Antisymmetric Tensor Gauge Fields on $S_4$

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1 Abstract

Antisymmetric tensor gauge fields $\phi_{ab}(\eta)$ are formulated on the surface of a sphere $S_4 (\eta^2 = a^2)$ embedded in five dimensions. The free field model is equivalent to a scalar model on this sphere. Interactions with gauge fields are discussed. It is feasible to formulate models for interactions with $U(1)$ gauge fields $A_a(\eta)$ that are akin to those of Freedman and Townsend in flat space. In addition, it proves possible to have a novel interaction of $\phi_{ab}$ with $A_a$ and a spinor field $\Psi(\eta)$ on $S_4$ with both Abelian and non-Abelian gauge invariance. In these models, $A_a$ plays the role of a Stueckelberg field.

2 Introduction

The antisymmetric field $\phi_{\mu\nu}(x)$ has been of interest for some time [1-4]. In flat space, the free field possesses the gauge invariance

$$\delta \phi_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu;$$

preserving this gauge invariance when $\phi_{\mu\nu}$ is coupled to a vector gauge field is a problem that has been addressed by Freedman and Townsend [5].

Vector gauge fields have also been considered on the surface of a sphere $S_4 (\eta^2 = \eta_a \eta_a = a^2)$ embedded in five dimensional Euclidean space [6-12]. In this paper we consider how an antisymmetric field $\phi_{ab}(\eta)$ can be defined on this surface and how it can be coupled to a vector gauge field $A_a(\eta)$. The gauge transformations that we will consider are of the form

$$\phi_{ab} \rightarrow \phi_{ab} + L_{ab}\theta$$

$$A_a \rightarrow A_a + \eta_b L_{ba}\omega$$

where $L_{ab} = \eta_a \partial_b - \eta_b \partial_a$. It will be shown how models similar to those of ref. [5] can be formulated on $S_4$.

It will be also shown how a vector gauge field $A_a$ can serve as a “Stueckelberg field” for the tensor field $\phi_{ab}$. This will make it possible to have self interactions for $\phi_{ab}$ and to have
it interact with a spinor field $\Psi$ without losing gauge invariance. These interactions have a non-Abelian generalization. They do not appear to have an analogue in flat space.

In Appendix A we review the results of ref. [5] and in Appendix B some features of gauge fields on $S_4$ are provided.

3 Antisymmetric Tensor Field on $S_4$

A suitable Lagrange density for a free antisymmetric tensor field $\phi_{ab}$ on $S_4$ is

$$L = \frac{1}{2} \Gamma_a \Gamma_a \quad (\Gamma_a = \epsilon_{abcde} L_{bc} \phi_{de}) . \quad (4)$$

It is evident that this is invariant under the gauge transformation of eq. (2) as well as the radial displacement

$$\phi_{ab} \rightarrow \phi_{ab} + \eta_a \sigma_b(\eta) - \eta_b \sigma_a(\eta). \quad (5)$$

It is possible to show that the Lagrange density in eq. (4) is equivalent to

$$L = -\frac{2}{5} \left[ \phi_{ab} L_{pq} L_{pq} \phi_{ab} + \phi_{ab} L_{ab} L_{cd} \phi_{cd} 
-4 \phi_{pa} L_{ab} L_{bc} \phi_{cp} + 8 \phi_{pa} L_{ab} \phi_{bp} \right]. \quad (6)$$

The Lagrange density of eq. (4) can be written in first order form

$$L = m \left( V_a \Gamma_a - \frac{1}{2} m V_a V_a \right); \quad (7)$$

this follows from substituting the equation of motion for $V_a$

$$V_a = \frac{1}{m} \Gamma_a \quad (8)$$

directly back into eq. (7). The equation of motion for $\phi_{ab}$ is

$$\epsilon_{abcde} L_{cd} V_e = 0 \quad (9)$$

which implies that $V_a$ is a pure gauge field

$$V_a = \eta_b L_{ba} \omega / m . \quad (10)$$
This reduces the Lagrange density of eq. (7) to

\[ L = -\frac{1}{2} (\eta_p L_{pa}\dot{\omega}) (\eta_q L_{qa}\dot{\omega}), \]  

