PP-wave GS superstring, polygon divergent structure and conformal field theory

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Abstract

In the semi-light cone gauge $g_{ab} = e^{2\phi} \delta_{ab}$, $\tilde{\gamma}^+ \theta = 0$, we evaluate the $\phi$-dependent effective action for the pp-wave Green-Schwarz (GS) superstring in both harmonic and group coordinates. When we compute the fermionic $\phi$-dependent effective action in harmonic coordinates, we find a new triangular one-loop Feynman diagram. We show that the bosonic $\phi$-dependent effective action cancels with the fermionic one, indicating that the pp-wave GS superstring is a conformal field theory. We introduce the group coordinates preserving $SO(4) \times SO(4)$ and conformal symmetry. Group coordinates are interesting because vertex operators take simple forms in them. The new feature in group coordinates is that there are logarithmic divergences from n-gons, so that the divergent structure is more complicated than in harmonic coordinates. After summing over all contributions from n-gons, we show that in group coordinates, the GS superstring on pp-wave RR background is still a conformal field theory.

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I. INTRODUCTION

Recently, Berenstein, Maldacena, and Nastase (BMN) [1] argued that the states of type IIB string theory on a plane-wave (pp-wave) Ramond-Ramond (RR) background [2] correspond to a class of operators in $\mathcal{N} = 4$ $SU(N)$ Yang-Mills with large $R$-charge $J$. The type IIB string worldsheet action on the pp-wave RR background was constructed using the Green-Schwarz (GS) formalism in [3, 4]. In the light-cone gauge, the exact solvability of the Green-Schwarz string theory on pp-wave RR background provides for an explicit realization of the pp-wave/SYM correspondence.

Although this light-cone formulation is very useful in calculating the string spectrum, it is not manifestly superconformally invariant. This makes it difficult to compute scattering amplitudes. Some covariant versions of type IIB string theory on pp-wave RR background were proposed [5, 6] by exploiting the hybrid and $U(4)$ formalisms, but since the most straightforward method for constructing the superstring action on pp-wave RR background is the Green-Schwarz formalism [3, 4], it is interesting to check the conformal invariance of the pp-wave GS superstring action explicitly. To calculate pp-wave string interaction in the context of the perturbative string theory, besides the conformal invariant action, one has to construct string vertex operators. In a pp-wave background, the vertex operators take a quite complicated form in harmonic coordinates, but they are simplified in group coordinates [7, 8]. As the transformation between harmonic coordinates and group coordinates is highly nonlinear, if we treat group coordinates as “fundamental fields” in the calculation, it is not a priori obvious whether the pp-wave GS superstring action in group coordinates is superconformal invariant or not.

In general backgrounds, because of the mixing of vertex operators, if one constructs the vertex operators like $G_{\mu\nu}\partial^n x^\mu \partial^n x^\nu$, one has to calculate n-loop Feynman diagrams for arbitrary n [9]. This is extremely difficult to carry out in practice. If we compute the pp-wave string interaction including BMN operators, it seems some n-loop Feynman diagrams have to be involved. Thus it is interesting to study the structure of the n-loop interaction.

1 The harmonic coordinates are convenient in calculating the string spectrum, but awkward for computing scattering amplitudes. Group coordinates have the opposite properties.
Feynman diagrams of the GS superstring theory on pp-wave RR background.

In this paper, we study the GS superstring in the semi-light cone gauge $g_{ab} = e^{2\phi} \delta_{ab}$, $\tilde{\gamma}^+ \theta = 0$ on pp-wave RR background and its conformal invariance. Following Kallosh-Morozov [10], we calculate the partition function of the pp-wave GS superstring in the semi-light cone gauge, and rewrite the original GS superstring action with its $SO(8)$ spinors in a concise form using $SU(4)$ spinors. When $m = 0$, the conformal anomaly from $SU(4)$ spinors has a coefficient $+8$ while the $x^\mu$ contribute $\frac{10}{2}$ and reparametrization ghost contribute $-\frac{26}{2}$, thus the total conformal anomaly is $\frac{10}{2} - \frac{26}{2} + 8 = 0$.

When $m \neq 0$, we first evaluate the $\phi$-dependent bosonic effective action in harmonic coordinates. Then we compute the fermionic $\phi$-dependent effective action, we find a new triangular one-loop Feynman diagram. We show that the bosonic $\phi$-dependent effective action cancels with the fermionic one, which indicates that the pp-wave GS superstring is an exact conformal field theory. The quartic interaction term in pp-wave background takes the special form $x^2 \partial_+ x^+ \partial_- x^+$. Since $x^+$ can only contract with $x^-$, there is no interacting term quadratic in $x^-$, thus the higher-loop Feynman diagrams can be decomposed as the products of the one-loop diagrams, which is an important feature when we discuss the mixing of the vertex operators.

We introduce the group coordinates preserving $SO(4) \times SO(4)$ and conformal symmetry. The new feature in group coordinates is that there are logarithmic divergences from n-gons whose divergent structure is more complicated than that in harmonic coordinates. After summing over all contributions from n-gons, however, we find that in group coordinates, the GS superstring on pp-wave RR background is still a conformal field theory.

The paper is organized as follows. In Section 2, we discuss the partition function of the GS superstring on the pp-wave RR background. In Section 3, we calculate the $\phi$-dependent effective action and quantum counterterms in harmonic coordinates, and find a new fermionic triangular one-loop diagram. In Section 4, we study the quantum counterterm and the $\phi$-dependent effective action in group coordinates. Some interesting logarithmic divergences from n-gons are found, which is a peculiar feature of the group coordinates. In Section 5, we present our summary and conclusion.
II. PARTITION FUNCTION OF GS SUPERSTRING ON PP-WAVE RR BACKGROUND

Let us start with the ten dimensional plane wave space with the metric and RR-flux [2]
\[
ds^2 = 2dx^+ dx^- - m^2 x^i x^i + dx^i dx^-,
F_{+1234} = F_{+5678} = 2m
\]
where \(i = 1, \cdots, 8\), and \(x^\pm = (x^9 \pm x^0)/\sqrt{2}\). The coordinates \(x^\pm, x^i\) are called harmonic coordinates and are physically convenient in calculating the string spectrum [7, 8].

The \(\kappa\)-symmetry gauge fixed type IIB GS superstring Lagrangian in the pp-wave RR background (1) with \(\bar{\gamma} + \theta = \bar{\gamma} + \bar{\theta} = 0\) is [3]
\[
\mathcal{L} = \frac{-1}{2} \sqrt{gg_{ab}} (2\partial_+ x^+ \partial_b x^- - m^2 x^i x^i \partial_+ \partial_b x^+ + \partial_+ x^i \partial_b x^i) \\
- \frac{i}{2} \sqrt{gg_{ab}} \partial_a x^+ (\bar{\theta}\bar{\gamma}^- \partial_b \theta + \theta\bar{\gamma}^- \partial_b \bar{\theta} + 2i m \partial_b x^+ \bar{\theta}\bar{\gamma}^- \Pi \theta) \\
+ i \epsilon^{ab} \partial_a x^+ (\theta\bar{\gamma}^- \partial_b \bar{\theta} + \bar{\theta}\bar{\gamma}^- \partial_b \theta)
\]

where \(\Pi = \gamma^1 \bar{\gamma}^2 \gamma^3 \bar{\gamma}^4\), \(\theta = (\theta^1 + i\theta^2)/\sqrt{2}\), \(\bar{\theta} = (\theta^1 - i\theta^2)/\sqrt{2}\), \(\theta^1\) and \(\theta^2\) are real Majorana-Weyl spinors or \(SO(8)\) spinors.

