in $L_2(\mathbb{R}, \mathcal{E}) \oplus E$ onto a proper domain; here $\mathcal{E}$ is the input space and $E$ is the inner space (with dim $(E) \geq 2$). The positive part of the spectrum $\sigma_\beta$ of the operator $A_\beta$ is absolutely-continuous and fills the positive half-axis $\lambda \geq 0$ with multiplicity dim $(\mathcal{E})$. The role of eigen-functions of the spectral point $p^2 = \lambda > 0$ is played by the scattered waves $\overrightarrow{\Psi}_\nu(p)$, labeled with vectors $\nu \in \mathcal{E}$. The components of the scattered waves $\overrightarrow{\Psi}_\nu(p)$ in the outer space $L_2(\mathbb{R})$ are presented as linear combinations of exponentials:

$$\overrightarrow{\Psi}_{\nu,p}(x) = \begin{cases} e^{-ipx} \nu + e^{ipx} \overrightarrow{T}(p)\nu, & x < 0, \\ e^{-ipx} \overrightarrow{T}(p)\nu, & x > 0, \end{cases} \quad x \in \mathbb{R}.$$ (5)

The matrix

$$S_\beta(p) = \begin{pmatrix} \overrightarrow{T}(p) \\ - \overrightarrow{R}(p) \end{pmatrix}$$ (6)

is called the scattering matrix of the operator $A_\beta$.

The evolution of the wave function of the quantum mechanical system with Hamiltonian $A_\beta$ given by the equation

$$\frac{1}{i} \frac{\partial \Psi}{\partial t} = A_\beta \Psi,$$ (7)

and proper initial condition

$$\Psi \bigg|_{t=0} = \Psi_0,$$

can be described by the corresponding evolution operator constructed from the above scattered waves and square-integrable bound states $\Psi_s$ which satisfy the homogeneous equation

$$A_\beta \Psi_s = \lambda_s \Psi_s,$$

with negative eigen-values $\lambda_s$. Bound states do not play an essential role in our construction, so we may assume that the initial state $\Psi_0$ is orthogonal to all bound states and may be expanded in an analog of Fourier integral over the scattered waves

$$\Psi_0 = \frac{1}{2\pi} \int_\mathbb{R} \sum_\nu \overrightarrow{\Psi}_{\nu,p}(\nu_0, \nu_0) dp.$$ (5)

Then the evolution described by the solution of the equation (7) and the above initial data can be presented as a (continuous) linear combination

$$\Psi(t) = \frac{1}{2\pi} \int_\mathbb{R} \sum_\nu e^{ip^2t} \overrightarrow{\Psi}_{\nu,p}(\nu_0, \nu_0) dp$$

of modes incoming from infinity on the left $(-\infty)$ and on the right $(+\infty)$, and outgoing modes scattered to both directions $\pm \infty$ according to the solution of the time-dependent Schrödinger equation $\frac{1}{i} \frac{\partial \Psi}{\partial t} = AU$:

$$S_\beta(p) : e^{ip^2t} \begin{pmatrix} e^{-ipx} \nu_{left} \\ e^{ipx} \nu_{right} \end{pmatrix} \longrightarrow e^{ip^2t} \begin{pmatrix} e^{-ipx} (\overrightarrow{T}(p)\nu_{left} + \overrightarrow{R}(p)\nu_{right}) \\ e^{ipx} (\overrightarrow{T}(p)\nu_{left} + \overrightarrow{R}(p)\nu_{right}) \end{pmatrix}. \quad (8)$$

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3The transmission coefficients appear on the main diagonal of the matrix to fit the physical meaning of the scattering matrix for small values of $|\beta|$, when it is reduced to the undisturbed transmission $S(p) = I$. 

6
In this case first the behaviour of the scattering matrix at the resonance energy and non-occupied levels in the “quantum dot” described by the inner Hamiltonian. We observe – in implementing the switching function.