(11)

which is equivalent to

\[ L = \frac{1}{2} \omega (\eta_p L_{ap}) (\eta_q L_{aq}) \omega \]

(12)

\[ = \omega (\eta^2 L_{ab} L_{ab}) \omega. \]

(13)

(This is close to the Lagrange density for a massless scalar field \( \tilde{\omega} \) on \( S_4 \) given in ref. [6]

\[ L = \eta^2 \tilde{\omega} (L_{ab} L_{ab} - 4) \tilde{\omega}. \]

(14)

We have thus demonstrated how the Lagrange densities of eq. (4) and (7) have properties analogous to those of eqs. (A.1) and (A.6).

We now consider a non-Abelian generalization of eq. (7),

\[ L = -m \left( \frac{1}{2} \epsilon_{abcde} \phi_{ab}^{i} F_{cde}^{i} (V) + \frac{m}{2} V_{a}^{i} V_{a}^{i} \right) \]

(15)

where now \( i \) is an \( SU(2) \) index and \( F_{cde}^{i} (V) \) is defined by eq. (B7). This Lagrange density is invariant under the transformation

\[ \phi_{ab}^{i} \rightarrow \phi_{ab}^{i} + L_{ab} \theta^{i} + \epsilon^{ijk} \phi_{ab}^{j} \theta^{k} \]

(16a)

\[ V_{a}^{i} \rightarrow V_{a}^{i} + \epsilon^{ijk} V_{a}^{j} \theta^{k}. \]

(16b)

Normally, for \( F_{abc}^{i} (V) \) to transform as an isovector, one requires the gauge transformation of \( V_{a}^{i} \) to take the form of eq. (B8). However, in (15) \( F_{abc}^{i} (V) \) enters only through \( \epsilon_{abcde} F_{cde}^{i} (V) \) and

\[ \epsilon_{abcde} L_{cd} \left( \eta_p L_{pe} \theta^{i} \right) = 0; \]

(17)

it is thus sufficient to consider eq. (16b) to ensure that \( L \) in eq. (15) is an isoscalar.

We now introduce a \( U(1) \) vector gauge field \( A_{a} \) and couple it to \( \phi_{ab} \) using a Lagrange density analogous to eq. (A.14). The Lagrange density

\[ L = m \left[ \epsilon_{abcde} (A_{a} + V_{a}) L_{bc} \phi_{de} - \frac{1}{2} m V_{a} V_{a} \right] - \frac{1}{12} F_{abc} (A) F_{abc} (A) \]

(18)
(with $F_{abc}(A)$ given by eq. (B2)) possesses the invariances of eqs. (2), (5), (B.4) and (B.5). The equation of motion for $\phi_{ab}$

$$\epsilon_{abcd}L_{cd}(V_e + A_e) = 0 \quad (19)$$

shows that $V_a + A_a$ is a pure gauge field so that

$$V_a = -A_a + \frac{1}{m}\eta_bL_{ba}\lambda. \quad (20)$$

Substitution (20) into (18) gives rise to the $S_4$ analogue of the Stueckelberg Lagrange density of eq. (A.17)

$$L = -\frac{1}{12}F_{abc}(A)F_{abc}(A) - \frac{m^2}{2}\left(A_a - \frac{1}{m}\eta_bL_{ba}\lambda\right). \quad (21)$$

The gauge invariance of eq. (B4) is retained in eq. (21) provided

$$\lambda \rightarrow \lambda + m\omega \quad (22)$$

though the invariance of eq. (B.5) is lost.

It is not immediately apparent how to generalize the Lagrange density of eq. (18) so that one has a non-Abelian gauge invariance.

