Twisted K–Theory of Lie Groups

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Abstract

I determine the twisted K–theory of all compact simply connected simple Lie groups. The computation reduces via the Freed–Hopkins–Teleman theorem [?] to the CFT prescription, and thus explains why it gives the correct result. Finally I analyze the exceptions noted by Bouwknegt et al [?].
1 Introduction

WZW models provide a nice playground of solvable conformal field theories. The key to their treatment is of course that the target space is a group manifold $G$, and everything in the theory should be expressed in terms of the representation theory of this group.

An example for this is the fusion ring of the CFT, which is determined by the representation theory of the loop group $LG$ at the corresponding level. A more sophisticated instance are the Cardy branes, special boundary states in the CFT labeled by the irreducible representations.

In [?, ?] this was combined with boundary RG flow methods to determine possible decay paths of boundary states. The authors then used this to guess the underlying conserved charges and compared it with the expected result, the twisted K–theory of $G$. 
This check was significantly improved in [?], where the predicted charge groups for all $G = SU(n)$ are compared with a purely K–theoretic computation. However the K–theory computation uses an explicit cell decomposition and thus neither generalizes to other Lie groups nor does it make use of the representation theory. So although one finds at the end that the charge groups match this is not a very satisfactory result.

My work closes this gap: I reduce the computation of the twisted K–theory to a calculation in representation theory, and the latter will be very similar to the CFT calculation. So there will be no mystery that the resulting charge groups match, and furthermore it works for all simply connected, compact, simple Lie groups in the same way. The crucial connection between CFT and K–theory is the theorem of Freed, Hopkins and Teleman [?].

The actual computation is closely related to the method in [?] and I would like to thank Sakura Schäfer–Nameki for sharing her draft with me. However the seemingly simpler case of a WZW model instead of a coset is technically quite involved, and this is what I will be dealing with.

During the preparation of this paper a comment of Hopkins was relayed through [?]. Although it suffered from the Chinese whispers phenomenon it clearly suggested to compute the twisted K–theory along the following lines.

2 An Example

Take the simplest case, the Lie group $G = SU(2) \simeq S^3$. The WZW model at level $k$ contains Cardy boundary states $|\lambda\rangle$, labeled by the first $k + 1$ irreps of $SU(2)$ (or alternatively by irreducible positive energy irreps of $\widetilde{LG}$). Call this index set $J_k$. Now it was argued in [?] that the charges $q_\lambda$ of the boundary states satisfy

$$\dim_{C}(\mu) q_\lambda = \sum_{\nu \in J_k} N_{\mu\lambda}^{\nu} q_\nu \quad (1)$$

where $N_{\mu\lambda}^{\nu}$ are the fusion coefficients. This has two immediate consequences:

- Take $\lambda = 0$ the trivial representation $\Rightarrow q_\mu = \dim_{C}(\mu) q_0$, every charge is a multiple of $q_0$.

- Any relation $\sum_i a_i \mu_i = 0$ in the fusion ring leads to

$$\left( \sum_{i \in J_k} a_i \dim_{C}(\mu_i) \right) q_0 = 0 \quad (2)$$

so the charge $q_0$ is torsion. If there are no further identifications than eq. (1) then the order of the torsion is the minimal dimension of a relation in the fusion ring.
Especially for $SU(2)$ at level $k$ the relations in the fusion ideal have dimensions $(k + 2)\mathbb{Z}$, therefore the charge group is $\mathbb{Z}_{k+2}$.

Let us compare this with the twisted K–theory of $SU(2)$. It is readily evaluated for any given twist class $\tau \in H^3(SU(2)) \simeq \mathbb{Z}$, for example Rosenberg’s spectral sequence \[\tau K^*(SU(2)) = \mathbb{Z}_\tau\] (3)

Although the result nicely matches the CFT if you identify $\tau = k + 2$ it does not explain why this should be so. Even from the physicist’s perspective this is something of a miracle since the WZW model only has to reproduce the classical geometry in the $k \to \infty$ limit.