The partition function in the path integral formalism can be computed by fixing the semi-light cone gauge, i.e., \(\bar{\gamma}^+ \theta = \bar{\gamma}^+ \bar{\theta} = 0\) and \(g_{ab} = e^{2\phi}\delta_{ab}\). The gauge fixed path integral takes the form
\[
Z = \int Dx^\mu D\theta^1 D\bar{\theta}^2 DbDc \left(\det \partial_+ x^+ \right)^{-4} \left(\det \partial_- x^+ \right)^{-4} e^{-S}
\]
with
\[
S = -\frac{1}{2\pi \alpha'} \int d^2 \sigma \left( \partial_+ x^- \partial_+ x^+ + \frac{1}{2} \partial_+ x^i \partial_+ x^i - \frac{1}{2} m^2 x^i x^i \partial_+ x^+ \partial_- x^+ \\
+ i \partial_- x^+ \theta^1 \gamma^- \partial_+ \theta^1 + i \partial_+ x^+ \theta^2 \gamma^- \partial_- \theta^2 - 2i m \partial_+ x^+ \partial_- x^+ \gamma^- \Pi \theta^2 + \mathcal{L}_{bc} \right)
\]

2 We adopt the same notation as in [3], where the coset superspace [11] has been exploited.
where $\mathcal{L}_{bc}$ is the usual conformal ghost Lagrangian, and in Euclidean worldsheet we have $\partial_\pm = i\partial_0 \pm \partial_1$. The origin of $(\det \partial_+ x^+)^{-4}(\det \partial_- x^-)^{-4}$ can be traced back to the Faddeev-Popov fermionic gauge symmetry ghost Lagrangian \( \sqrt{g^+ - c^{(1)}} \Gamma^+ \Gamma^\mu L^+ \Gamma^\mu + \sqrt{g^0 - c^{(2)}} \Gamma^0 \Gamma^\mu L^0 \Gamma^\mu \) → (det $\partial_+ x^+$)$^{-8}$(det $\partial_- x^-$)$^{-8}$ and to $(\det \partial_+ x^+)^4(\det \partial_- x^-)^4$ due to the existence of second class constraints [10].

Using the fact that the $SO(8)$ spinor $\theta$ with $\bar{\gamma}\theta = 0$ can be presented as two $SU(4)$ spinors $\psi_\alpha, \eta_\alpha, \alpha = 1, \cdots, 4$, the partition function can be written as

$$
Z = \int Dx^\mu D\psi^1 D\eta^1 D\psi^2 D\eta^2 DbDc \,(\det \partial_+ x^+)^{-4}(\det \partial_- x^-)^{-4} e^{-S}
$$

with

$$
S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left( \partial_+ x^- \partial_- x^+ + \frac{1}{2} \partial_+ x^i \partial_- x^i - \frac{1}{2} m^2 x^i \partial_+ x^+ \partial_- x^+ \\
- i \partial_+ x^+ \psi^1 \partial_- \eta^1 - i \partial_+ x^+ \psi^2 \partial_- \eta^2 + i m \partial_+ x^+ \partial_- x^+ (\psi^1 \psi^2 + \eta^1 \eta^2) + \mathcal{L}_{bc} \right)
$$

(6)

where we have exploited $\partial_+ x^+ \eta^1 \partial_+ \psi^1 \psi^1 \partial_- \eta^1 + \psi^1 \eta^1 \partial_+ \partial_- x^+ \partial_+ x^+ \partial_- \eta^2 \\
\partial_+ x^+ \psi^2 \partial_- \eta^2 + \psi^2 \eta^2 \partial_+ \partial_- x^+$, and $x^- \rightarrow \frac{1}{2} \psi^1 \eta^1 + \frac{1}{2} \psi^2 \eta^2 \rightarrow x^-$, that is, $(\frac{1}{2} \psi^1 \eta^1 + \frac{1}{2} \psi^2 \eta^2) \partial_+ \partial_- x^+$ can be absorbed into the redefinition of $x^-$. After rescaling $\partial_- x^+ \psi^1 \rightarrow -\psi^1$ and $\partial_+ x^+ \psi^2 \rightarrow -\psi^2$, we have

$$
Z = \int Dx^\mu D\psi^1 D\eta^1 D\psi^2 D\eta^2 DbDc e^{-S}
$$

(7)

with

$$
S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left( \partial_+ x^- \partial_- x^+ + \frac{1}{2} \partial_+ x^i \partial_- x^i + i \psi^1 \partial_+ \eta^1 + i \psi^2 \partial_- \eta^2 + \mathcal{L}_{bc} \\
- \frac{1}{2} m^2 x^i \partial_+ x^+ \partial_- x^+ + i m \psi^1 \psi^2 + i m \eta^1 \eta^2 \partial_+ x^+ \partial_- x^+ \right)
$$

(8)

which consists of 10 ordinary scalar fields $x^\mu$, reparametrization ghost $b, c$, and four $SU(4)$ fermions $\psi^1_\alpha, \eta^1_\alpha, \psi^2_\alpha, \eta^2_\alpha$ with $\alpha = 1, \cdots, 4$. The first line shows the kinetic terms, and the second line is the interaction Lagrangian. The action (8) is nothing but the type IIB GS superstring action in the semi-light cone gauge on the pp-wave RR background, where its original $SO(8)$ spinors have been presented as $SU(4)$ spinors. Here we point out that the action (8) is much simpler than the superstring action derived using the $U(4)$ formalism in [6].
III. CONFORMAL INVARIANCE OF GS SUPERSTRING IN HARMONIC COORDINATES

To show the action (8) is conformally invariant, let us first consider the \( m = 0 \) case. In (8), there are no worldsheet fermions, but instead some 0-differential spacetime fermions \( \eta^1, \eta^2 \) and 1-differential spacetime fermions \( \psi^1, \psi^2 \). The 1-differential fermions \( \psi^1 \) do not have an ordinary norm \( \| \psi^1 \|^2 = \int d^2 \sigma \sqrt{g} \psi^1 \bar{\psi}^1 \), instead their norm is induced by that of \( \theta^1 \) and \( \theta^2 \). From the rescaling \( \partial_- x^\pm \psi^1 \to -\psi^1 \), the appropriate norm for \( \psi^1 \) is

\[
\| \psi^1 \|^2 = \int d^2 \sigma \sqrt{g} \psi^1 \bar{\psi}^1 \frac{1}{|\partial_- x^+|^2}
\]

and the regulator integral should be

\[
\int |D\eta^1|^2 |D\psi^1|^2 \exp \left( -\int d^2 \sigma (\psi^1 \partial_+ \eta^1 + \bar{\psi}^1 \partial_- \bar{\eta}^1 + M^2 \frac{\sqrt{g}}{|\partial_- x^+|^2} \psi^1 \bar{\psi}^1) \right).
\]