There is an essential difference between the gauge transformation for the antisymmetric tensor field in flat space (eq. (11)) and on $S_4$ (eq. (2)). In the former case, the gauge function is itself a vector while in the latter case it is a scalar. This suggests that the gauge parameters for $A_a$ and $\phi_{ab}$ could be the same, and that the field $A_a$ could itself play the role of the function $\sigma_a$ in eq. (5). Indeed, the quantity

$$\Lambda_a = \epsilon_{abcd}\left(\eta_bA_c - \frac{1}{4}\eta^2\phi_{bc}\right)\phi_{de} \quad (23)$$

is invariant under the gauge transformation,

$$A_a \rightarrow A_a + \eta_bL_{ba}\theta \quad (24a)$$

$$\phi_{ab} \rightarrow \phi_{ab} + L_{ab}\theta \quad (24b)$$

as is

$$\Pi_{abc} = \epsilon_{abcde}\left(\eta_dA_a - \frac{1}{2}\eta^2\phi_{de}\right). \quad (25)$$

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As a result, the Lagrange density

\[ L = -\frac{1}{12} F_{abc}(A) F^{abc}(A) + m \left( V^a \Gamma_a - \frac{1}{2} m V^a V_a \right) + g_\Lambda \Lambda_a \Lambda_a + g_\Pi (\Pi_{abc} \Pi_{abc})^2 + g_\Lambda \Lambda_a \Gamma_a \]

is invariant under the transformation of eq. (24). We also note that eq. (26) is invariant under shifts of eq. (B.5), but that of eq. (5) is lost for any term involving \( \Lambda_a \) or \( \Pi_{abc} \).

Coupling to a spinor field \( \Psi \) is possible if we supplement the \( U(1) \) version of the Lagrange density of eq. (B.10) with interactions of the form

\[ L = \lambda \Psi^\dagger \alpha \cdot \eta (\alpha_a \alpha_b \alpha_c \Pi_{abc}) \Psi. \]  

(27)

The algebra of the matrices \( \alpha_a \) implies that \( \epsilon_{abcde} \alpha_a \alpha_b \alpha_c \) is proportional to \( \gamma_{ab} \), and hence eq. (27) is equivalent to

\[ L = \lambda \Psi^\dagger \alpha \cdot \eta \gamma_{ab} \left( \eta^a A^b - \frac{1}{2} \eta^2 \phi_{ab} \right) \Psi. \]

(28)

One could of course use other combinations of \( \Lambda_a \) and \( \Pi_{abc} \) to devise gauge invariant interactions; the ones considered above are likely the only ones consistent with renormalizability.

It is easy to generalize (26) and (27) to the non-Abelian case. For the group \( SU(2) \), eqs. (15) and (B.6) can be used in conjunction with the isoscalar

\[ \Lambda_a = \epsilon_{abcde} \left( \eta_b A^i_c \phi^i_{de} - \frac{1}{4} \eta^2 \phi^i_{bc} \phi^i_{de} \right) \]

(29a)

and the isovector

\[ \Pi^i_{abc} = \epsilon_{abcde} \left( \eta^i d^a A^b - \frac{1}{2} \eta^2 \phi^i_{de} \right) \]

(29b)

to construct the Lagrange density

\[ L = -\frac{1}{12} F^i_{abc}(A) F^{abc}(A) - m \left( \frac{1}{3} \epsilon_{abcde} \phi^i_{ab} F^i_{cde}(V) + \frac{m}{2} V^i_a V^i_a \right) + g_\Lambda \Lambda_a \Lambda_a + g_\Pi (\Pi^i_{abc} \Pi^i_{abc})^2. \]

(30)

This is invariant under the gauge transformations of eqs. (16) and (B.8). Furthermore, eq. (28) can be generalized to give

\[ L = \lambda \Psi^\dagger \alpha \cdot \eta \gamma_{ab} \gamma^i \left( \eta^i A^j_b - \frac{1}{2} \eta^2 \phi^i_{ab} \right) \Psi \]

(31)
while maintaining $SU(2)$ non-Abelian gauge invariance. We note that in (26), (28), (30) and (31), $A_a$ and $\phi_{ab}$ can be decoupled by making the shift

$$\phi_{ab} \to \phi_{ab} + \frac{1}{\eta^2} \left( \eta_a A_b - \eta_b A_a \right),$$

leaving us with self interactions for the field $\phi_{ab}$.