I will now explain why the CFT formula works, details of the computation will be explained in the following sections. First introduce $G$ equivariant K–theory via \[\tau K^*(G) = \tau K^*_G(G_{\text{Tr}} \times G_L)^{-\tau K^*_G(G_{\text{Ad}} \times G_L)}\] (4)

where the subscripts Tr, L, Ad denote the group action on that factor as Trivial action, Left multiplication, Adjoint action. Now the twisted K–theory for a Cartesian product can be computed via a K"unneth spectral sequence, the result is that \[\tau K^*_G(G_{\text{Ad}} \times G_L) = \text{Tor}^*_R(G_{\text{Ad}}(\{\text{pt.}\}), K^*_G(G_L))\] (5)

with the Verlinde algebra\footnote{Verlinde algebra will always denote the algebra over $\mathbb{Z}$, i.e. $RG \simeq \mathbb{Z}[x_1, \ldots, x_{\text{rank}(G)}]$ modulo the ideal generated by the fusion rules. The associated algebra over $\mathbb{Q}$ will be called rational Verlinde algebra, similarly complexified Verlinde algebra. In the language of [?], the Verlinde algebra is the fusion ring and the complexified Verlinde algebra is the fusion rule algebra.} $RG/I_k$ at level $k = \tau - \tilde{h}(G)$, where $\tilde{h}(G)$ is the dual Coxeter number. Especially for $G = SU(2)$ the representation ring is \[RSU(2) = \mathbb{Z}[\Lambda]\] (6)

where $\Lambda$ is the fundamental 2-dimensional representation. In this case the Verlinde algebra is particularly easy to write down. The ideal of relations is generated by $(k + 1)$–th symmetric power of $\Lambda$, i.e. the irreducible representation in dimension
3 The general computation

Fix once and for all a simply connected compact simple Lie group $G$. In the following we will work with topological spaces, together with group actions on them and twist
classes on them (the category $t$-$G$-$\text{Top}$). To facilitate this I will use the following conventions:

For a given $X \in \text{Ob}(t$-$G$-$\text{Top})$ let $X^{G$-$\text{Top}}$ be the underlying $G$-space and $t_X \in H^3_G(X^{G$-$\text{Top}})$ the twist class. A map $h : X \to Y$ is a $G$-map $h^{G$-$\text{Top}} : X^{G$-$\text{Top}} \to Y^{G$-$\text{Top}}$ such that $t_Y = h^*(t_X)$; In general there need neither be a map from nor to a point.

The Cartesian product $X \times Y$ is the ordinary product with twist class $t_{X \times Y} = \pi_1^*(t_X) + \pi_2^*(t_Y)$, where $\pi_{1,2}$ is the projection (they are not maps in $t$-$G$-$\text{Top}$) to the first and second factor.

The $t_X$-twisted equivariant K–theory will be denoted $\mathcal{K}_G(X)$.

### 3.1 The action map

Write $G_L$ for $G$ acting on itself by left multiplication and no twist, $t_{G_L} = 0$. Furthermore let $G_{Tr}$ be $G$ acting trivially on itself, but with arbitrary twist class $t_{G_{Tr}} \in H^3_G(\mathbb{Z})$.

The all-important observation is the following [?]: For $G_{Ad}$ ($G$ with the adjoint action on itself) one can pick a twist class such that there is an isomorphism

$$f : G_{Ad} \times G_L \cong G_{Tr} \times G_L$$

This is the map I used previously in eq. (4).

On the underlying $G$-spaces, $f$ is of course the $G$-diffeomorphism

$$f^{G$-$\text{Top}} : G_{Ad}^{G$-$\text{Top}} \times G_L^{G$-$\text{Top}} \cong G_{Tr}^{G$-$\text{Top}} \times G_L^{G$-$\text{Top}}, \ (g,\gamma) \mapsto \ (\gamma^{-1} g \gamma, \gamma)$$

and since it identifies the cohomology groups we can pick a suitable twist class on $G_{Ad}^{G$-$\text{Top}} \times G_L^{G$-$\text{Top}}$, it remains to show that it comes from the projection on the first factor.

So (dropping the superscripts $G$-$\text{Top}$ for readability) consider

$$p : G_{Ad} \times G_L \to G_{Ad}$$
We want to show that $p^{\ast}: H^3_G(G_{\text{Ad}}) \rightarrow H^3_G(G_{\text{Ad}} \times G_L) \simeq \mathbb{Z}$ is an isomorphism. By the universal coefficient theorem and the Hurewicz isomorphism it suffices to show that

$$z_{G_{\text{Ad}}} \pi_i \left( \frac{G_{\text{Ad}} \times G_L}{G_{\text{Ad}} \times EG \times G_L} \right) \simeq \mathbb{Z}$$

is an isomorphism for $i = 2, 3$. This follows from the long exact homotopy sequence of the fibration $p$:

$$\pi_i(G) \rightarrow \pi_i(G_{\text{Ad}}) \xrightarrow{p^{\ast}} \pi_i \left( \frac{G_{\text{Ad}} \times G_L}{G_{\text{Ad}} \times EG \times G_L} \right) \rightarrow \pi_{i-1}(G) \rightarrow \cdots$$

(15)

Following the image of the generating $S^3 \subset G$ one sees that the image of the leftmost map is contractible (zero) since the $G$-action on $G_{\text{Ad}}$ had fixed points.

