A relativistic variant of the Wigner function

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The conventional Wigner function is inappropriate in a quantum field theory setting because, as a quasiprobability density over phase space, it is not manifestly Lorentz covariant. A manifestly relativistic variant is constructed as a quasiprobability density over trajectories instead of over phase space.

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1 Introduction

The Wigner function is a quasiprobability density over phase space (for reviews, see Lee[1], Cohen[2], or Hillery et al.[3]; a quasiprobability is generally not positive semi-definite), so in a quantum field theory setting it is not manifestly Lorentz covariant. Quasiprobabilities are conceptually interesting partly because Arthurs-Kelly type interpretations of positive definite Husimi functions derived by smoothing from Wigner functions are available[4–6], but the lack of manifest Lorentz covariance in the quantum field theory setting makes Wigner functions, in this respect, a poor relation to Feynman path integrals.

A relativistic variant of the Wigner function that is manifestly Lorentz covariant in a quantum field theory setting is introduced in section 2, as a quasiprobability density over trajectories instead of over phase space. As a formalism, the relativistic variant Wigner function is conceptually similar to the Feynman integral in its use of trajectories, but is conceptually different in its use of quasiprobabilities instead of phase integrals. A helpful alternative name for the “relativistic variant Wigner function” is the “trajectory Wigner function”, just because it is a quasiprobability density over trajectories instead
of over phase space. The emphasis on trajectories here can be compared with the phase space approach to quantum field theory of Zachos and Curtright[7].

The relativistic variant Wigner function is discussed for the specific case of the quantized real Klein-Gordon field (called here QKG) in section 3. It turns out that the relativistic variant Wigner function does not exist for QKG, prompting the introduction of a modified quantized real Klein-Gordon field (mQKG). QKG is a singular limit of mQKG. The emphasis on QKG as a field theory (instead of as a second quantized particle theory, for example as in [3, §2.5] and references therein) allows a nonlocality that is present in QKG to be characterized clearly in terms of the concepts of classical statistical field theory, and also allows a clear characterization of both the similarity and the group theoretic difference between quantum fluctuations and classical thermal fluctuations. The extension of the relativistic variant Wigner function to other non-interacting fields is discussed in section 4.

2 The relativistic variant Wigner function in general

The conventional Wigner function can be presented in non-relativistic quantum mechanics as the inverse fourier transform of

\[ \langle \psi | e^{i\hat{\phi} + i\hat{\pi}} | \psi \rangle, \text{ that is, as } \int d\theta d\omega e^{-ix\theta - ip\omega} \langle \psi | e^{i\hat{\phi} + i\hat{\pi}} | \psi \rangle; \]  

(1)

the Wigner function is a quasiprobability density. For quantum field theory the Wigner function is the inverse fourier transform of

\[ \langle \psi | e^{i\hat{\phi}_f + i\hat{\pi}_f} | \psi \rangle, \]  

(2)

where

\[ \hat{\phi}_f = \int_S \hat{\phi}(x)f(x)d^3x \quad \text{and} \quad \hat{\pi}_f = \int_S \hat{\pi}(x)f(x)d^3x \]  

(3)

are smeared field operators on a hypersurface S and \( \hat{\phi}(x) \) and \( \hat{\pi}(x) \) are operator-valued distributions. The emphasis on phase space is inappropriate for relativistic quantum field theory because a phase space approach is not manifestly covariant, so we will instead consider the inverse fourier transform of the c-number functional

\[ Q_{\psi}[f] = \langle \psi | e^{i\hat{\phi}_f} | \psi \rangle \]  

(4)
as the starting point for this paper, where \( \hat{\phi}_f \) is now associated with all of space-time,

\[
\hat{\phi}_f = \int \hat{\phi}(x)f(x)d^4x.
\] (5)

We can then construct our relativistic variant of the Wigner function as

\[
\chi_\psi[w] = \int \mathcal{D}f e^{-i \int f(x)w(x)d^4x} Q_\psi[f] = \int \mathcal{D}f e^{-i \int f(x)w(x)d^4x} \langle \psi | e^{i \hat{\phi}_f} | \psi \rangle
\] (6)

(the fourier transform functional measure \( \mathcal{D}f \) includes a factor \((2\pi)^{-1}\) for each of the infinite number of degrees of freedom that is fourier transformed). This definition is equally applicable for interacting and for non-interacting fields.