The constructed solvable model reveals the role of zeroes of the scattering matrix – the where $|\alpha\rangle$ attached. We may assume that the circuit lies on the surface of a semiconductor with Fermi-level $e = e_i \in \mathcal{E}$ (see [2]) and introduce the scalar function

$$\mathcal{M} = \frac{I + \lambda A}{A - \lambda p I + e},$$

then using the interaction defined by the boundary conditions (9) imposed on the boundary values (the jump $[u'](0)$ and the value $u(0)$ at the origin) of the component of the wave-function in the outer space and the symplectic coordinates $\xi_\pm \in \mathcal{E}$ of the inner component of the wave-function:

$$\begin{pmatrix} [u'](0) \\ -\xi_- \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \beta^* & 0 \end{pmatrix} \begin{pmatrix} u(0) \\ \xi_+ \end{pmatrix}, \tag{9}$$

we obtain the scattering matrix in the form

$$S_\beta(p) = \begin{pmatrix} \overrightarrow{T}(p) & \overrightarrow{R}(p) \\ \overleftarrow{R}(p) & \overleftarrow{T}(p) \end{pmatrix},$$

with equal transmission and reflection coefficients $\overrightarrow{T}, \overrightarrow{T}, \overleftarrow{R}, \overleftarrow{R}$:

$$\overrightarrow{T}(p) = \overrightarrow{T}(p) = \frac{2ip}{2ip + |\beta|^2 \mathcal{M}^{-1}}, \overrightarrow{R}(p) = \overrightarrow{R}(p) = -\frac{|\beta|^2 \mathcal{M}^{-1}}{2ip + |\beta|^2 \mathcal{M}^{-1}}.$$

The constructed solvable model reveals the role of zeroes of the scattering matrix – the resonances – in implementing the switching function.

Next we explore the properties of the scattering matrix depending on distribution of occupied and non-occupied levels in the “quantum dot” described by the inner Hamiltonian. We observe first the behaviour of the scattering matrix at the resonance energy $\alpha^2 > 0$ in case the resonance level $\alpha^2$ in the quantum dot is vacant, but $\alpha^2$ is occupied, hence eliminated from the quantum picture (due to Pauli’s principle, as discussed below). In this case we have

$$\mathcal{M}_1 = \frac{1 + \alpha^2 \lambda}{\alpha^2 - \lambda} |e_1|^2 + \sum_{l=3}^N \frac{1 + \alpha^2 \lambda}{\alpha^2 - \lambda} |e_l|^2 = \frac{1 + \alpha^2 \lambda}{\alpha^2 - \lambda} |e_1|^2 + \mathcal{M}_3,$$

where $|e_l|^2$ are the squares of the Fourier coefficients of the deficiency vector $e$ with respect to the eign-vectors of the operator $A_1$.

Next we consider the case when the resonance level $\alpha^2$ is occupied, but the level $\alpha^2$ is vacant. In this case

$$\mathcal{M}_2 = \frac{1 + \alpha^2 \lambda}{\alpha^2 - \lambda} |e_2|^2 + \sum_{l=3}^N \frac{1 + \alpha^2 \lambda}{\alpha^2 - \lambda} |e_l|^2 = \frac{1 + \alpha^2 \lambda}{\alpha^2 - \lambda} |e_2|^2 + \mathcal{M}_3,$$

where $|e_l|^2$ are the squares of the Fourier coefficients of the deficiency vector with respect to the eign-vectors of the operator $A_2$.

The above constructed model corresponds to “spin-less” electrons (electrons with the constant spin in absence of the magnetic field) on the quantum circuit $\mathbf{R}_- \cup \mathbf{R}_+$, with the quantum dot attached. We may assume that the circuit lies on the surface of a semiconductor with Fermi-level $\alpha^2_\uparrow$ (see [25]); the levels $\alpha^2_\uparrow, \alpha^2_\downarrow$ are (due to Pauli’s principle) alternatively occupied by electrons
traveling on the circuit or by electrons transferred from one level to another inside the quantum dot under the resonance laser shining.\textsuperscript{4}

We assume that the position of the electron on the level $\alpha_2$ with the level $\alpha_1$ vacant corresponds to $I_{in} = I_{out} = 0$ and the position of the electron on the level $\alpha_2$ with the level $\alpha_1$ vacant corresponds to $I_{in} = I_{out} = 1$. We identify these states of the system as the state $S_1$ and state $S_2$, respectively.

In case $\dim \mathcal{E} = 1$, for every value $\beta$ the transmission coefficients on the resonance electron’s energy $\lambda = \alpha_1^2$ can be expressed as (see [2]):

$$
\overline{T}(p) = \overline{T}(p) = \frac{2ip}{2ip + |\beta|^2 \mathcal{M}_1^{-1}} = 1, \quad \overline{R}(p) = \overline{R}(p) = 0,
$$

(at the resonance energy we have $\mathcal{M}_1^{-1} = 0$).