Obviously isomorphic spaces have the same cohomology, thus

$$\mathcal{K}^G(G) = \mathcal{K}^G(G_{\text{Ad}} \times G_L) \simeq \mathbb{Z}$$

with $RG$-action

$$\rho \cdot x = \dim_{\mathbb{C}}(\rho)x \quad \forall \rho \in RG, x \in K^i_G(G_L) \simeq \mathbb{Z}$$

(18)

The twisted equivariant K-theory of $G_{\text{Ad}}$ is much more complicated and given by

**Theorem 1 (Freed–Hopkins–Teleman=FHT).**

$$tK^{\dim_G}_{G}(G_{\text{Ad}}) = RG/I_k$$

(19)

is an isomorphism of $RG$-modules, where the level $k = t_{G_{\text{Ad}}-h}$. 

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### 3.2 Making contact with FHT

In the next section I will discuss how the K-theory of the factors determines the K-theory of $G_{\text{Ad}} \times G_L$, for now I will summarize the K-theory of the factors, and their $RG$-module structure (i.e. how tensoring with a ordinary $G$-representation acts).

The easy part is

$$K^i_{G}(G) = K^i_{G}(G_{\text{Ad}} \times G_L)$$

with $RG$-action

$$\rho \cdot x = \dim_{\mathbb{C}}(\rho)x \quad \forall \rho \in RG, x \in K^i_G(G_L) \simeq \mathbb{Z}$$

The twisted equivariant K-theory of $G_{\text{Ad}}$ is much more complicated and given by

**Theorem 1 (Freed–Hopkins–Teleman=FHT).**

$$tK^{\dim_G}_{G}(G_{\text{Ad}}) = RG/I_k$$

is an isomorphism of $RG$-modules, where the level $k = t_{G_{\text{Ad}}-h}$. 
3.3 The Küneth Spectral Sequence

A Küneth theorem in general computes the cohomology of a Cartesian product $X \times Y$ from the cohomology of $X$ and $Y$. Now if you are really lucky you find that $h^*(X \times Y) = h^*(X) \otimes h^*(Y)$, for example if $h^*$ is the usual de Rham cohomology. But in general this is far more difficult.

A rather exhaustive account for untwisted equivariant K–theory is given in [?]. One finds a Küneth spectral sequence which — to make things even worse — does not always yield the K–theory of the product. However for sufficiently nice groups (like $G$ compact simply connected Lie groups, as is the case we are interested in) this spectral sequence does compute the desired K–theory of the product. Unfortunately the proof in [?] uses knowledge of the non-equivariant K–theory of $G$, which is what we are after in the twisted case.

Fortunately the Küneth spectral sequence was later extended in [?] to K–theory for $C^*$–algebras. This is useful since we can think of twisted K–theory as the K–theory of some (possibly noncommutative) $C^*$–algebra, see [?], and we get the following

Theorem 2. Let $G$ be a simply connected compact Lie group, $X \in \text{Ob}(t\text{-}G\text{-}\text{Top})$ and $Y \in \text{Ob}(G\text{-}\text{Top})$ (i.e. there is no twist on $Y$). Then there is a spectral sequence with

$$E_2^{p,} = \text{Tor}_R^p \left( {}^tK^*_G(X), K^*_G(Y) \right)$$

converging to $^tK^*_G(X \times Y)$.

In section 3.5 I will determine the $E_2$ term for the case at hand. Of course we then have to worry about higher differentials and extension ambiguities, but we will see in section 5 that

$$^tK^*_G(G_\text{Ad} \times G_L) \cong E_2^{*,} = \text{Tor}^*_R(RG/I_k, \mathbb{Z}).$$

3.4 Properties of the Verlinde algebra

Of course it is possible to work out the Verlinde algebra for any given group and level. However to actually compute the Tor in general it would be very helpful if $RG/I_k$ were a complete intersection, that is $I_k$ generated by a regular sequence (this will be defined in the following).