When \( \chi_\psi[w] \) exists, a set of marginal density functionals can be constructed from it by averaging over degrees of freedom, which includes probability density functionals that can be constructed from mutually commuting sets of field observables \( \hat{\phi}_f \). A paradigm case of a set of mutually commuting field observables is obtained when we restrict functions \( f' \) to be defined on a space-like hyperplane \( S \). Then,

\[
\rho^S_\psi[v|S] = \int \mathcal{D}f' e^{-i \int f'(x)v(x)d^3x} \langle \psi | e^{i \hat{\phi}_{f'}} | \psi \rangle
\] (7)

is manifestly a probability density functional, since \( \{ \hat{\phi}_{f'} \} \) is effectively a set of classical commuting observables. Straightforwardly, but heuristically,

\[
\rho^S_\psi[v|S] = \int \mathcal{D}f' e^{-i \int f'(x)v(x)d^3x} \langle \psi | e^{i \hat{\phi}_{f'}} | \psi \rangle = \int \mathcal{D}f' \langle \psi | e^{i \int (\hat{\phi}(x) - v(x))f'(x)d^3x} | \psi \rangle = \langle \psi | \prod_{x \in S} \delta(\hat{\phi}(x) - v(x)) | \psi \rangle \geq 0,
\] (8)

where \( \stackrel{N}{=} \) represents equality up to normalization.

3 The relativistic variant Wigner function for mQKG

For QKG, the algebraic structure of the field is specified by the commutation relation \([a_g, a_f^\dagger] = (f, g)\), where \( a_f^\dagger \) and \( a_f \) are smeared creation and annihilation components of the QKG field, \( \hat{\phi}_f = a_f^\dagger + a_f \), and \((f, g)\) is a Lorentz invariant positive semi-definite inner product,
\[ (f, g) = \hbar \int \frac{d^3 k}{(2\pi)^3} \frac{\tilde{f}^*(k)\tilde{g}(k)}{2\sqrt{k^2 + m^2}} \]
\[ = \hbar \int \frac{d^4 k}{(2\pi)^4} 2\pi \delta(k_\mu k^\mu - m^2) \vartheta(k_0) \tilde{f}^*(k)\tilde{g}(k). \]

Together, \[ \hat{\phi}_f = a_f^\dagger + a_f, [a_g, a_f^\dagger] = (f, g), \] and the action of annihilation operators on the vacuum, \[ a_g |0\rangle = 0, \] fix all the Wightman functions of the vacuum sector of QKG, so they are equivalent to other specifications of QKG, including specifications that require, in part, that \[ \hat{\phi}(x) \] satisfies the Klein-Gordon equation.

A 3-dimensional inverse functional fourier transform for the QKG vacuum does exist,

\[
\rho_0^S[v|s] = \int \hat{D}f' e^{-i \int f'(x)v(x)d^3x} \langle 0| e^{i\hat{\phi}'|0\rangle} \\
= \int \hat{D}f' e^{-i \int f'(x)v(x)d^3x} \langle 0| e^{ia\int_i \frac{1}{2}(f',f')e^{i\alpha_{f'}} |0\rangle} \\
= \int \hat{D}f' e^{-i \int f'(x)v(x)d^3x} e^{-\frac{1}{2}(f',f')} \\
= \int \hat{D}f' e^{-i \int f'(x)v(x)d^3x} \exp \left[ \frac{-\hbar}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{\tilde{f}^*(k)\tilde{g}(k)}{2\sqrt{k^2 + m^2}} \right] \\
\cong \exp \left[ -\frac{1}{\hbar} \int \frac{d^3 k}{(2\pi)^3} \frac{\tilde{v}^*(k)\sqrt{k^2 + m^2}\tilde{v}(k)}{2} \right],
\]