In the second case, when the resonance level $p^2 = \lambda = \alpha_1^2$ is occupied, we obtain (due again to Pauli’s principle) the following expression for the transmission coefficients of passing electrons with resonance energy:

$$
\overline{T}(p) = \overline{T}(p) = \frac{2ip\mathcal{M}_2(\lambda)}{2ip\mathcal{M}_2(\lambda) - |\beta|^2} = \frac{2ip \left( \frac{1+\alpha_2^2}{\alpha_2^2 - \alpha_1^2} |e_2|^2 + \sum_{l=3}^{N} \frac{1+\alpha_2^2}{\alpha_2^2 - \alpha_l^2} |e_l|^2 \right)}{2ip \left( \frac{1+\alpha_2^2}{\alpha_2^2 - \alpha_1^2} |e_2|^2 + \sum_{l=3}^{N} \frac{1+\alpha_2^2}{\alpha_2^2 - \alpha_l^2} |e_l|^2 \right) + |\beta|^2},
$$

and the corresponding expressions for the reflection coefficients

$$
\overline{R}(p) = \overline{R}(p) = -\frac{|\beta|^2}{2ip\mathcal{M}_2 + |\beta|^2},
$$

which can be approximated, for large enough $\beta$, as

$$
\overline{T}(\alpha_1) = \overline{T}(\alpha_1) \approx 0, \quad \overline{R}(\alpha_1) = \overline{R}(\alpha_1) \approx -1.
$$

Hence the scattering matrix is equal to

$$
\mathbf{S}_\beta(\alpha_1^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{for any relatively large enough } \beta, \text{ if the resonance level } \alpha_1^2 \text{ is not occupied, and is equal to}
$$

$$
\mathbf{S}_\beta(\alpha_1^2) \approx \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \text{if the resonance level } \alpha_1^2 \text{ is occupied, and the above conditions on the interaction of the quantum dot with environment are fulfilled.}
$$

We continue by showing how the probabilistic approach discussed in the previous section can be realised. We consider now an imaginable quantum scattering system with an infinitely-dimensional input-space and, in particular, with the infinite dimensional space $\mathcal{E}$ for which a solvable model was described above as an extension of the orthogonal sum $l_0 \oplus A_0$ with boundary condition (9), in which, however, the $\beta$-channel connecting the outer subspace $L_2(\mathcal{R}, \mathcal{E})$ with the inner subspace $E$ is as before two-dimensional. We associate these extensions with two states ($S_1$, $S_2$) of the total quantum system combined of the inner and outer components, with the interaction respectively switched on via the boundary condition (9), $\beta \neq 0$, or switched off, $\beta = 0$, and we interpret the Halting Problem in a probabilistic setting as the problem of distinguishing of the states ($S_1$, $S_2$) of the quantum system via a scattering experiment with a random input.

\textsuperscript{4}The manipulation of the resonance quantum dot is a sophisticated few-body problem of quantum scattering. A “solvable” model for it was discussed in [26]. The model involving the resonance laser shining as a tool of manipulation of the current through the quantum dot was introduced in [4].
Following the probabilistic strategy in Section 4, we compare the scattering matrices $S_\beta(\alpha_1^2)$ in states $S_1$ and $S_2$. In the first state this matrix coincides with identity, hence

$$\frac{1}{2}(I + S_\beta(\alpha_1^2)) = I.$$ 

In the second state, by (11) we have

$$\frac{1}{2}(I + S_\beta(\alpha_1^2)) = I - P_\beta,$$

where $P_\beta$ is an orthogonal projector on the subspace of unity dimension, which is collinear to the vector $\beta$ in the input space. Therefore in the second case for arbitrary test vector $e$ we have

$$\frac{1}{2} (\langle e, e \rangle + \langle S_\beta(\alpha_1^2) e, e \rangle) = |e|^2 - |P_\beta e|^2.$$

The expectation is that if the probability of the event $P_\beta e = 0$ is zero, then by choosing a random test-vector, with probability 1 the above correlation is strictly less than 1. To obtain the corresponding quantitative result we will assume that we have a testing “device” distinguishing between the two states of the system, which “clicks” if