## 4 Discussion

We have been able to demonstrate the first order formalism in flat space of ref. [5] for coupling a $U(1)$ tensor gauge field $\phi_{ab}$ to a vector gauge field $A_a$ has a direct analogue on a sphere $S_4$. Furthermore, it has proved possible to couple $\phi_{ab}$ to $A_a$ on $S_4$ in a manner that has no immediate flat space analogue. In the latter model, symmetry of $\phi_{ab}$ under the radial shift of eq. (5) is lost and there is but a single gauge parameter shared by $\phi_{ab}$ and $A_a$ (see eq. (24)). By shifting $\phi_{ab}$, $\phi_{ab}$ and $A_a$ can be decoupled. A non-Abelian extension of eq. (18) has not (as yet) been devised.

The problems of quantizing and renormalizing these models are currently being addressed.

## 5 Acknowledgements

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## References


6 Appendix A The Tensor Field

In flat space, the tensor field $\phi_{\mu\nu}(x) = -\phi_{\mu\nu}(x)$ (diag $g_{\mu\nu} = (+, -, -, -)$) with the Lagrange density [1-4]

$$L = -\frac{1}{8}G_\mu G^\mu \left( G^\mu = \epsilon^{\mu\alpha\beta\gamma} \partial_\alpha \phi_{\beta\gamma} \right)$$ (A.1)

possesses the gauge invariance of eq. (1). The equation of motion that follows from (A.1) is

$$G_{\mu,\nu} - G_{\nu,\mu} = 0;$$ (A.2)

div, combined with the identity

$$G^\mu_{\mu} = 0$$ (A.3)

implies that we can write

$$G_{\mu} = \partial_\mu \sigma$$ (A.4)

with the scalar $\sigma$ satisfying the massless Klein-Gordon equation

$$\partial^2 \sigma = 0.$$ (A.5)
The action of eq. (A.1) can be rewritten using a first order formalism in which $G_\mu$ is directly coupled to an auxiliary field $V_\mu$ [5]. If we have

$$L = \frac{1}{2}m (G_\mu V^\mu + m V_\mu V^\mu), \quad (A.6)$$

then the equation of motion for $\phi_{\alpha\beta}$ implies that

$$\epsilon^{\alpha\beta\gamma\delta} \partial_\gamma V_\delta = 0 \quad (A.7)$$

so that

$$V_\mu = \partial_\mu \lambda/m^2 \quad (A.8)$$

reducing (A.6) to

$$L = +\frac{1}{2} (\partial_\mu \lambda)^2; \quad (A.9)$$

again demonstrating that the model of eq. (A.1) is just that of a free spin zero field. (A non-Abelian generalization of eq. (A.6) leads [5] to a non-linear sigma model in place of eq. (A.9).)

If we use the equation of motion of $V_\mu$

$$V_\mu = -\frac{1}{2m} G_\mu \quad (A.10)$$

to eliminate $V_\mu$ in eq. (A.6) then we recover eq. (A.1).

We can also supplement eq. (A.6) with a term proportional to $\phi_{\mu\nu} \phi^{\mu\nu}$ so that

$$L = \frac{m}{2} (G_\mu V^\mu + m V_\mu V^\mu) - \kappa \phi_{\mu\nu} \phi^{\mu\nu}. \quad (A.11)$$

(This breaks the gauge invariance of eq. (1).) In place of eq. (A.7) we now find

$$\phi_{\mu\nu} \sim \epsilon_{\mu\nu\alpha\beta} \partial^\alpha V^\beta \quad (A.12)$$

and with an appropriate choice of $\kappa$, eliminating $\phi_{\mu\nu}$ in eq. (A.11) leaves us with the Lagrange density for a massive vector (Proca) field