where the inversion of the factor \[ \sqrt{k^2 + m^2} \] at the last line is the standard consequence of the Fourier transform of a Gaussian. The fourier-mode kernel \[ \sqrt{k^2 + m^2} \] is nonlocal; \[ \rho_0^S[v|s] \] can be converted to a nonlocal real-space description,

\[
\rho_0^S[v|s] \cong \exp \left[ -\frac{1}{\hbar} \int d^3x d^3y v(x) \frac{m^2 K_2(m|x-y|)}{\sqrt{\frac{\pi}{2}}|x-y|^2} v(y) \right],
\]

where \[ K_2(m|x-y|) \] is a modified Bessel function. In terms of the concepts of classical statistical field theory, this probability density functional characterizes a nonlocality that is present in QKG. The dynamical nonlocality is manifest in the appearance of the fourier mode operator \[ \tilde{f}(k) \rightarrow \sqrt{k^2 + m^2} \tilde{f}(k), \] the nonlocality of which is described by Segal and Goodman[8]. This nonlocality is qualitatively the same as the nonlocality of the heat equation in classical physics, in that it has exponentially reducing effects at increasing space-like separation, so it should be understood to be similar to Hegerfeldt-type nonlocality[10], rather than similar to Bell-type nonlocality, which can be a significant effect at arbitrary space-like separation. Faster-than-light signals cannot be sent using this nonlocality, as always in quantum field theory,
as long as we insist that measurement operators commute at space-like separation, which can be understood to be because the initial states that would allow signals to be sent would require infinite energy to set up\[9\]. The quantum fluctuations of the QKG field vacuum state described by equation (11) are compared with the thermal fluctuations of a classical Klein-Gordon field at equilibrium in Appendix A.

Unfortunately, a 4-dimensional inverse functional fourier transform for the QKG vacuum is not obviously well-defined, because of the appearance of a delta function in a denominator,

\[
\int \mathcal{D}f e^{-i \int f(x)\omega(x)dx} \langle 0 | e^{i \hat{\phi} f} | 0 \rangle = \int \mathcal{D}f e^{-i \int f(x)\omega(x)dx} e^{-\frac{1}{2} (f, f)}
\]

which is undefined if the delta function is understood in a distributional sense. To construct a modified quantized real Klein-Gordon field (mQKG), for which the relativistic variant Wigner function is well-defined, in contrast to QKG, we replace \(\delta(k_\mu k^\mu - m^2)\) by \(F(k_\mu k^\mu)\), a positive semi-definite function (that is, no longer a distribution) of measure 1, where \(F(x) > 0\) only if \(x \geq 0\). mQKG\(^1\) is defined by the Lorentz invariant inner product

\[
(f, g) = \hbar \int \frac{d^4k}{(2\pi)^4} 2\pi F(k_\mu k^\mu) \theta(k_0) \tilde{f}^*(k) \tilde{g}(k);
\]

then for the mQKG vacuum, we obtain a well-defined relativistic variant Wigner function,

\[
\chi_0[w] \overset{N}{=} \exp \left[ -\frac{1}{2\hbar} \int \frac{d^4k}{(2\pi)^4} 2\pi F(k_\mu k^\mu) \theta(k_0) \tilde{w}^*(k) \tilde{w}(k) \right] \tag{15}
\]

(or, rather, see appendix B and appendix C for how it can be made well-defined).