Although it seems to be true, there is no general proof so far. I will show that it would be a consequence of the existence of fusion potentials.
3.4.1 Fusion potentials

A fusion potential is a polynomial
\[ \phi \in RG \otimes \mathbb{Q} = \mathbb{Q}[x_1, \ldots, x_n], \quad n = \text{rank } G \] (22)
such that the Verlinde ideal \( I_k \subset RG \) is generated by the partial derivatives:
\[ I_k = \left\langle \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n} \right\rangle_{RG} \] (23)

Such potentials have been determined for the \( A_n = SU(n+1) \) and \( C_n = Sp(2n) \) Lie groups for all \( k \). Gepner [?] conjectured that such a potential exists in general, but I do not know of any proof so far. In the following I will assume this to be true.

Actually I will only be using that the Verlinde ideal is generated by \( n \) elements. I do not know any proof for this weaker statement either.

3.4.2 Regular sequences

First let us recall the following

**Definition 1.** Let \( R \) be a ring, then an ordered sequence of elements \( y_1, y_2, \ldots, y_n \in R \) is a **regular sequence** on \( R \) if

1. They do not generate the whole ring: \( \langle y_1, \ldots, y_n \rangle \neq R \)
2. For \( i = 1, \ldots, n \), \( y_i \) is a nonzerodivisor in \( R/ \langle y_1, \ldots, y_{i-1} \rangle \).

In general an ideal will have different sets of generators, some might form a regular sequence and some will not. So to make everything explicit (for \( R = RG \)) one would have to write down generators of the Verlinde ideal for each Lie group \( G \) and each level \( k \), and then show that the chosen generators in the chosen ordering form a regular sequence.

Instead, I will take a more high–powered approach and only show the existence of a regular sequence which generates the Verlinde ideal. This is a rather elementary application of the theory of commutative rings.

First we note that the representation ring of \( G \) is a polynomial ring generated by the \( n \) fundamental representations, \( RG \simeq \mathbb{Z}[x_1, \ldots, x_n] \). Since it is a polynomial ring over \( \mathbb{Z} \) it is Cohen–Macaulay. This means that for each ideal \( I \in RG \) we have\(^2\) \( \text{codim}(I) = \text{depth}(I) \). The same holds for the rational representation ring \( RG \otimes \mathbb{Q} \).

\(^2\)Here (Co)Dimension always refers to the Krull dimension, i.e. lengths of chains of prime ideals. The Cohen–Macaulay property means that this definition retains some of the intuitive properties of “dimension”.

}\]
To simplify notation I will use a subscript $\mathbb{Q}$ to denote the change of base ring from $\mathbb{Z}$ to $\mathbb{Q}$, so if $RG/I$ is the Verlinde algebra then $RG_{\mathbb{Q}}/I_{\mathbb{Q}}$ is the rational Verlinde algebra (I will suppress the level to avoid double subscripts).

The second ingredient is the fact that we are dealing with a RCFT, so the Verlinde algebra is a finite rank torsion free $\mathbb{Z}$–module, i.e. $RG/I \simeq \mathbb{Z}^d$ for some $d \in \mathbb{Z}_>$. Especially $\dim (RG_{\mathbb{Q}}/I_{\mathbb{Q}}) = 0$: From the point of view of algebraic geometry, the spectrum of the rational Verlinde algebra is just a finite set of points. Put differently, the minimal primes over $I_{\mathbb{Q}}$ are maximal ideals.

Recall that the codimension of $I_{\mathbb{Q}}$ is the minimum over all prime ideals $p \supset I_{\mathbb{Q}}$ over the maximum of lengths of chains of prime ideals descending from $p$. If $p$ is maximal then those chains have maximal length and we get $\text{codim} I_{\mathbb{Q}} = \text{dim} RG_{\mathbb{Q}} = n$, the rank of $G$.

Now we actually want the codimension of $I$, and not its rational version $I_{\mathbb{Q}}$. A fancy way of going from $\mathbb{Z}$ to the quotient field $\mathbb{Q}$ is localization at the nonzero integers. The advantage of this description is that it yields a nice relation between the prime ideals in the corresponding polynomial rings, and we get (see e.g. Theorem 36 of [?])

$$\text{codim} I = \text{codim} I_{\mathbb{Q}} = n$$

The last piece of information we need is the only point specific to WZW models: the Verlinde ideal $I$ can be generated by $n$ elements.

By the above remarks we know that $n = \text{depth} I$, so we can apply Theorem 125 of [?]: If an ideal $I$ can be generated by $n = \text{depth} I$ elements then it can be generated by a regular sequence (of $n$ elements).

**Theorem 3.** There exist a regular sequence $y_1, \ldots, y_n \in RG$ of length $n = \text{rank}(G)$ that generates the Verlinde ideal: $\langle y_1, \ldots, y_n \rangle = I_k$.