Integrating out \( \psi^1, \bar{\psi}^1 \) fields gives \( \int |D\eta^1|^2 \exp \int \sqrt{g} \bar{\eta}^1 \Delta \eta^1 / M^2 \) with Laplace operator of the form

\[
\Delta = \frac{1}{\sqrt{g}} \partial_- |\partial_- x^+|^2 \partial_+.
\]

The conformal anomaly for the Laplace operator \( \Delta_{p,q} = p(z) \partial q(z) \bar{\partial} \) is described by Liouville action [10]

\[
\exp \left( -\frac{1}{48 \pi} \int \left( \frac{|\partial p|^2}{p^2} - \frac{4 \partial p \partial q}{pq} + \frac{|\partial q|^2}{q^2} \right) \right)
\]

In conformal gauge \( g_{ab} = e^{2\phi} \delta_{ab} \), i.e., \( p = e^{-2\phi}, q = e^{-2\phi} |\partial_- x^+|^2 \), then (11) becomes

\[
\exp \left( -\frac{1}{48 \pi} \int \left[ -8 \partial \bar{\partial} \partial \bar{\partial} \Phi - (4 \Phi + \ln |\partial_- x^+|^2) \partial \bar{\partial} \ln |\partial_- x^+|^2 \right] \right).
\]

The first term in the bracket shows that the conformal anomaly from \( \psi^1, \eta^1 \) has a coefficient +8 while the \( x^\mu \) contribute \( \frac{10}{2} \) and reparametrization ghosts contribute \( -\frac{26}{2} \), so the total Polyakov anomaly in (8) with \( m = 0 \) is \( \frac{10}{2} - \frac{26}{2} + 8 = 0 \). The second term in the bracket represents the additional anomaly, which is proportional to \( \partial \bar{\partial} x^+ \) and can be removed by a shift of the \( x^- \) field [10]. For \( \psi^2, \eta^2 \), we have the same result.

When \( m \neq 0 \), the dimensional continuation of the action (8) is

\[
S = -\frac{1}{2\pi \alpha'} \int d^{2+\epsilon} \sigma e^{i \phi} \left( \partial_+ x^- \partial_- x^+ + \frac{1}{2} \partial_+ x^i \partial_- x^i + i \psi^1 \partial_+ \eta^1 + i \psi^2 \partial_- \eta^2 + \mathcal{L}_{bc} \right.
\]

\[
-\frac{1}{2} m^2 x^i \partial_+ x^i \partial_- x^+ + i m \psi^1 \psi^2 + i m \eta^1 \eta^2 \partial_+ x^i \partial_- x^i \bigg)
\]

6
where we perform all the index algebra in two dimensions and continue the volume element to $2 + \epsilon$ dimensions [12, 13]. The trace of the stress-energy tensor can be obtained by variation of the effective action with respect to $\phi$ [12].

If $m$ is small, we can calculate the $\phi$-dependence of the effective action by perturbation theory. Since in the nonlinear sigma model the quadratic divergences like the contact term $\delta(0)$ can be omitted in dimensional regularization [13], in the following calculation we only consider the logarithmical divergences.\(^3\) When $\epsilon$ is small, the action (13) can be written as

$$S = -\frac{1}{2\pi\alpha'} \int d^{2+\epsilon} \sigma \left( \partial_+ x^- \partial_- x^+ + \frac{1}{2} \partial_+ x^i \partial_- x^i + i \psi^1 \partial_+ \eta^1 + i \psi^2 \partial_- \eta^2 + \mathcal{L}_{bc} 
+ \partial_+ x^- \partial_- x^+ \epsilon \phi + \frac{1}{2} \partial_+ x^i \partial_- x^i \epsilon \phi + i \psi^1 \partial_+ \eta^1 \epsilon \phi + i \psi^2 \partial_- \eta^2 \epsilon \phi + \mathcal{L}_{bc} \epsilon \phi 
- \frac{1}{2} m^2 x^2 \partial_+ x^+ \partial_- x^- - \frac{1}{2} m^2 x^2 \partial_+ x^+ \partial_- x^- \epsilon \phi + i m \psi^1 \psi^2 + i m \psi^1 \psi^2 \epsilon \phi 
+ i m \eta^1 \eta^2 \partial_+ x^+ \partial_- x^- + i m \eta^1 \eta^2 \partial_+ x^+ \partial_- x^- \epsilon \phi \right)$$

(14)

where the quadratic terms are usual kinetic terms, the cubic and quartic terms are interacting ones. From (14), we can read off the propagators as

$$\begin{align*}
\langle x^- (\sigma_1) x^+ (\sigma_2) \rangle & = 2\pi\alpha' \int \frac{d^{2+\epsilon} k}{(2\pi)^2 k^2} e^{ik \cdot (\sigma_1 - \sigma_2)} \\
\langle x^i (\sigma_1) x^j (\sigma_2) \rangle & = 2\pi\alpha' \delta^{ij} \int \frac{d^{2+\epsilon} k}{(2\pi)^2 k^2} e^{ik \cdot (\sigma_1 - \sigma_2)} \\
\langle \psi^1_\alpha (\sigma_1) \eta^1_\beta (\sigma_2) \rangle & = 2\pi\alpha' \delta^{\alpha\beta} \int \frac{d^{2+\epsilon} k}{(2\pi)^2 i k^+} e^{ik \cdot (\sigma_1 - \sigma_2)} \\
\langle \psi^2_\alpha (\sigma_1) \eta^2_\beta (\sigma_2) \rangle & = 2\pi\alpha' \delta^{\alpha\beta} \int \frac{d^{2+\epsilon} k}{(2\pi)^2 i k_-} e^{ik \cdot (\sigma_1 - \sigma_2)}
\end{align*}$$

(15)

where $k_\pm = k_0 \mp i k_1$. For $x^+$ and $x^-$, only the propagator $\langle x^+ x^- \rangle$ is nonzero, while averages of any number of $x^+$ without $x^-$ vanish.