QKG is in this approach a singular, and not obviously well-defined, limit of mQKG, in which the function \(F(k_\mu k^\mu)\) approaches \(\delta(k_\mu k^\mu - m^2)\). If we regard QKG as only an effective field theory, we can equally effectively describe a system using mQKG, provided \(F(\cdot)\) is as small off mass-shell as is necessary to reproduce results of experiments. In general, quantum field theories that are

\(^1\) mQKG still satisfies the Wightman axioms. In addition, it conforms to the requirements of the cluster decomposition theorem\[11, \S 4.4\], since the algebraic and Lorentzian structure and the Hamiltonian of the theory are all unchanged, so that the S-matrix satisfies the cluster decomposition principle.
delta-function concentrated to on mass-shell will be singular limits of quantum field theories like mQKG, as far as the relativistic variant Wigner function discussed in this paper is concerned. Taking trajectories to be sharply confined to a smooth classical dynamics is not consistent with a measure-theoretic approach to fields defined on space-time (in contrast, for phase space methods trajectories have to be confined to a single classical dynamics).

We can construct \( \chi_\psi[w] \) straightforwardly for arbitrary mQKG states in a Fock space generated from the vacuum. For the mQKG state \( a^\dagger_g |0\rangle \), for example, we obtain, by applying the commutation relations and the action of the annihilation operators on the vacuum,

\[
\chi_1[w] = \int \mathcal{D}f e^{-i \int f(x)w(x)dx} \frac{\langle 0 | a_g e^{i\phi f} a_g^\dagger |0\rangle}{(g,g)} \\
= \int \mathcal{D}f e^{-i \int f(x)w(x)dx} \left[ 1 - \frac{|(g,f)|^2}{(g,g)} \right] e^{-\frac{1}{2}(f,f)} \\
\overset{N}{=} \left[ -(g,g) + \left| \left\langle g, w \right\rangle \right|^2 \right] \chi_0[w],
\]

where we have written \( \left\langle g, w \right\rangle \) for the neutral inner product. For a superposition \( (v + u a^\dagger_g) |0\rangle \) of the vacuum with \( a^\dagger_g |0\rangle \), we obtain

\[
\chi_{01}[w] \overset{N}{=} \left[ -|u|^2(g,g) + |v + u\left\langle g, w \right\rangle|^2 \right] \chi_0[w],
\]

while for the state \( a^\dagger_{g_1} a^\dagger_{g_2} |0\rangle \), we obtain

\[
\chi_{2}[w] \overset{N}{=} \begin{pmatrix}
(g_1, g_2)(g_2, g_1) + (g_1, g_1)(g_2, g_2) \\
-\left| \left\langle g_1, w \right\rangle \right|^2(g_2, g_2) - \left\langle g_1, w \right\rangle \left( g_2, g_2 \right) (g_2, g_1) \\
-\left\langle g_2, w \right\rangle \left( g_1, g_1 \right)(g_1, g_2) - \left| \left\langle g_2, w \right\rangle \right|^2(g_1, g_1) \\
+\left| \left\langle g_1, w \right\rangle \right|^2 \left| \left\langle g_2, w \right\rangle \right|^2
\end{pmatrix} \chi_0[w];
\]

when \( g_1 \) and \( g_2 \) are orthogonal, \( (g_1, g_2) = 0 \), this reduces to

\[
\left[ -(g_1, g_1) + \left| \left\langle g_1, w \right\rangle \right|^2 \right] \left[ -(g_2, g_2) + \left| \left\langle g_2, w \right\rangle \right|^2 \right] \chi_0[w];
\]
when \( g_1 = g_2 = g \), it reduces to
\[
[-(2 + \sqrt{2})(g,g) + |(g,w)|^2][-(2 - \sqrt{2})(g,g) + |(g,w)|^2] \chi_0[w].
\] (20)