### 3.4.3 Koszul resolutions

Given any sequence $y_1, \ldots, y_n \in RG$ the **Koszul complex** $\mathcal{K}(y_1, \ldots, y_n)$ is a complex of length $n + 1$ with the $i$th entry the degree $i$ piece of the exterior algebra $\bigwedge_n RG$ (see [?] for a nice introduction). Another useful way of thinking about the Koszul complex would be the following: First for one element, $\mathcal{K}(y_1)$ is the length two complex analogous to the one depicted in eq. (8). Then $\mathcal{K}(y_1, \ldots, y_{i+1}) = \mathcal{K}(y_1, \ldots, y_i) \otimes \mathcal{K}(y_{i+1})$ (of course now most subtleties are hidden in the tensor product of complexes).

The whole point of this construction is that if $y_1, \ldots, y_n$ form a regular sequence in $RG$, then $\mathcal{K}(y_1, \ldots, y_n)$ is a resolution of the quotient ring $RG/\langle y_1, \ldots, y_n \rangle$. Especially we can choose (by the preceding section) a regular sequence generating the Verlinde ideal and thus obtain a resolution of the Verlinde algebra.
3.5 Deriving Tensor

Finally we have everything to compute the $E_2$ term of the spectral sequence in theorem 2. We know a projective resolution of the Verlinde algebra, and now it is a rather simple algebraic task to compute the Tor.

Using the Koszul resolution we readily compute

$$\text{Tor}_{RG}^*(RG/I_k, \mathbb{Z}) = \text{Tor}_{RG}^*(RG/\langle y_1, \ldots, y_n \rangle, \mathbb{Z}) =$$

$$= H_*\left( \bigotimes_{i=1}^{n} \mathcal{K}(y_i) \otimes_{RG} \mathbb{Z} \right)$$

$$= H_*\left( \bigotimes_{i=1}^{n} \left[ 0 \to \mathbb{Z} \xrightarrow{d_i} \mathbb{Z} \to 0 \right] \otimes_{RG} \mathbb{Z} \right)$$

$$= H_*\left( \bigotimes_{i=1}^{n} \left[ 0 \to \mathbb{Z} \xrightarrow{d_i} \mathbb{Z} \to 0 \right] \right)$$

where multiplication by $d_i = \text{dim}(y_i)$ is the map induced by $y_i$ as explained in section 2.

The homology of each factor is of course

$$H_* \left( \left[ 0 \to \mathbb{Z} \xrightarrow{d_i} \mathbb{Z} \to 0 \right] \right) = \begin{cases} \mathbb{Z}_{d_i} & r = 0 \\ 0 & \text{else} \end{cases}$$

and all that remains is to apply the usual Künneth formula for chain complexes, the result is

$$H \left( \bigotimes_{i=1}^{n} \left[ 0 \to \mathbb{Z} \xrightarrow{d_i} \mathbb{Z} \to 0 \right] \right) = \bigoplus_{j=1}^{2^n-1} \mathbb{Z}_{\gcd(d_1, \ldots, d_n)}$$

Proof. Induction: It is true for $n = 1$ and

$$H_* \left( \bigotimes_{i=1}^{n+1} \left[ 0 \to \mathbb{Z} \xrightarrow{d_i} \mathbb{Z} \to 0 \right] \right) =$$

$$= \bigoplus_{j=1}^{2^n-1} \left( \mathbb{Z}_{\gcd(d_1, \ldots, d_n)} \otimes \mathbb{Z}_{d_{n+1}} \right) \oplus \bigoplus_{j=1}^{2^n-1} \text{Tor}_{\mathbb{Z}} \left( \mathbb{Z}_{\gcd(d_1, \ldots, d_n)}, \mathbb{Z}_{d_{n+1}} \right)$$

$$= \left( \bigoplus_{j=1}^{2^n-1} \mathbb{Z}_{\gcd(d_1, \ldots, d_{n+1})} \right) \oplus \left( \bigoplus_{j=1}^{2^n-1} \mathbb{Z}_{\gcd(d_1, \ldots, d_{n+1})} \right)$$

$\square$
Finally note that \( \gcd(d_1, \ldots, d_n) \) does not depend on the choice of generators since it is the generator of the image \( \dim(I_k) \subset \mathbb{Z} \) under the dimension homomorphism \( \dim : RG \to \mathbb{Z} \). Explicitly computing \( \gcd(d_1, \ldots, d_n) \) in any special case is straightforward but tedious. Fortunately general expressions were determined in [?], and I will use their results (although they do not prove every formula).