We first consider the bosonic $\phi$-dependent effective action, that is, we switch off the fermionic fields. From (14), we find that the first logarithmically divergent term $B_1$ is

\(^3\) Because of the equal total numbers of bosons and fermions, the trivial quadratic divergences cancel.
given by

\[
B_1 = \frac{1}{2\pi\alpha'} \int d^{2+}\sigma \left( -\frac{1}{2} m^2 \sum x_i^i \partial_+ x^+ \partial_- x^- \epsilon \phi \right) \\
= -\frac{2m^2}{\pi} \int d^2 \sigma \phi \partial_+ x^+ \partial_- x^+ 
\]

which is obtained by the contraction \( < x^i x^i > \) and described by Fig.1. In (16), the limit \( \epsilon \to 0 \) has been taken.

FIG. 1: The one-loop logarithmically divergent term coming from \( < x^i x^i > \).

The second one comes from the contraction between the \( \phi \)-dependent kinetic term and the quartic term

\[
B_2 = 2 \left( \frac{2}{(2\pi\alpha')^2} \int d^{2+} \sigma_1 d^{2+} \sigma_2 \frac{1}{2} \left( \sum x_i^i (\sigma_1) \partial_- x^+(\sigma_1) \epsilon \phi(\sigma_1) [-m^2 x^j(\sigma_2) x^j(\sigma_2)] \right) \cdot \frac{1}{2} \partial_+ x^+(\sigma_2) \partial_- x^+(\sigma_2) \right) \\
= \frac{2m^2}{\pi} \int d^2 \sigma \phi \partial_+ x^+ \partial_- x^+. 
\]

The factor 2 in the first line comes from two sorts of the contraction between \( \partial_+ x^i \partial_- x^i \) and \( x^j x^j \). The Feynman diagram for \( B_2 \) is shown in Fig.2.

The third logarithmically divergent term \( B_3 \) is

\[
B_3 = \frac{2}{(2\pi\alpha')^2} \int d^{2+} \sigma_1 d^{2+} \sigma_2 \left( -\frac{1}{2} m^2 \sum x_i^i (\sigma_1) \partial_+ x^+(\sigma_1) \partial_- x^+(\sigma_1) \right) \sum x_i^i (\sigma_2) \partial_- x^+(\sigma_2) \epsilon \phi \\
= \frac{4m^2}{\pi} \int d^2 \sigma \phi \partial_+ x^+ \partial_- x^+. 
\]
 FIG. 2: The logarithmic divergence coming from the contraction between the \( \phi \)-dependent kinetic term and the quartic term, the cross represents the insertion of \( \epsilon \phi \).

which is obtained by the contraction between the part of \( B_1 \) and \( \partial_+ x^- \partial_- x^+ \epsilon \phi \), and can be described similarly by Fig.1. We can check from the partition function and the action (14) that there are some higher-loop divergent contributions to the total bosonic \( \phi \)-dependent effective action given by

\[
B_\phi = \ln W - \frac{1}{2\pi \alpha'} \int d^{2+\epsilon} \sigma \left( -\frac{1}{2} m^2 x^i \partial_+ x^+ \partial_- x^+ \right)
\]

with

\[
W = \lim_{\epsilon \to 0} \frac{T}{\epsilon} \left\{ \exp \left[ \frac{1}{2\pi \alpha'} \int d^{2+\epsilon} \sigma \left( \partial_+ x^- \partial_- x^+ \epsilon \phi + \frac{1}{2} \partial_+ x^i \partial_- x^i \epsilon \phi - \frac{1}{2} m^2 x^i \partial_+ x^+ \partial_- x^+ \right) \right] \right\}
\]

\[
= \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{1}{2\pi \alpha'} \int d^{2+\epsilon} \sigma \left( -\frac{1}{2} m^2 x^i \partial_+ x^+ \partial_- x^+ \epsilon \phi \right) \right]^n \sum_{k,l,s=0}^{\infty} \frac{1}{k! l! s!} \left( \begin{array}{c} l \\ k \end{array} \right) \left( \begin{array}{c} l-k \\ s \end{array} \right)
\]

\[
\left[ \frac{2}{(2\pi \alpha')^2} \int d^{2+\epsilon} \sigma_1 \sigma_2 \frac{1}{2} \partial_+ x^i (\sigma_1) \partial_- x^j (\sigma_2) \epsilon \phi (\sigma_1) \left[ -m^2 x^j (\sigma_2) x^i (\sigma_2 \partial_+ x^+ (\sigma_2) \right] \right]^k
\]

\[
\cdot \left[ \frac{2}{(2\pi \alpha')^2} \int d^{2+\epsilon} \sigma_1 \sigma_2 \left( -\frac{1}{2} m^2 x^i \partial_+ x^+ (\sigma_1) \partial_- x^j (\sigma_1) \partial_+ x^+ (\sigma_2) \partial_- x^j (\sigma_2) \right) \right]^s
\]

\[
\cdot \left[ \frac{1}{2\pi \alpha'} \int d^{2+\epsilon} \sigma \left( -\frac{1}{2} m^2 x^i \partial_+ x^+ \partial_- x^+ \right) \right]^{l-k-s}
\]

and the diagramatic illustration for the total bosonic \( \phi \)-dependent effective action is drawn in Fig.3. Fig.3 shows that the higher-loop diagrams can be decomposed into the products of one-loop diagrams, which will be a crucial feature when we discuss the mixing of the vertex operators on pp-wave RR background.
Inserting (16), (17) and (18) into (20), \( W \) is reduced to

\[
W = e^{-\frac{2m^2}{\pi}} \int d^2 \sigma \partial_+ x^+ \partial_+ x^- \cdot e^{\frac{2m^2}{\pi}} \int d^2 \sigma \partial_+ x^+ \partial_- x^- \cdot e^{\frac{4m^2}{\pi}} \int d^2 \sigma \partial_+ x^+ \partial_- x^- \\
\cdot e^{\frac{1}{2\pi \alpha'}} \int d^{2+} \sigma (-\frac{1}{2} m^2 i^2 \partial_+ x^+ \partial_- x^-)
\]

\[
= e^{\frac{4m^2}{\pi}} \int d^2 \sigma \partial_+ x^+ \partial_- x^- \cdot e^{\frac{1}{2\pi \alpha'}} \int d^{2+} \sigma (-\frac{1}{2} m^2 i^2 \partial_+ x^+ \partial_- x^-).
\]

(21)

Plugging (21) into (19), we have

\[
B_\phi = \frac{4m^2}{\pi} \int d^2 \sigma \partial_+ x^+ \partial_- x^+
\]

(22)

which indicates that the total bosonic \( \phi \)-dependent effective action does not vanish on pp-wave background. In other words, the \( \beta \)-function \( \beta_G \) is not zero [14]. It has been shown from other approaches that the \( D = 26 \) bosonic string on pp-wave defines a good string background [15] only when \( \sum A_i = 0 \), but in our case it is \( 8m^2 \).