\( \chi_1[w], \chi_{01}[w] \), and \( \chi_2[w] \) are not positive semi-definite, as we expect for such a close variant of the Wigner function. For the coherent state \( e^{a_s^d}|0\rangle \), we obtain
\[
\chi_c[w] \overset{N}{=} e^{(g,w) + (w,g)} \chi_0[w],
\] (21)

which is positive semi-definite, as the conventional Wigner function also is for coherent states. For arbitrary mixtures of coherent states we obtain positive semi-definite relativistic variant Wigner functions, but for a superposition \( (c_1 e^{a_{s1}^d} + c_2 e^{a_{s2}^d})|0\rangle \) of coherent states we obtain
\[
\chi_{sc}[w] \overset{N}{=} \left( c_1^* c_1 e^{[g_1,w] + [w,g_1]} + c_2^* c_2 e^{[g_2,w] + [w,g_2]} \right) \chi_0[w],
\] (22)

which again is not positive semi-definite (unless it is trivial, \( g_1 = g_2 \)). Note that all these relativistic variant Wigner functions are finite order multinomials in the field \( w \) times \( \chi_0[w] \), with a closure induced by the Fock space norm that includes \( \chi_c[w] \) and \( \chi_{sc}[w] \).

We can also present a thermal state as a positive semi-definite relativistic Wigner function, invariant under the little group of a unit time-like 4-vector \( T^\mu \) (see Appendix D),
\[
\chi_T[w] = \int \mathcal{D}f \, e^{-i \int f(x) w(x) d^4x} \frac{\text{Tr}[e^{-\beta \hat{H}} e^{i \hat{\phi}^j}]}{\text{Tr}[e^{-\beta \hat{H}}]} \overset{N}{=} \exp \left[ -\frac{1}{2\hbar} \int \frac{d^4k}{(2\pi)^4} \tanh \left( \frac{\hbar k_\mu T_\mu}{2kT} \right) \frac{\tilde{w}^*(k) \tilde{w}(k)}{2\pi F(k_\mu k^\mu) \theta(k_0)} \right].
\] (23)

This presentation of a thermal quantum field state clarifies its relationship to the vacuum quantum field state in relatively elementary, albeit also relatively intractable, terms. A Hilbert space norm is mathematically effective largely because it is a tight constraint on theory, but the constraint is tight enough that the vacuum state and a thermal state cannot be presented in the same Hilbert space. If we instead use a function space that does not have a separable Hilbert space structure, the vacuum state and a thermal state can be presented in a uniform way. It should also be possible to present the vacuum of an interacting quantum field in the same function space formalism, by evaluating equation (6).
There are no particles as such in this field approach, but there is a countable basis for the Fock space, which can lead to the conventional particle interpretation. A particle interpretation for quantum field theory is not possible in general, however, when not only Fock space representations are considered.

4 Other quantum fields

For real interaction-free spin-1 quantum fields, relativistic variant Wigner functions over real classical spin-1 fields are identical to the results in section 3 — the only change is in the inner product, between real classical spin-1 test functions, representing the commutation bracket between smeared operator valued distributions as a c-number. We can introduce, for example,

$$[a_g, a_f^\dagger] = (f, g)$$

$$= \hbar \int \frac{d^4k}{(2\pi)^4} 2\pi F(k_\mu k^\mu) \theta(k_0) \left[ \tilde{f}_\mu^*(k) \tilde{g}^\mu(k) - \frac{k_\mu \tilde{f}_\nu^*(k)k_\nu \tilde{g}^\nu(k)}{k_\alpha k^\alpha} \right]. \quad (24)$$

It will be interesting to see whether the development given for spin-0 quantum fields can also be extended to spin-1/2 quantum fields. For spin-1/2 quantum fields we are of course faced with the additional difficulty of anticommutation relations, but we have several choices in considering them:

- we can consider fermion fields to be an essentially formal way to describe a perturbation of boson fields;
- we can try to develop a bosonization approach in 1+3 dimensions;
- we can take fermion fields to satisfy commutation relations instead of anticommutation relations, but modify interactions with the gauge fields to make the spin-1/2 quantum fields stable nonetheless.