Combining eqns. (16), (21) and (27) we determined \( t_K^G \). One can also determine the contributions to \( t_{K^0,1} \) separately by keeping track of even/odd degree terms in \( \text{Tor}^*_{RG} \). This is in principle clear but gets somewhat messy to write down, so I avoided it so far. The final result is that

\[
\begin{align*}
\nu^\dim G(G) & = (\mathbb{Z}_x)^{r_0}, \\
\nu^{1+\dim G(G)} & = (\mathbb{Z}_x)^{r_1}, \\
\nu^* G(G) & = (\mathbb{Z}_x)^r
\end{align*}
\]

where

\[
x \overset{\text{def}}{=} \frac{t_G}{\gcd(t_G, y)}, \quad t_G = k + \tilde{h}(G)
\]

with all numerical coefficients determined by table 1.

<table>
<thead>
<tr>
<th>( G )</th>
<th>( y )</th>
<th>( r_0 )</th>
<th>( r_1 )</th>
<th>( r )</th>
<th>( \tilde{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 = SU(2) )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( A_n, n \geq 2 )</td>
<td>lcm(1, 2, \ldots, n)</td>
<td>2^{n-2}</td>
<td>2^{n-2}</td>
<td>2^{n-1}</td>
<td>n + 1</td>
</tr>
<tr>
<td>( B_n, n \geq 2 )</td>
<td>lcm(1, 2, \ldots, 2n - 1)</td>
<td>2^{n-2}</td>
<td>2^{n-2}</td>
<td>2^{n-1}</td>
<td>2n - 1</td>
</tr>
<tr>
<td>( C_n, n \geq 3 )</td>
<td>lcm(1, 2, \ldots, n, 1, 3, 5, \ldots, 2n - 1)</td>
<td>2^{n-2}</td>
<td>2^{n-2}</td>
<td>2^{n-1}</td>
<td>n + 1</td>
</tr>
<tr>
<td>( D_n, n \geq 4 )</td>
<td>lcm(1, 2, \ldots, 2n - 3)</td>
<td>2^{n-2}</td>
<td>2^{n-2}</td>
<td>2^{n-1}</td>
<td>2n - 2</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>lcm(1, 2, \ldots, 5) = 60</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>lcm(1, 2, \ldots, 11) = 27720</td>
<td>2^2</td>
<td>2^2</td>
<td>2^3</td>
<td>9</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>lcm(1, 2, \ldots, 11) = 27720</td>
<td>2^4</td>
<td>2^4</td>
<td>2^5</td>
<td>12</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>lcm(1, 2, \ldots, 17)</td>
<td>2^5</td>
<td>2^5</td>
<td>2^6</td>
<td>18</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>lcm(1, 2, \ldots, 29) = 12252240</td>
<td>2^6</td>
<td>2^6</td>
<td>2^7</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 1: Coefficients determining the twisted K–groups, see eq. (29)
4 There are no Exceptions

For certain Lie groups and low levels \((k = 1 \text{ or } 2)\) it was noted in [?] that some of the fundamental representations are no longer in the Verlinde algebra. So they proposed that the corresponding generator should be removed, and the CFT rule for computing the D–brane charges should be applied to this presentation of the algebra. Of course this then depends on the explicit presentation of the Verlinde algebra, i.e. the choice of generators and relations.

However (although far from obvious) in the K–theory computation above there is no ambiguity since \(\text{Tor}\) is independent of the chosen resolution. There is only a technical problem of computing the \(\mathcal{R}G\)–module action of the Verlinde Algebra in the exceptional cases of [?], which I want to comment on.

So, for example, consider \(G_2\) at level \(k = 1\) as in [?]. The Verlinde algebra is

\[
V_k(G_2) = \mathbb{Z}[x_1, x_2] / \langle x_1, x_2^2 - x_1 - x_2 - 1 \rangle \\
= \mathbb{Z}[x_2] / \langle x_2^2 - x_2 - 1 \rangle
\]

(31a)

(31b)

where \(x_1\) and \(x_2\) are the two fundamental representations of dimensions 14 and 7.

The generators for the relations in eq. (31a) are a regular sequence in \(\mathcal{R}G_2\) and we obviously recover the result of table 1 if we compute \(\mathcal{K}(G_2)\) that way.