We consider the fermionic contribution to the \( \phi \)-dependent effective action. The action (14) shows that there are two logarithmically divergent terms, the first is

\[
F_1 = \frac{2}{(2\pi \alpha')^2} \int d^{2+} \sigma_1 d^{2+} \sigma_2 \text{im} \psi^1(\sigma_1)\psi^2(\sigma_1)\epsilon^1(\sigma_1)\eta^2(\sigma_2)\\
\cdot \partial_+ x^+(\sigma_2)\partial_- x^-(\sigma_2)
\]

\[
= \frac{4m^2}{\pi} \int d^2 \sigma \partial_+ x^+ \partial_- x^+
\]

(23)

where the factor 2 in the first line comes from the contraction between \( \text{im} \psi^1 \psi^2 \) and \( \text{im} \eta^1 \eta^2 \epsilon \phi \). The Feynman diagram for \( F_1 \) is shown by Fig.4.
The second logarithmically divergent term is given by

\[
F_2 = \frac{2}{(2\pi\alpha')^3} \int d^{2+\epsilon} \sigma_1 d^{2+\epsilon} \sigma_2 d^{2+\epsilon} \sigma_3 \, i \psi^1(\sigma_1) \partial_+ \eta^1(\sigma_1) \epsilon \phi(\sigma_1) \text{Im} \psi^2(\sigma_2) \psi^2(\sigma_2) \text{Im} \eta^1(\sigma_3) \eta^1(\sigma_3) \\
\cdot \partial_+ x^+(\sigma_1) \partial_- x^+(\sigma_2) \\
= -\frac{4m^2}{\pi} \int d^2 \sigma \phi \partial_+ x^+ \partial_- x^+ 
\]

where the factor 2 in the first line is obtained by replacing \( \psi^1 \partial_+ \eta^1 \) with \( \psi^2 \partial_+ \eta^2 \). The interesting feature of \( F_2 \) is that it can be described by the one-loop triangle diagram shown in Fig.5.

The third one can be obtained from the contraction between the part of \( F_1 \) and \( \eta^1 \eta^2 \partial_+ x^+ \partial_- x^+ \).
\[ \partial_+ x^- \partial_- x^+ \epsilon \phi \]

\[ F_3 = \frac{2}{(2 \pi \alpha')^3} \int d^2 \epsilon \sigma_1 d^2 + \epsilon \sigma_2 d^2 + \epsilon \sigma_3 \text{im} \psi^1(\sigma_1) \psi^2(\sigma_1) \text{im} \eta^1(\sigma_2) \eta^2(\sigma_2) \]

\[ \cdot \partial_+ x^+ (\sigma_2) \partial_- x^+ (\sigma_2) \partial_+ x^- (\sigma_3) \partial_- x^+ (\sigma_3) \epsilon \phi \]

\[ = -\frac{4m^2}{\pi} \int d^2 \sigma \phi \partial_+ x^+ \partial_- x^+ \tag{25} \]

which is shown similarly as Fig.4.

The total fermionic \( \phi \)-dependent effective action can be obtained the same way as in the bosonic case with the similar structure of Fig.3, where Fig.4 and Fig.5 instead of Fig.1 and Fig.2 are used as the “fundamental blocks”.

Eqs. (16), (17), (18), (23), (24) and (25) show that the \( B_1 + B_2 + B_3 + F_1 + F_2 + F_2 = 0 \), which indicates that the total \( \phi \)-dependent effective action vanishes. We therefore conclude that the type IIB GS superstring in semi-light cone gauge on the pp-wave RR background is described by a superconformal field theory.

Here we emphasize that there are some higher-loop diagrams, but they can be decomposed as the products of the one-loop diagrams. This is because the quartic interaction term takes the special form \( x^2 \partial_+ x^+ \partial_- x^+ \) and \( x^+ \) can only contract with \( x^- \), while there is no interacting term quadratic in \( x^- \).

Let us study the quantum counterterm in (8) instead of the \( \phi \)-dependent effective action. The only logarithmic divergence in bosonic part is

\[ b_\epsilon = \frac{1}{2 \pi \alpha'} \int d^2 + \epsilon \sigma \left( -\frac{1}{2} m^2 \bar{x}^i \partial_+ x^+ \partial_- x^+ \right) \]

\[ = -\frac{2m^2}{\pi \epsilon} \int d^2 \sigma \partial_+ x^+ \partial_- x^+ \tag{26} \]

which can be illustrated by Fig.1.

The logarithmically divergent term in fermionic action is

\[ f_\epsilon = \frac{1}{(2 \pi \alpha')^2} \int d^2 + \epsilon \sigma_1 d^2 + \epsilon \sigma_2 \text{im} \psi^1(\sigma_1) \psi^2(\sigma_1) \text{im} \eta^1(\sigma_2) \eta^2(\sigma_2) \]

\[ \cdot \partial_+ x^+ (\sigma_2) \partial_- x^+ (\sigma_2) \]

\[ = \frac{2m^2}{\pi \epsilon} \int d^2 \sigma \partial_+ x^+ \partial_- x^+ \tag{27} \]
which can be shown by Fig.4, but the cross represents the factor $im$.

Like the $\phi$-dependent effective action, the counterterms in bosonic and fermionic parts are not zero respectively, but the total counterterm for the action (8) vanishes: $b_\epsilon + f_\epsilon = 0$.

**IV. POLYGON DIVERGENT STRUCTURE IN GROUP COORDINATES**

The coordinates $(x^\pm, x^i)$ are called harmonic coordinates and are physically convenient in calculating the string spectrum. To display the symmetries of the geometry and compute scattering amplitudes [17], the so called “group coordinates” are more suitable [7, 8]. In group coordinates, the metric takes the form [7, 8]

$$ds^2 = 2dy^+ dy^- + e^{\pm 2imy^+} dy^2$$  \hspace{1cm} (28)

where the group coordinates are related to the harmonic coordinates by

$$x^+ = y^+,$$
$$x^- = y^- \mp \frac{im}{2} e^{\pm 2imy^+} y^i y^i,$$
$$x^i = e^{\pm imy^+} y^i.$$  \hspace{1cm} (29)

In [7, 8], no criterion is given to determine the sign $\pm$. Here we assign $\pm$ in the way that preserves conformal invariance, and the metric is then

$$ds^2 = 2dy^+ dy^- + e^{2imy^+} \sum_{i=1}^{4} dy^2 + e^{-2imy^+} \sum_{i'=5}^{8} dy^{i'2}$$  \hspace{1cm} (30)

with

$$x^+ = y^+,$$
$$x^- = y^- - \frac{im}{2} e^{2imy^+} \sum_{i=1}^{4} y^i y^i + \frac{im}{2} e^{-2imy^+} \sum_{i'=5}^{8} y^{i'} y^{i'},$$
$$x^i = e^{imy^+} y^i,$$
$$x^{i'} = e^{-imy^+} y^{i'}$$  \hspace{1cm} (31)

where $i = 1, \cdots, 4$ and $i' = 5, \cdots, 8.$
Notice that our choice of signs has the property of preserving the volume of the coordinate system. The Jacobian $J(\frac{\partial x^\mu}{\partial y^\nu})$ gives the measure factor $e^{i(4-4)m y^+} = 1$. This assignment of $\pm$ also makes manifest the $SO(4) \times SO(4)$ symmetry.