Once we represent quantum field theory in terms of quasiprobability densities over trajectories, we can use different fields as coordinates in our classical description of trajectories — so we are free to eliminate some variables in favour of others — where the more formal structure of other representations discourages such freedom. There may of course be other ways of approaching the question of fermion fields. If we adopt the last choice above, of commutation relations for spin-1/2 quantum fields, there is an obvious vacuum probability density functional over trajectories of a classical Dirac field $\zeta(x)$,

$$P_0[\zeta] \propto \exp \left[ -\frac{1}{\hbar} \int \frac{d^4k}{(2\pi)^4} \frac{\sqrt{k_\mu k^\mu}}{2\pi F(k_\mu k^\mu) \theta(k_0)} \left[ \zeta^\dagger(k) \left[ k_\mu \gamma^\mu + \sqrt{k_\mu k^\mu} \right]^{-1} \zeta(k) \right] \right]. \quad (25)$$
The requirement for anticommutation relations for spin-$\frac{1}{2}$ quantum fields can be understood to be relative to a requirement for positive energy, which is only needed for stability when interactions are introduced. We can ensure stability even if we adopt commutation relations for spin-$\frac{1}{2}$ quantum fields, provided we introduce interactions in such a way that the Feynman diagrams in the new description are as they would have been if we had made the usual choice of anticommutation relations for spin-$\frac{1}{2}$ fields.

It will also be interesting to see whether relativistic variant Wigner function representations of vacuum states of interacting relativistic quantum field theories can be constructed as relativistically invariant modifications of relativistic variant Wigner functions for the vacuum states of non-interacting quantum fields. If we can construct a vacuum state $I[w, \zeta]$ of an interacting theory as a positive semi-definite relativistically invariant modification of a product $\chi_0[w]P_0[\zeta]$ of non-interacting QKG and spin-$\frac{1}{2}$ vacuums, for example, then other coherent-like states can immediately be written as $P(w, \zeta)I[w, \zeta]$, where $P(w, \zeta)$ is an arbitrary positive semi-definite multinomial in components of the fields $w$ and $\zeta$.

5 Conclusion

We have constructed a relativistic variant of the Wigner function for quantum field states, which is conceptually preferable to the conventional Wigner function. In particular, as a Lorentz covariant formalism, the relativistic variant Wigner function is an alternative to the Feynman path integral formalism. We have seen some of the properties of the relativistic variant Wigner function for the quantized real Klein-Gordon field, or at least we have for the slightly modified theory, mQKG, and also for other quantum fields. The distinction between QKG and mQKG is not very great, but the fact that QKG is singular in terms of the relativistic variant Wigner function is interesting in itself.

The striking similarity between quantum fluctuations and thermal fluctuations in a Wigner function formulation of quantum field theory in terms of fields (whether in the conventional phase space formulation or in the relativistic variant formulation), and the clarity with which the difference can be identified, suggests a description of quantum measurement in which quantum fluctuations are described explicitly. Equally striking is the nonlocal kernel that in classical statistical field theory terms is necessary to reproduce the QKG vacuum state.

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A Quantum and classical thermal fluctuations

In contrast to equation (11), the probability density for the classical Klein-Gordon field at equilibrium on a hyperplane $S$ is

$$\rho_{CKG}^S[v|s] \equiv \exp \left[ -\frac{1}{kT} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \bar{v}^\ast(k)(k^2 + m^2)\bar{v}(k) \right],$$

where the fourier-mode kernel $(k^2 + m^2)$ is local. Both quantum and classical probability densities restrict non-zero probability to solutions of the classical Klein-Gordon equation, but with different densities. The probability density for the QKG vacuum state is Poincaré invariant, in contrast to the Galilean invariance of the probability density for the classical Klein-Gordon equilibrium state.

Despite the difference in units and associated functional forms, Planck’s constant of action plays a very similar role in $\chi_0[w]$ to the role played by the Boltzmann energy $kT$ in a Gibbs probability density $\exp [-H[v]/kT]$. Both determine the amplitude of fluctuations. We have to be careful to remember the difference between the 3-dimensional Galilean symmetry of an equilibrium state and the (3+1)-dimensional Poincaré symmetry of the quantum field theory vacuum, but the Boltzmann energy and Planck’s constant are nonetheless closely analogous in their effect.