$$|P_\beta e|^2 > \varepsilon |e|^2. \quad (12)$$

Unfortunately, the above “device” is not sensitive enough to derive proper estimates for probabilities and we need another norm in the right-hand side of the last inequality. In our case the input space $E$ is $l_2$ with the standard orthogonal basis $x \rightarrow \{x_m\}_{m=0}^{\infty}$. Following ([9]) we consider the discrete Sobolev classes and norms introduced in Section 4, (1) and (2) in order to define the case when the “device” clicks. Next we assume that the (complex) increments $x_m - x_{m-1}$ are independent. We are going to use, together with $l_2$ two more spaces of test-vectors. Both are stochastic spaces of all trajectories $x(t)$ of a Brownian particle on the complex plane along different discrete sequences of intermediate moments of time (“stops”): the equidistant sequence $t_l = \sum_{m=1}^l \delta_m$, for the first space, and the perturbed sequence $\tilde{t}_l = \sum_{m=1}^l \Delta_m$, for the second space. Both spaces are equipped with proper Wiener measures $W$, $W$ (see [28]). The measure $W$ is defined on the algebra of all finite-dimensional cylindrical sets $C_{\Delta_1, \Delta_2, \ldots, \Delta_N}$ of trajectories with fixed initial point $x_0 = 0$ and “gates” $\Delta_l, l = 1, \ldots, N$ (which are open discs in the complex plane):

$$C_{\Delta_1, \Delta_2, \ldots, \Delta_N} = \{x \mid x_{t_l} \in \Delta_l, l = 1, 2, \ldots, N\},$$

via multiple convolutions of the Green functions $G(x_{t_l+1}, t_{l+1} | x_l, t_l)$ corresponding to the sequence $\delta_{l+1} = t_{l+1} - t_l$:

$$W(C_{\Delta_1, \Delta_2, \ldots, \Delta_N}) =$$

$$\int \cdots \int_{\Delta_N, \Delta_{N-1}, \ldots, \Delta_1} \frac{dx_1 dx_2 \ldots dx_N}{\sqrt{\pi^N N!}} e^{\frac{|x_N - y_N|^2}{N}} \cdots e^{\frac{|x_1 - y_1|^2}{N}}$$

$$\int \cdots \int_{R_N, R_{N-1}, \ldots, R_1} \frac{dx_1 dx_2 \ldots dx_N}{\sqrt{\pi^N N!}} e^{\frac{|x_N - y_N|^2}{N}} \cdots e^{\frac{|x_1 - y_1|^2}{N}}, \quad (13)$$

where $R_N = R_{N-1} = \cdots = R_1 = R$. Using the convolution formula, the denominator of (13) can be reduced to the Green function $G(x_N, t_N | 0, 0)$, for any $\tau \in (s, t)$:

$$G(x, t | y, s) = \int_{-\infty}^{\infty} G(x, t | \xi, \tau) G(\xi, \tau | y, s) d\xi.$$ 

In a similar way we can define the Wiener measure for trajectories corresponding to the “perturbed” sequence $\tilde{t}_l$. 

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In what follows we are going to use the absolute continuity of the perturbed Wiener measure $\tilde{W}$ with respect to the non-perturbed one $W$: for every $W$–measurable set $\Omega$,

$$
\tilde{W}(\Omega) = \frac{1}{\prod_{l=1}^{\infty} \sqrt{\Delta_l}} \int_{\Omega} e^{-\sum_{m=1}^{\infty} \frac{1 - \Delta_m}{\Delta_m} |x_m - x_{m-1}|^2} dW.
$$

(14)

Further we consider the class of quasi-loops, that is the class of all trajectories of the “perturbed process” which begin from $(x_0, t_0) = (0, 0)$ and for any $t$, $\max_{0 < s < t} |x_s|^2 < C t$. We note that

- every $x \in l_2^1$ is a quasi-loop (with $C = |x|^2$),
- due to the reflection principle (see [28], p. 221), the class of all quasi-loops has Wiener measure one, both in respect of $W$, $\tilde{W}$.

We assume that the device clics, if the result of averaging exceeds a certain level defined by the above norm $\| \cdot \|^2_1$:

$$
|P_{\beta}x|^2 > \varepsilon \|x\|^2_1.
$$

This device cannot identify the state of the system from the observation of the Breit-Wigner averaged correlation between the input and output of a single act of scattering when presented a randomly chosen input $x \in E$ if $|P_{\beta}x|^2 < \varepsilon \|x\|^2_1$. This means that the test-vector $x$ belongs to the indistinguishable set

$$
\mathcal{F}_\varepsilon = \left\{ x \in l_2 \cap l_2^1, |P_{\beta}x|^2 < \varepsilon \left( \sum_{m=1}^{\infty} \frac{1 - \Delta_m}{\Delta_m} |x_m - x_{m-1}|^2 \right) \right\}
$$

(15)

= \{ x \in l_2 \cap l_2^1, |P_{\beta}x|^2 < \varepsilon \|x\|^2_1 \}.