Though the coordinates and metric are complex, the vertex operators in group coordinates are quite simple [7, 8]. The massless scalar field solution of the Laplacian [18] in harmonic coordinates is

$$V_T(x) = e^{ik_+x^+ + ik_-x^-} \prod_{j=1}^{8} e^{-\alpha_j x_j^2/4} H_{n_j} \left( \sqrt{\alpha_j} x_j \right)$$  \hspace{1cm} (32)

with $k_+ = \sum_j \alpha_j(n_j + \frac{1}{2})$, and $\alpha_j = \pm mk_-$. For $\alpha_j > 0$, the $H_{n_j}$ are Hermite polynomials. In group coordinates, the solution is simplified to [7, 8]

$$V_T(y) = e^{i \int_0^{y^+} dy^+ k_+(y^+) + ik_-y^- + ik_+ y_1}$$  \hspace{1cm} (33)

where $k_-, k_i$ are constants and play the role of components of the momentum.4

In terms of group coordinates, the partition function (7) turns into

$$Z = \int D\gamma D\psi D\eta D\psi^2 D\eta^2 DbDc e^{-S}$$  \hspace{1cm} (34)

with

$$S = -\frac{1}{2\pi \alpha'} \int d^2 \sigma \left( -y^- \partial_+ \partial_- y^+ + \frac{1}{2} \sum_{i=1}^{4} \partial_+ y_i^i \partial_- y_i^i + \frac{1}{2} \sum_{i'=5}^{8} \partial_+ y_i^i \partial_- y_i^{i'} + i \psi^1 \partial_+ \eta^1 \\
+ i \psi^2 \partial_- \eta^2 + \mathcal{L}_{bc} + \frac{1}{2} (e^{2imy^+} - 1) \sum_{i=1}^{4} \partial_+ y_i^i \partial_- y_i^i + \frac{1}{2} (e^{-2imy^+} - 1) \sum_{i'=5}^{8} \partial_+ y_i^{i'} \partial_- y_i^{i'} \\
+ i m \psi^1 \psi^2 + i m \eta^1 \eta^2 \partial_+ y^+ \partial_- y^+ \right)$$  \hspace{1cm} (35)

Here the fermionic part is the same as that in harmonic coordinates, and the fermionic quantum counterterm does not change in group coordinates, so in group coordinates we only need to consider the bosonic quantum counterterm.

---

4 After some modification, the solution can be identified as tachyon vertex operator in string theory [8].
The bosonic interaction is now changed into \( \frac{1}{2}(e^{2imy^+} - 1) \sum_{i=1}^4 \partial_+ y^i \partial_- y^i = -\frac{1}{2}(e^{2imy^+} - 1) \sum_{i=1}^4 \partial_a y^i \partial_a y^i \), and \(-\frac{1}{2}(e^{-2imy^+} - 1) \sum_{i'=5}^8 \partial_a y^{i'} \partial_a y^{i'} \). In group coordinates, the vertex operators and calculation of scattering amplitudes are simple, but the action (35) and calculation of string spectrum are more complicated than in harmonic coordinates.

Though the \( y^i \) interacts with \( y^{i'} \) through \( y^+ \) and \( y^- \), when we calculate the bosonic quantum counterterm and \( \phi \)-dependent effective action, \( y^i \) still decouples effectively from \( y^{i'} \). Thus in the following calculation, we first consider \( y^i \), then add the contribution from \( y^{i'} \) by replacing \( m \) by \(-m \). Since quadratic divergences like the contact term \( \delta^2 (0) \) can be omitted in dimensional regularization, the contraction between \( y^i \) and \( y^i \) does not create any counterterm. To extract the logarithmic divergence, we expand

\[
e^{2imy^+}(\sigma_2) = e^{2imy^+}(\sigma_1) + (\sigma_2 - \sigma_1)_a \cdot \partial_a e^{2imy^+}(\sigma_1) + \frac{1}{2!} (\sigma_2 - \sigma_1)_a (\sigma_2 - \sigma_1)_b \partial_a \partial_b e^{2imy^+}(\sigma_1) + \cdots
\]

where the higher-order terms only contribute to the finite term, so they can be omitted.

The first logarithmic divergence comes from the contraction

\[
\mathcal{A}_2 = \frac{1}{2!} \cdot (-1)^2 \cdot (\frac{1}{2})^2 \cdot 2 \cdot \frac{1}{(2\pi\alpha')^2} \int d^{2+\epsilon} \sigma_1 d^{2+\epsilon} \sigma_2 (e^{2imy^+} - 1)(\sigma_1) \cdot (e^{2imy^+} - 1)(\sigma_2) \cdot i \partial_a y^i(\sigma_1) \partial_a y^i(\sigma_2) \partial_b y^j(\sigma_2) \partial_b y^j(\sigma_2)
\]

\[
= -\frac{1}{4\pi\epsilon} \int d^2 \sigma (e^{2imy^+} - 1) \partial_+ \partial_- e^{2imy^+}
\]

where the factor 2 is from two sorts of contraction between \( \partial_a y^i \partial_a y^i \) and \( \partial_b y^j \partial_b y^j \), and the finite terms have been omitted. The Feynman diagram for \( \mathcal{A}_2 \) is drawn in Fig.6.

Besides \( \mathcal{A}_2 \), there are polygon contributions to the logarithmic divergences. The \( n \)-gon logarithmic divergence \( \mathcal{A}_n \) \( (n = 2, 3, 4, \ldots) \) is

\[
\mathcal{A}_n = \frac{1}{n!} \cdot (-1)^n \cdot (\frac{1}{2})^n \cdot 2^{n-1} \cdot (n-1)! \cdot \frac{1}{(2\pi\alpha')^n} \int d^{2+\epsilon} \sigma_1 \cdots d^{2+\epsilon} \sigma_n (e^{2imy^+} - 1)(\sigma_1) \cdots (e^{2imy^+} - 1)(\sigma_n) \cdot \partial_{a_1} y^{i_1}(\sigma_1) \partial_{a_2} y^{i_2}(\sigma_2) \partial_{a_3} y^{i_3}(\sigma_3) \ldots \partial_{a_n} y^{i_n}(\sigma_n) \partial_{a_n} y^{i_n}(\sigma_n)
\]
\[
\frac{2(-1)^n}{(2\pi)^{2n}} \int d^{2+\epsilon} \sigma_1 \ldots d^{2+\epsilon} \sigma_n d^{2+\epsilon} k_1 \ldots d^{2+\epsilon} k_n \frac{(-1)a_1(k_1)}{k_1^2} \frac{a_2(k_2)}{k_2^2} \frac{a_3(k_3)}{k_3^2} \ldots \frac{a_{n-1}(k_{n-1})}{k_{n-1}^2} \frac{a_n(k_n)}{k_n^2}
\]
\[
\cdot \frac{1}{2} (e^{2imy^+} - 1)^{n-1}(\sigma_1) \left\{ \frac{n}{2} \sum_{i=2}^{n} (\sigma_i - \sigma_1)^a (\sigma_i - \sigma_1)^b \right\} \partial_a \partial_b e^{2imy^+}(\sigma_1)
\]
\[
+ (e^{2imy^+} - 1)^{n-2}(\sigma_1) \left( \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} (\sigma_i - \sigma_1)^a (\sigma_j - \sigma_1)^b \right) \partial_a e^{2imy^+} \partial_b e^{2imy^+}(\sigma_1) \right\} \tag{38}
\]

where the quadratic divergence \( \sim \delta(0) \) and finite terms have been omitted. The Feynman diagram for this n-gon contribution is shown in Fig.7.