The difference between the functional forms of quantum fluctuations and thermal fluctuations is critical for understanding quantum measurement. Although quantum fluctuations and thermal fluctuations are both just fluctuations, we are apparently unable to reduce the “q-temperature” of a measurement device below $\hbar$ to reduce the effects of quantum fluctuations on measurement, whereas we routinely reduce the temperature of measurement devices to reduce the effects of thermal fluctuations. That we cannot reduce the q-temperature of a measurement device at all is an empirical principle at the heart of quantum theory, without which the distinction that quantum theory makes in principle between quantum fluctuations and thermal fluctuations becomes tendentious: if we could in practice reduce quantum fluctuations even a little, we would have to admit the possibility that quantum fluctuations can be reduced arbitrarily close to zero, just as we admit for thermal fluctuations. Even without a present possibility of actually reducing quantum fluctuations, however, we can nonetheless formulate a description of quantum measurement in which
quantum fluctuations of a measurement device are explicitly described, just as we explicitly describe thermal fluctuations of a measurement device. Note that it is only because quantum fluctuations cannot be eliminated that measurements using different devices have to be represented by noncommuting operators in quantum theory.

B Inverse functional fourier transform of a positive semi-definite Gaussian

In a finite dimensional case, it is well-defined to take the inverse fourier transform of a Gaussian $e^{-q(x)}$, where $q(x)$ is a positive semi-definite quadratic form, since $q(x)$ splits the space $X \ni x$ into orthogonal subspaces $X_0, q(x_0) = 0,$ and $X_1, q(x_1) > 0.$ For the inverse fourier transform we have

$$
\int_X e^{-iy.x} e^{-q(x)} = \int_{X_0} e^{-iy_0.x_0} \int_{X_1} e^{-iy_1.x_1} e^{-q(x_1)} \overset{\mathbb{N}}{=} \delta(y_0) e^{-q^{-1}(y_1)}, \tag{B.1}
$$

where the inverse quadratic form $q^{-1}$ exists on $X_1.$ This simple method extends to mQKG, but, given only a definition of $\delta(x)$ as a distribution, it does not extend to QKG. If we define $\delta(x)$ as a Colombeau generalized function[12], this simple method may possibly extend to QKG.

C Regularization of Gaussian integrals

For the functional

$$
\rho_D[w] \overset{\mathbb{N}}{=} \exp \left[ -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{w}^*(k) D(k) \tilde{w}(k) \right], \tag{C.1}
$$

where $D(k)$ determines the dynamics of a classical statistical field theory, the functional integral $\int D[w \rho_D[w]$ only exists in general if we restrict the range of the functional integration to functions that are smooth below a chosen scale (for a straightforward discussion, see [13, §8.1 and Appendix L]). This integral must be finite for us to regard $\rho_D[w]$ as a probability density functional (implicitly assuming normalization), as must the moments of the distribution. A simple way to ensure finiteness is to introduce a wave number cutoff, $|k| < \Lambda,$ for some Euclidean metric on $k.$

The Gaussian model of classical statistical field theory takes $D(k) = |k|^2 + m^2$, which progressively reduces the probability of higher frequency components.
of \( \bar{w}(k) \) (but not sufficiently to give a finite functional integral when \( \Lambda \to \infty \) except for one dimensional systems). In contrast, mQKG takes \( D(k) = [2\pi F(k, \kappa^\mu)\theta (k_0)]^{-1} \), where \( F(k, \kappa^\mu) \) has support, say, only for \( m^2 < k_\mu k^\mu < m^2 + \delta \), near the hyperboloid \( k_\mu k^\mu = m^2 \), so the functional integrals of mQKG are already constrained to functions \( \bar{w}(k) \) having support only where \( k_\mu \) is in the support of \( F(k_\mu k^\mu) \theta (k_0) \). mQKG can be treated in the same way as the well-understood Gaussian model, and the functional integral \( \int D \omega_{\mu} D [w] \) and the moments of the probability density are all finite for \( |k| < \Lambda \) (but not for \( \Lambda \to \infty \)). One difficulty is that this regularization breaks Lorentz invariance, but this is always a difficulty for simple regularizations of relativistic quantum field theory.

