Quantum Projectors and Local Operators in Lattice Integrable Models

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Abstract

In the framework of the quantum inverse scattering method, we consider a problem of constructing local operators for two-dimensional quantum integrable models, especially for the lattice versions of the nonlinear Schrödinger and sine-Gordon models. We show that a certain class of local operators can be constructed from the matrix elements of the monodromy matrix in a simple way. They are closely related to the quantum projectors and have nice commutation relations with the half of the matrix elements of the elementary monodromy matrix. The form factors of these operators can be calculated by using the standard algebraic Bethe ansatz techniques.

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1 Introduction

In two-dimensional quantum integrable models, the quantum inverse scattering method (QISM) [1–5] provides a powerful tool for investigating physical quantities. Among them, the correlation functions have been studied extensively. In order to calculate the correlation functions, it is necessary to deal with states and local operators. At the early stages of the development of QISM, the problem of constructing states was solved by means of the Bethe ansatz.

Recently, a great progress was made for constructing local operators in a large class of spin chain models [6, 8, 7] which contain the XXX and XXZ spin chains with spin 1/2. Simple inverse mappings from the matrix elements of the monodromy matrix to the local spin variables were found [8, 7]. The inverse mappings help the calculations of form factors and correlation functions of the spin variables in the framework of QISM [8–10].

The XXX and XXZ spin chains with spin 1/2 are fundamental models, i.e. the auxiliary space and the quantum space at a site are isomorphic and the elementary monodromy matrix has a special point at which it becomes the permutation operator for the auxiliary and quantum spaces. The construction of the inverse mapping depends deeply on the existence of the permutation operators. Some non-fundamental models such as higher spin XXX chains were solved by means of the fusion procedure [7]. One characteristic property of these models is that the matrix elements of the elementary monodromy matrix are numerical.

But for some non-fundamental integrable models, such as the lattice nonlinear Schrödinger (LNS) models [11–19] and the lattice sine-Gordon (LSG) model [2, 20, 21, 14], the problem of constructing the inverse mapping is not solved. The LNS and LSG models are closely related to the XXX and XXZ spin chains respectively. Their elementary monodromy matrices are realized by quantum operators and the quantum space at each site is related to an infinite-dimensional representation of the Lie algebra $sl_2$ or its quantum deformation $U_q(sl_2)$. Only at very specific values of the coupling constant, the infinite dimensional representations are truncated into finite-dimensional ones. The approach by the fusion procedure is possible only at these special points and is very artificial. Moreover, we should take the infinite-dimensional representation limit which has many difficulties.

Therefore, it is better to consider the inverse mapping in more direct way. This paper is an attempt toward the construction of the inverse mapping.

The form factor bootstrap [22] is one of approaches to obtain the correlation functions and was applied to the (continuum limit of) LNS models and LSG models. In this approach, creation operators of the states are Zamolodchikov-Faddeev (ZF) creation operators. The ZF creation-annihilation operators are constructed by using the quantum reflection operators $B(\lambda)A^{-1}(\lambda)$ and their conjugate $D^{-1}(\lambda)C(\lambda)$. The local operators are treated by means of the quantum Gel’fand-Levitan equations [23, 24]. The calculation procedure for the form
factors are summarized into the axioms by Smirnov [22]. (See also [25–28] for the approach by the quantum Gel’fand-Levitan equation in case of quantum nonlinear Schrödinger model).

In contrast to the Gel’fand-Levitan method, we use the reflection operators to construct local operators. The elementary monodromy matrices of LNS and LSG models have special points at which they factorize into quantum projectors. The constructed operators are closely related to these quantum projectors. In this paper, a basis of states is chosen to be the Bethe eigenstates. We show that the form factors of the operators can be calculated by using the algebraic relations in the framework of QISM.

This paper is organized as follows. In section 2, the main idea for constructing the local operators is explained. In section 3, we show that form factors of these local operators can be calculated in the framework of the standard algebraic Bethe ansatz method. Some properties of these local operators are discussed in section 4. The explicit form of the operators are given for LNS and LSG models in sections 5 and 6 respectively. Section 7 is devoted to discussion.

2 Local operators from quantum projectors

Let $L_n(\lambda)$ ($n = 1, 2, \ldots, N$) be an infinitesimal monodromy matrix of lattice models with the intertwining property:

The monodromy matrix of the lattice model is given by

At the points $\lambda = \nu$ where the quantum determinant vanishes, the elementary monodromy matrix factorizes into quantum projectors:

We call a state $\prod_k B(\lambda_k)\Omega$ *Bethe state* for generic $\{\lambda_k\}$. When we emphasize that $\{\lambda_k\}$ satisfy the Bethe equations, we call the state *Bethe eigenstate*.

Let us denote the eigenvalues of diagonal part of the monodromy matrix on the reference state by

Thus, the form factors of $Q_n$ (resp. $P_n$) are easily represented by the form factors of $D^{-1}(\nu)C(\nu)$ (resp. $B(\nu)D^{-1}(\nu)$). These form factors can be calculated by using the algebraic commutation relations.
The time evolution of these operators is controlled by the Hamiltonian operators of the models. The Hamiltonian operator is also diagonalized on the Bethe eigenstates. The form factors of operators at any time can be easily expressed by those of the operators at a time (e.g. at $t = 0$). We do not discuss the time evolution in this paper.

Consideration for other points at which $A(\nu)$ is invertible is quite similar. Therefore we omit these cases.

3 Form Factors

In this section, we calculate the form factors of $D^{-1}(\nu)C(\nu)$ in general setting. The calculation for $B(\nu)D^{-1}(\nu)$ is similar. So we omit the case of $B(\nu)D^{-1}(\nu)$.