FIG. 7: The n(=6)-gon logarithmic divergence where the circles stand for the factor \( e^{2imy^+} - 1 \).

The terms like \( (\sigma_i - \sigma_1)^a (\sigma_j - \sigma_k)^b \) with \( j > k \geq i + 1 \) vanish after integration, so that (38) is reduced to

\[
\mathcal{A}_n = \frac{2(-1)^n}{(2\pi)^{2n}} \int d^{2+\epsilon} \sigma_1 \ldots d^{2+\epsilon} \sigma_n d^{2+\epsilon} k_1 \ldots d^{2+\epsilon} k_n \frac{(-1)a_1(k_1)}{k_1^2} \frac{a_2(k_2)}{k_2^2} \frac{a_3(k_3)}{k_3^2} \ldots \frac{a_{n-1}(k_{n-1})}{k_{n-1}^2} \frac{a_n(k_n)}{k_n^2}
\]
The terms $(\sigma_i - \sigma_1)^a (\sigma_{i+1} - \sigma_1)^b$ and $(\sigma_i - \sigma_1)^a (\sigma_i - \sigma_1)^b$ can be rewritten as

\[
(\sigma_i - \sigma_1)^a (\sigma_{i+1} - \sigma_1)^b = \sum_{i=2}^{n-1} (\sigma_i - \sigma_{i-1})^a (\sigma_i - \sigma_{i-1})^b + \sum_{i=2}^{n-1} (\sigma_i - \sigma_{i-1})^a (\sigma_{i+1} - \sigma_i)^b \\
+ \sum_{i=3}^n (\sigma_i - \sigma_{i-1})^a (\sigma_{i-1} - \sigma_{i-2})^b + \sum_{j \neq l-1,l,l+1} (\sigma_l - \sigma_{l-1})^a (\sigma_j - \sigma_{j-1})^b
\]

After integration, only the terms like $(\sigma_l - \sigma_{l-1})^a (\sigma_j - \sigma_{j-1})^b$ with $j = l-1, l, l+1$ are not zero, so we have

\[
\mathcal{A}_n = \frac{2(-1)^n}{(2\pi)^{2n}} \int d^2+\sigma_1 \cdots d^2+\sigma_n d^2+k_1 \cdots d^2+k_n \frac{(k_1)_{a_1} (k_2)_{a_2} (k_3)_{a_3}}{k_1^2 k_2^2 k_3^2} \cdot e^{ik_1 (\sigma_1 - \sigma_2)} e^{ik_2 (\sigma_2 - \sigma_3)} \cdots e^{ik_n (\sigma_{n-1} - \sigma_n)} e^{ik_n (\sigma_n - \sigma_1)}
\]

\[
\cdot \left\{ \frac{1}{2} (e^{2imy^+} - 1)^{n-1} (\sigma_1) \left[ (\sigma_n - \sigma_1)^a (\sigma_n - \sigma_1)^b + \sum_{i=2}^{n-1} (\sigma_i - \sigma_{i-1})^a (\sigma_i - \sigma_{i-1})^b \right] + \sum_{i=3}^n (\sigma_i - \sigma_{i-1})^a (\sigma_{i-1} - \sigma_{i-2})^b \right\} \partial_a \partial_b e^{2imy^+} (\sigma_1)
\]

\[
+ (e^{2imy^+} - 1)^{n-2} (\sigma_1) \left[ - (n-2) (\sigma_2 - \sigma_1)^a (\sigma_1 - \sigma_n)^b + \sum_{i=2}^{n-1} (n-1-i) \right]
\]
\[
\left( \sum_{l=2}^{i} (\sigma_l - \sigma_{l-1})^a (\sigma_l - \sigma_{l-1})^b + \sum_{l=2}^{i} (\sigma_l - \sigma_{l-1})^a (\sigma_{l+1} - \sigma_l)^b + \sum_{l=3}^{i} (\sigma_l - \sigma_{l-1})^a (\sigma_{l-1} - \sigma_{l-2})^b \right) \cdot \partial_a e^{2imy^+} \partial_b e^{2imy^+} (\sigma_1).
\]

To further calculate (41), we consider the typical term \( P_l \)

\[
P_l = \frac{2(-1)^n}{(2\pi)^{2n} \epsilon} \int d^{2+\epsilon} \sigma_1 \cdots d^{2+\epsilon} \sigma_n d^{2+\epsilon} k_1 \cdots d^{2+\epsilon} k_n \frac{(k_1)_{a_1}(k_1)_{a_2}(k_2)_{a_2}(k_2)_{a_3}}{k_1^2 k_2^2} \cdots \frac{(k_{n-1})_{a_{n-1}}(k_{n-1})_{a_n}(k_n)_{a_n}(k_n)_{a_1}}{k_{n-1}^2 k_n^2} \cdot e^{ik_1(\sigma_1 - \sigma_2)} e^{ik_2(\sigma_2 - \sigma_3)} \cdots e^{ik_{n-1}(\sigma_{n-1} - \sigma_n)} e^{ik_n(\sigma_n - \sigma_1)} (\sigma_l - \sigma_{l-1})^a (\sigma_{l+1} - \sigma_l)^b \cdot F_{ab}(\sigma) \]

\[
= \frac{2(-1)^n}{(2\pi)^{2n} \epsilon} \int d^{2+\epsilon} \sigma_1 \cdots d^{2+\epsilon} \sigma_n d^{2+\epsilon} k_1 \cdots d^{2+\epsilon} k_n F_{ab}(\sigma) \frac{(k_1)_{a_1}(k_1)_{a_2}(k_2)_{a_2}(k_2)_{a_3}}{k_1^2 k_2^2} \cdots \frac{(k_{n-1})_{a_{n-1}}(k_{n-1})_{a_n}(k_n)_{a_n}(k_n)_{a_1}}{k_{n-1}^2 k_n^2} \cdot e^{ik_1(\sigma_1 - \sigma_2)} e^{ik_2(\sigma_2 - \sigma_3)} \cdots e^{ik_{n-1}(\sigma_{n-1} - \sigma_n)} e^{ik_n(\sigma_n - \sigma_1)} \]