But we are certainly allowed to use the simpler presentation eq. (31b) to compute \(E_2\) in the Küneth spectral sequence. We find that

\[
E_2^{-p,*} = \text{Tor}_{\mathcal{R}G_2} \left( \mathbb{Z}[x_2] / \langle x_2^2 - x_2 - 1 \rangle , \mathbb{Z} \right) = H^{-p} \left( \begin{array}{c}
0 \\
\mathbb{Z}[x_2] \otimes_{\mathcal{R}G_2} \mathbb{Z} \\
\mathbb{Z}[x_2] \otimes_{\mathcal{R}G_2} \mathbb{Z}
\end{array} \right)
\]

(32)

But now \(\mathbb{Z}[x_2] \otimes_{\mathcal{R}G_2} \mathbb{Z} \simeq \mathbb{Z}_7\) since

\[
7(q \otimes n) = q \otimes (7n) = (x_1 q) \otimes n = 0 \otimes n = 0
\]

(33)

But now \(\mathbb{Z}[x_2] \otimes_{\mathcal{R}G_2} \mathbb{Z} \simeq \mathbb{Z}_7\) since

\[
7(q \otimes n) = q \otimes (7n) = (x_1 q) \otimes n = 0 \otimes n = 0
\]

(33)

Since 41 is invertible in \(\mathbb{Z}_7\) we get

\[
E_2^{-p,*} = H^{-p} \left( \begin{array}{c}
0 \\
\mathbb{Z}_7 \\
\mathbb{Z}_7
\end{array} \right) = 0 \quad \forall p
\]

(34)

and thus we reproduce the result of table 1.

This fits together nicely with the physical RG flow picture. Recall that the flows that lead to eq. (1) come from perturbations

\[
S \rightarrow S + \text{Tr} P \exp \left( \int_{\partial \Sigma} J \right)
\]

(35)
by the holonomy of the spin current $J = \Lambda_\mu J^\mu$ in some representation. Here you may choose any representation of $G$, regardless of what its image in the Verlinde algebra would be. We have to take all possible flows into account when we determine the D–brane charge group, i.e. we would get the wrong answer if we artificially restricted ourselves to representations which would be nonzero in the Verlinde algebra.

5 An algebra of BPS states

It remains to show that the K"unneth spectral sequence for $tK^*_G(G_{Ad \times G_{L}})$ does actually collapse at the 2nd term, and that we can solve the extension problems. Both can be dealt with by considering multiplicative structures$^3$.

Of course twisted K–theory does not come with a multiplication: The twist class adds when you tensor twisted bundles. Instead I will work in K–homology and use the fact that all spaces we consider are actually groups, so they come with multiplication maps $\mu : X \times X \to X$. Push forward induces the Pontryagin product

$$tK^*_G(X) \otimes_{RG} tK^*_G(Y) \to tK^*_G(X \times Y) \xrightarrow{\mu_*} tK^*_G(X)$$

For $tK^*_G(G_{Ad})$ it was noted in $[?]$ that the Pontryagin product is simply the fusion product in the Verlinde algebra. Since it is easy to identify the Pontryagin product for the trivial group $\{\text{pt.}\}$ we see that

$$tK^*_G(G_{Ad}) = RG/I_k$$
$$K^*_G(G_{L}) = K_*(\{\text{pt.}\}) = \mathbb{Z}$$
$$K^*_G(\{\text{pt.}\}) = RG$$

as rings.

There is a dual version of the K"unneth spectral sequence (the bar spectral sequence of $[?]$) with

$$E^2_{p,*} = \text{Tor}_p^{RG}(tK^*_G(X), tK^*_G(Y)) \Rightarrow tK^*_G(X \times Y)$$

so as before we use the trick with the action map of section 3.1 to get a spectral sequence

$$E^2_{p,*} = \text{Tor}_p^{RG}(RG/I_k, \mathbb{Z}) \Rightarrow tK^*_G(G_{Ad \times G_{L}})$$

The advantage here is that it is a spectral sequence of $RG$-algebras.