Though technically we may easily consider, with Breit-Wigner averaging, the iterated scattering processes described by the powers $S^m$ of the scattering matrix, we will analyze now the independent single acts of scattering. In this case the indistinguishable set depends only upon the positive number $\varepsilon$, the vector $\beta \in E$ defining the interaction in the quantum system, and the sequence $\Delta$. Without loss of generality we may assume that the vector $\beta$ has all non-zero components $\beta_l \neq 0$.

We assume that the vector $b = \{b_l\}_{l=1}^{\infty}, b_l = \sum_{m=l}^{\infty} \beta_m$ belongs to $l_2$:

$$
|b|^2_2 = \sum_{m=1}^{\infty} m^2 |\beta_m|^2 < \infty.
$$

(16)

Our main result reads: If the condition (16) is satisfied, then the Wiener probability $\tilde{W}(\mathcal{F}_{\varepsilon,1})$ of the indistinguishable set $\mathcal{F}_{\varepsilon,1}$ corresponding to a single act of scattering is finite and is estimated as

$$
\tilde{W}(\mathcal{F}_\varepsilon) < \frac{\sqrt{\varepsilon |\beta|}}{\prod_{l=1}^{\infty} \sqrt{\Delta_l} \sqrt{\varepsilon |\beta|^2 + |b|^2}}.
$$

(17)

Following the calculation presented in [9], we approximate the indistinguishable set with finite-dimensional cylinder sets and reduce the estimation of $\tilde{W}(\mathcal{F}_{\varepsilon,1})$ to the calculation of a Wiener integral with respect to the $W$ measure on trajectories associated with “equidistant stops”. We have:

$$
\tilde{W}(\mathcal{F}_{\varepsilon,1})$$

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Finally, we obtain the announced result by taking into account the omitted factor which can be estimated due to (15) by the exponential:

\[
π\int \cdots \int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_N e^{-\frac{\|x_N-x_0\|^2}{\Delta_N}} \frac{\|x_{N-1}\|^2}{\Delta_{N-1}} \cdots \frac{\|x_1-x_0\|^2}{\Delta_1}.
\]

The integrand of the inner integral in the numerator contains the exponential factor

\[e^B = e^{-(1-\Delta_N)\frac{x_{N-1}-x_{N-2}}{\Delta_{N-1}} -(1-\Delta_{N-1})\frac{x_{N-2}-x_{N-3}}{\Delta_{N-2}} -(1-\Delta_{N-2})\frac{x_{N-3}-x_{N-4}}{\Delta_{N-3}}},\]

which can be estimated due to (15) by the exponential:

\[e^{-\frac{1}{2}\|p_{\beta}\|^2} = e^{-\frac{1}{2}\|x_{\beta}\|^2}.\]

Using this equality, the exponential in the numerator can be estimated from below by the quadratic form

\[-\sum_{m=1}^{N} |x_m - x_{m-1}|^2 - \frac{1}{\epsilon\|\beta\|^2} |(x, \beta)|^2.\]  

\[\text{(18)}\]

This quadratic form can be simplified using new vector variables \(\xi_m = x_m - x_{m-1}\):

\[\langle \beta, x \rangle = \sum_{m=1}^{\infty} x_m \bar{\beta}_m = \sum_{m=1}^{\infty} \xi_m \sum_{l=m}^{\infty} \beta_l.\]

Recall that the vector \(b, b_m = \sum_{l=m}^{\infty} \beta_l\) belongs to \(l_2\). Then the quadratic form in the exponent of the numerator can be presented as a quadratic form of an operator

\[\langle \xi, A_{\xi}\xi \rangle = |\xi|^2 + \frac{1}{\|\beta\|^2 \epsilon} |(b, \xi)| = \langle \xi, \left(I + \frac{\|b\|^2}{\|\beta\|^2 \epsilon} P_b\right) \xi \rangle,\]

where \(P_b\) is the orthogonal projection onto the one-dimensional subspace in \(l_2\) spanned by the vector \(b\). The ratio of the \(N\)-dimensional Gaussian integral in the numerator, normalized by the factor \(\pi^{-N/2}\) and the Gaussian integral in the denominator can be expressed as

\[\frac{1}{\pi^{N/2}} \int \cdots \int e^{-\langle \xi, A_{\xi}\xi \rangle} d\xi_1 d\xi_2 \cdots d\xi_N = \frac{1}{\sqrt{\det A_{\xi}}} = \frac{\sqrt{\|\beta\|^2 \epsilon}}{\sqrt{\|\epsilon\|^{2} + \|b\|^2}}.\]

Finally, we obtain the announced result by taking into account the omitted factor \(\prod_l \sqrt{\Delta_l}\).
References


[23] R. Landauer. Information is inevitably physical, in [18], 76–92.