We forget the lattice structure (??) for a while and treat the matrix elements $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ as abstract objects. Let

The action of $A(\lambda)$ on the Bethe states is well known:

For generic $\mu$, $\lambda$, we have the following lemma:

After some calculations which are a slight modification of [30,31], we have

In the following, we will show that the sum in the right hand side of eq.(??) can be rewritten by using a single determinant.

By using the Cauchy determinant identity or by evaluating the residues, we can prove the following identity:

With help of this identity, it is possible to check that the $(M + 1)$-dimensional vector $\xi_k$ is a left null vector of the matrix $S$: $\sum_{k=1}^{M+1} \xi_k S_{kl} = 0$. The substitution of $S_{M+1,l} = -\sum_{k=1}^{M} (\xi_k/\xi_{M+1}) S_{kl}$ into $\det_{M} S^{(j)}$ leads to

By virtue of eq.(??) and eq.(??) for $\eta = \nu_A$, we have the final result for the form factors:
4 Some properties of $Q_n$ and $P_n$

In this section, we discuss some properties of the local operators $Q_n$ and $P_n$.

For a spectral parameter $\mu$, let us define $\mu^\vee := \mu + i\kappa$ for the rational case and $\mu^\vee := \mu - i\gamma$ for the trigonometric case. Then

The lemma (??) can be rewritten as follows:

The following relation comes from the intertwining property:

Compare to the action of $Q_1 = D^{-1}(\nu)C(\nu)$ on the right Bethe states (??), the action on the left Bethe states are complicated. For a spectral parameter $\mu$, let $\mu^{(m)} := \mu + im\kappa$ for LNS model and $\mu^{(m)} := \mu - im\gamma$ for LSG model. (Note that $\mu^{(1)} = \mu^\vee$). From

The origin of these complicated action is the following commutation relation

To conclude, the local operator $Q_n$ has nice commutation relations with the half of the matrix elements of the infinitesimal monodromy matrix.

Similarly, from

5 Lattice Nonlinear Schrödinger model

The Hamiltonian of the quantum nonlinear Schrödinger model is given by

For simplicity, we use the LNS model of [13,14] as an example. Let put the system in a box of length $2L$: $(-L < x \leq L)$, and discretize it to the lattice with $N$-sites: $x_n = -L + n\Delta$, $(n = 1, 2, \ldots, N)$. Here the lattice spacing is given by $\Delta = 2L/N$. The elementary operators for this lattice model are constructed from original fields as follows:
The infinitesimal monodromy matrix is given by [13,14]

The corresponding local operators are

6 Application to the lattice sine-Gordon model

The Hamiltonian of the quantum sine-Gordon model is given by

\[
|\Omega\rangle
\]

In order to construct the reference state \( |\Omega\rangle \), the elementary monodromy matrix should be taken as a composite of the infinitesimal monodromy matrices of two-adjacent sites [2]:

Thus, the shift operator is defined by

Let us introduce a positive “momentum cutoff” parameter \( \Lambda \) by \( 2r \cosh \Lambda = 1 \).
At \( \lambda = \nu_{A}^{(\epsilon,\epsilon')} := (1/2)(i\gamma + \epsilon\Lambda + i\epsilon'\pi), (\epsilon, \epsilon' = \pm 1), a_1(\lambda) \) vanishes and the infinitesimal monodromy matrix factorizes into the quantum projectors:

Although \( D^{-1}(\nu_{A}^{(\epsilon,\epsilon')}) \) does not exist, we can define the following unitary operators:

Let us introduce the following unitary operators:

By using the explicit expressions, we can check that these operators satisfy eqs.\((??)\) and \((??)\).

Therefore, in place of \( D^{-1}(\nu_{A})C(\nu_{A}) \), we can use the well-defined operator \( Q_{1}^{(\epsilon)} \). Because the unitary operator \( Q_{1}^{(\epsilon)} \) does not annihilate the reference state, the action of \( Q_{1}^{(\epsilon)} \) on the Bethe state has an “anomalous” term:
It may seem that the anomalous term would contribute to the form factors for the same numbers of $C(\mu)$ and $B(\lambda)$. If we recall

We conclude that although $D^{-1}(\nu_A^{(e)})$ does not exist, the result of section 3 is still correct for the LSG model. Thus, as a mnemonic, we can write:

7 Discussion

In this paper, we showed that a certain class of local operators can be constructed by using the quantum projectors. The form factors of these operators were calculated by using the techniques of the algebraic Bethe ansatz.

For LNS model, these local operators ($\psi^e_n(x)$ and $\psi^e_n(x)$) are lattice analogues of the continuum nonlinear Schrödinger fields $\psi(x)$ and $\psi^*(x)$. Because the inputs are the elementary monodromy matrix $L_n(\lambda)$ ($\nu_e^n$), the “dressing” of the output by a factor $(1 + (\kappa/4)\psi_n^e\psi_n^e)^{-1/2}$ seems unavoidable if one try to keep simplicity of the inverse mapping.

For LSG model, we considered the local operators $\mathcal{O}_n^{(e)}$ in the sector of the zero topological charge $Q$. Consideration for the sector $Q \neq 0$ is necessary. Moreover, in order to make connection with the quantum sine-Gordon model, we should consider the thermodynamic limit. Notice that

Acknowledgments
The author would like to thank M. Bellacosa and F. Ravanini for helpful discussions and INFN for financial support. This work is partially supported by the EU network EUCLID, no. HPRN-CT-2002-00325.

References