D Thermal state characteristic function

For the characteristic function of a thermal state of a simple harmonic oscillator, we have:

\[
Q_T[z] = \frac{\text{Tr}[e^{-\lambda a^{\dagger} a} e^{i(\alpha z^* + a z)}]}{\text{Tr}[e^{-\lambda a^{\dagger} a}]} = \frac{\text{Tr}[e^{-\lambda a^{\dagger} a} e^{i a^{\dagger} z^*} e^{i a z}]}{\text{Tr}[e^{-\lambda a^{\dagger} a}]} e^{-\frac{1}{2} |\alpha|^2} ;
\]

(D.1)

where we will suppose that \([a, a^{\dagger}] = \alpha\). Then

\[
\text{Tr}[e^{-\lambda a^{\dagger} a}] = \frac{1}{1 - e^{-\lambda \alpha}} ,
\]

(D.2)

and

\[
\text{Tr}[e^{-\lambda a^{\dagger} a} e^{i a^{\dagger} z^*} e^{i a z}] = \text{Tr}\left[ 1 - \frac{|z|^2 a^{\dagger} a}{1!} + \frac{|z|^4 a^{\dagger} a^2}{2!} - \frac{|z|^6 a^{\dagger} a^3}{3!} + \ldots \right] e^{-\lambda a^{\dagger} a} \\
= \text{Tr}\left[ 1 + \frac{|z|^2 (a^{\dagger} a - \alpha)}{2!} + \ldots \right] e^{-\lambda a^{\dagger} a} \\
= \left( 1 + \frac{|z|^2}{2!} \frac{d}{d\lambda} \left( 1 + \alpha \right) \left[ 1 + \frac{|z|^2 (a^{\dagger} a - 2\alpha)}{3!} \ldots \right] \right) e^{-\lambda a^{\dagger} a} \\
= \left( 1 - \frac{\alpha |z|^2 e^{-\lambda \alpha}}{1!(1 - e^{-\lambda \alpha})} + \frac{\alpha^2 |z|^4 e^{-2\lambda \alpha}}{2!(1 - e^{-\lambda \alpha})^2} - \frac{\alpha^3 |z|^6 e^{-3\lambda \alpha}}{3!(1 - e^{-\lambda \alpha})^3} + \ldots \right) e^{-\lambda a^{\dagger} a} \\
= \frac{1}{1 - e^{-\lambda \alpha}} \exp \left[ -\frac{\alpha |z|^2 e^{-\lambda \alpha}}{1 - e^{-\lambda \alpha}} \right],
\]

(D.3)
where we have used
\[
a^{\dagger n}a^n = a^{\dagger}a(a^{\dagger}a - \alpha)(a^{\dagger}a - 2\alpha) \ldots (a^{\dagger}a - (n - 1)\alpha)
\]
and
\[
\text{Tr}[(a^{\dagger}a)^ne^{-\lambda a^{\dagger}a}] = (-1)^n \frac{d^n}{d\lambda^n} \text{Tr}[e^{-\lambda a^{\dagger}a}],
\]
so that
\[
Q_T[z] = \exp \left[ -\frac{\alpha |z|^2}{2 \tanh \lambda \alpha} \right].
\]

For mQKG, we take
\[
\hat{H} = \int \frac{a^{\dagger}(k)a(k)k_\mu T_\mu}{2\pi F(k_\mu k_\mu)\theta(k_0)} d^4k
\]
to obtain equation 23.

References