$^3$Of course this has nothing to do with the algebra of BPS states as in $[?]$. 
6 LEVEL–RANK NONDUALITY

Let’s have a closer look at the $E^2$ term. The $\text{Tor}(\cdot, \cdot)$ of two algebras is again an algebra: Again I will use the Koszul resolution of the Verlinde algebra (see section 3.4.3), which can be described as the exterior algebra on $n = \text{rank}(G)$ generators of degree 1.

$$K(y_1, \ldots, y_n) = \Lambda_{RG}(\eta_1, \ldots, \eta_n)$$

(40)

The Verlinde algebra is then the homology of this exterior algebra with respect to the differential $d(\eta_i) = y_i$. Applying $\otimes \mathbb{Z}$ and taking the homology with respect to $d(\eta_i) = \text{dim}_\mathbb{C}(y_i)$ we determine the algebra structure of the $E^2$ term as

$$\text{Tor}_p^{RG}(RG/I_k, \mathbb{Z}) = \Lambda_{\mathbb{Z}_x}(\eta_1, \ldots, \eta_n)$$

(41)

So the $E^2$ term looks like this

$$E^2_{p, *} = \begin{array}{cccc}
0 & d_2 & d_2 & \\
p = -1 & \mathbb{Z}_x & \bigoplus_{i=1}^n \eta_i \mathbb{Z}_x & \bigoplus_{1 \leq i < j \leq n} \eta_i \wedge \eta_j \mathbb{Z}_x \\
p = 0 & & & \\
p = 1 & & & \\
p = 2 & & & 
\end{array}$$

(42)

and $d_2$ obviously vanishes on $E^2_{1,*}$. Since all the algebra generators are at degree 1, the differential $d_2$ vanishes identically. By the same reason all higher differentials are zero, and $E^2 = E^\infty$.

It remains to see that $K_*(G)$ is only $x$–torsion, in principle there could be nontrivial extensions $0 \to \mathbb{Z}_x \to \mathbb{Z}_x \to \mathbb{Z}_x \to 0$ when we try to recover the homology from the associated graded groups. The $E^\infty$ term of the spectral sequence are the successive quotients $E^\infty_{p,*} = F^p/F^{p-1}$ of a filtration

$$0 \subset \cdots \subset F^{p-1} \subset F^p \subset F^{p+1} \subset \cdots \subset K_*(G).$$

(43)

The first quotient is $\mathbb{Z}_x = E^\infty_{0,*} = F^0/0 = F^0$, and from the multiplicative structure of the exterior algebra we see that this is a ring with unit. Since $1 \in \mathbb{Z}_x \subset K_*(G)$ acts as the identity on the successive quotients one can show by a simple induction that it is actually the identity in the bigger ring $K_*(G)$. But 1 is $x$–torsion, and thus everything:

$$x\kappa = (x \cdot 1)\kappa = 0\kappa = 0 \quad \forall \kappa \in K_*(G)$$

(44)

By similar arguments one can show that $K_*(G) = \text{Tor}_*^{RG}(RG/I_k, \mathbb{Z})$ as a ring.

6 Level–Rank Nonduality

For WZW models there are various level–rank dualities, see [?]. Those are distinct WZW models whose fusion rings happens to be the same, for example $B_2$ at level
$k = 1$ and $E_8$ at level $k = 2$. The corresponding Verlinde algebras are

\begin{align}
(B_2)_1 & : \quad RB_2/V_1 = \mathbb{Z}[x_1, x_2]/\langle x_2 = 1, \ x_2x_1 = x_1, \ x_1^2 = 1 + x_2 \rangle \\
(E_8)_2 & : \quad RE_8/V_2 = \mathbb{Z}[y_1, \ldots, y_8]/ \langle y_8^2 = 1, \ y_8 y_1 = y_1, \ y_1^2 = 1 + y_8, \ y_2, \ldots, y_7 \rangle
\end{align}

(45a) (45b)

where the nontrivial generators

$$\dim_{\mathbb{C}}(x_1) = 4 \quad \dim_{\mathbb{C}}(x_2) = 5 \quad \dim_{\mathbb{C}}(y_1) = 248 \quad \dim_{\mathbb{C}}(y_8) = 3875$$

(46)

correspond to the outermost nodes in the Dynkin diagram.

Now the two Verlinde algebras eq. (45a), (45b) are obviously isomorphic algebras. These algebras appear in the computation of the twisted K–theory for $B_2$ with $t_{B_2} = 1 + \tilde{h}(B_2) = 4$, $E_8$ with $t_{E_8} = 2 + \tilde{h}(E_8) = 32$.

(47)

From table 1 we readily find

$$'K(B_2) = \mathbb{Z}_2 \neq \mathbb{Z}_{128}^{128} = 'K(E_8)$$

(48)

Of course this should not be too much of a surprise, as the computation of the twisted K–theory depends on the $RB_2/RE_8$ module structure of the Verlinde algebra. There is no reason why level–rank duality should hold at the K–theory level.

This is in contrast to the supersymmetric Kazama–Suzuki models [?] where level–rank duality is believed to be an exact duality.

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