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 V_\mu V^\mu. \quad (A.13)$$
Coupling $\phi_{\mu\nu}$ to a $U(1)$ gauge field $A_\mu$ by complexifying $\phi_{\mu\nu}$ and replacing $\partial_\mu \phi_{\alpha\beta}$ by $(\partial_\mu - ieA_\mu) \phi_{\alpha\beta}$ in eq. (A.1) breaks the symmetry of eq. (1). Freedman and Townsend circumvent this by directly coupling $G_\mu$ to $A_\mu$, using the Lagrangian density

$$L = \frac{1}{2}m \left[ G_\mu \left( V^\mu + A^\mu \right) + mV_\mu V^\mu \right]$$  \hspace{1cm} (A.14)

$$-\frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A)$$

where $F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu$. (A non-Abelian generalization of eq. (A.14) also appears in ref. [5].) This Lagrangian is invariant under the transformations of eq. (1) as well as

$$A_\mu \rightarrow A_\mu + \partial_\mu \omega.$$  \hspace{1cm} (A.15)

The equation of motion for $\phi_{\mu\nu}$ gives, in place of eq. (A.8),

$$V_\mu = -A_\mu + \frac{1}{m} \partial_\mu \lambda$$  \hspace{1cm} (A.16)

so that elimination of $V_\mu$ in eq. (A.14) results in the Stueckelberg Lagrange density for a massive, gauge invariant vector field

$$L = -\frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A) + m^2 \left( A_\mu - \frac{1}{m} \partial_\mu \lambda \right)^2.$$  \hspace{1cm} (A.17)

Invariance under eq. (A.15) is retained, provided

$$\lambda \rightarrow \lambda + m\omega.$$  

7 Appendix B  Gauge Fields on $S_4$

In refs. [7-12], vector gauge fields are formulated on $S_4$. In the $U(1)$ case, the Lagrange density for the vector $A_a$ is

$$L = -\frac{1}{12}F_{abc}(A)F^{abc}(A)$$  \hspace{1cm} (B.1a)

or equivalently,

$$L = \frac{1}{2}A_a P_{ab} A_b$$  \hspace{1cm} (B.1b)
where
\[
F_{abc}(A) = L_{ab}A_c + L_{bc}A_a + L_{ca}A_b \tag{B.2}
\]
and
\[
P_{ab} = \frac{1}{2} F_{pq} \delta_{ab} + L_{ap} L_{pb} - 3 L_{ab}. \tag{B.3}
\]
The action of eq. (B.1a) is invariant under the gauge transformation
\[
A_a \to A_a + \eta_b L_{ba} \omega \tag{B.4}
\]
as well as the radial shift
\[
A_a \to A_a + \eta_a \lambda . \tag{B.5}
\]
A non-Abelian generalization of (B.1a) is \[9, 10, 12\]
\[
L = -\frac{1}{12} F_{abc}^i F_{abc}^i \tag{B.6}
\]
where now for the \( SU(2) \) case
\[
F_{abc}^i = \left( L_{ab} A_c^i + \epsilon^{ijk} \eta_a A_b^j A_c^k \right) + \text{(cyc. perm. of abc)}. \tag{B.7}
\]
Eq. (B.6) is invariant under the gauge transformation
\[
A_a^i \to A_a^i + \eta_b L_{ba} \theta^i + \epsilon^{ijk} A_a^j \theta^k \tag{B.8}
\]
as well as under the radial shift
\[
A_a^i \to A_a^i + \eta_a \lambda^i . \tag{B.9}
\]
The gauge field \( A_a^i \) can also be coupled to a spinor field \( \Psi \[9, 12\], using the Lagrange density
\[
L = \Psi^i \alpha_a \eta_a \left[ -\gamma_{pq} L_{pq} + 2 + \gamma_{pq} \left( \eta_p A_q^i - \eta_q A_p^i \right) \tau^i \right] \Psi \tag{B.10}
\]
where
\[
\{ \alpha_a, \alpha_b \} = 2 \delta_{ab} \tag{B.11a}
\]
\[
\gamma_{ab} = \frac{1}{4} [\alpha_a, \alpha_b] , \tag{B.11b}
\]
and \( 2 \tau^i \) is an \( SU(2) \) Pauli matrix. Gauge invariance is retained in eq. (B.10).