First integrating \( k_{l-1} \) and \( k_l \) by parts, then integrating out \( \sigma_2, \ldots, \sigma_n, k_1, k_{l-1}, k_{l+1}, \ldots, k_n \), and renaming \( \sigma_1 \rightarrow \sigma, k_l \rightarrow k \), we have

\[
P_l = -\frac{2(-1)^n}{(2\pi)^{2n} \epsilon} \int d^2 \sigma F_{ab}(\sigma) \delta^{ab}
\]

where \( P_l \) is independent of the index \( l \). Similarly, the other typical term \( Q_l \) is

\[
Q_l = \frac{2(-1)^n}{(2\pi)^{2n} \epsilon} \int d^{2+\epsilon} \sigma_1 \cdots d^{2+\epsilon} \sigma_n d^{2+\epsilon} k_1 \cdots d^{2+\epsilon} k_n \frac{(k_1)_{a_1}(k_1)_{a_2}(k_2)_{a_2}(k_2)_{a_3}}{k_1^2 k_2^2} \cdots \frac{(k_{n-1})_{a_{n-1}}(k_{n-1})_{a_n}(k_n)_{a_n}(k_n)_{a_1}}{k_{n-1}^2 k_n^2} \cdot e^{ik_1(\sigma_1 - \sigma_2)} e^{ik_2(\sigma_2 - \sigma_3)} \cdots e^{ik_{n-1}(\sigma_{n-1} - \sigma_n)} e^{ik_n(\sigma_n - \sigma_1)} (\sigma_l - \sigma_{l-1})^a (\sigma_l - \sigma_{l-1})^b \cdot F_{ab}(\sigma)
\]

\[
= \frac{(-1)^n}{n \pi \epsilon} \int d^2 \sigma F_{ab}(\sigma) \delta^{ab}.
\]
Inserting (43) and (44) into (41), we obtain

\[
\mathcal{A}_n = \frac{(-1)^n}{2n\pi\epsilon} \int d^2\sigma \frac{1}{2} \left[ 2 + \sum_{i=2}^{n-1} \left( 2(i-1) - (i-2) - (i-2) \right) \right] (e^{2i\epsilon \gamma^+} - 1)^{n-1} \partial_+ \partial_- e^{2i\epsilon \gamma^+} \\
+ \left[ (n-2) + \sum_{i=2}^{n-2} (n-i) \left( 2(i-1) - (i-1) - (i-2) \right) \right] (e^{2i\epsilon \gamma^+} - 1)^{n-2} \partial_+ e^{2i\epsilon \gamma^+} \partial_- e^{2i\epsilon \gamma^+} \\
= \frac{(-1)^{n-2}}{2\pi\epsilon} \frac{1}{n} \int d^2\sigma \left\{ \frac{(n-1)(n-2)}{2} (e^{2i\epsilon \gamma^+} - 1)^{n-2} \partial_+ e^{2i\epsilon \gamma^+} \partial_- e^{2i\epsilon \gamma^+} \\
+ (n-1)(e^{2i\epsilon \gamma^+} - 1)^{n-1} \partial_+ \partial_- e^{2i\epsilon \gamma^+} \right\}.
\]

(45)

After integration by parts, \( \mathcal{A}_n \) is reduced to

\[
\mathcal{A}_n = \frac{(-1)^{n-2}}{4\pi\epsilon} \int d^2\sigma (e^{2i\epsilon \gamma^+} - 1)^{n-1} \partial_+ \partial_- e^{2i\epsilon \gamma^+}
\]

(46)

which is the logarithmically divergent term from the n-gon.

The total logarithmically divergent part \( \mathcal{A} \) from \( y^i \) is

\[
\mathcal{A} = \sum_{n=2}^{\infty} \mathcal{A}_n = -\frac{m^2}{\pi\epsilon} \int d^2\sigma \partial_+ y^+ \partial_- y^+.
\]

(47)

Thus the total logarithmically divergent term from \( y^i \) and \( y^{i'} \) is obtained by adding the same term, replacing \( m \) by \( -m \)

\[
\mathcal{A}_T = -\frac{2m^2}{\pi\epsilon} \int d^2\sigma \partial_+ y^+ \partial_- y^+
\]

(48)

This is exactly the same as in harmonic coordinates (26), and can be illustrated by Fig.8.

![Fig. 8: The total logarithmic divergence from the sum of the n-gons.](image)

Now let us consider the \( \phi \)-dependent effective action in group coordinates. When \( \epsilon \) is small, the dimensional continuation of the action (35) is

\[
S = -\frac{1}{2\pi\alpha'} \int d^{2+\epsilon}\sigma \left( - y^+ \partial_+ \partial_- y^+ + \frac{1}{2} \sum_{i=1}^{4} \partial_+ y^i \partial_- y^i + \frac{1}{2} \sum_{i'=5}^{8} \partial_+ y^{i'} \partial_- y^{i'} \right)
\]

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where the relevant bosonic interactive Lagrangian is
\[ L_{bc} - y^- \partial_+ y^+ \epsilon \phi + \frac{1}{2} \sum_{i=1}^{4} \partial_+ y^i \partial_- y^i \epsilon \phi \]
\[ + \frac{1}{2} \sum_{i'=5}^{8} \partial_+ y^i \partial_- y^{i'} \epsilon \phi + i \psi^1 \partial_+ \eta^1 \epsilon \phi + i \psi^2 \partial_- \eta^2 \epsilon \phi + L_{bc} \epsilon \phi \]
\[ + \frac{1}{2} (e^{2imy^+} - 1) \sum_{i=1}^{4} \partial_+ y^i \partial_+ y^i + \frac{1}{2} (e^{2imy^+} - 1) \sum_{i'=5}^{8} \partial_+ y^i \partial_+ y^{i'} \epsilon \phi \]
\[ + \frac{1}{2} (e^{-2imy^+} - 1) \sum_{i'=5}^{8} \partial_+ y^{i'} \partial_+ y^i + \frac{1}{2} (e^{-2imy^+} - 1) \sum_{i'=5}^{8} \partial_+ y^{i'} \partial_+ y^i \epsilon \phi + i m \psi^1 \psi^2 \]
\[ + i m \psi^1 \psi^2 \epsilon \phi + i m \eta^1 \eta^2 \partial_+ y^+ \partial_- y^+ + i m \eta^1 \eta^2 \partial_+ y^+ \partial_- y^+ \epsilon \phi \] (49)
where the relevant bosonic interactive Lagrangian is \( \frac{1}{2} \partial_+ y^i \partial_- y^i \epsilon \phi + \frac{1}{2} (e^{2imy^+} - 1) \partial_+ y^i \partial_+ y^i + \frac{1}{2} (e^{2imy^+} - 1) \partial_+ y^i \partial_+ y^i \epsilon \phi \), similar for \( y^{i'} \).

Following the procedure of the calculation of the logarithmic divergence and exploiting (45), the first part of the \( \phi \)-dependent effective action from n-gon is
\[ L^\phi_{(n)} = \frac{(-1)^{n-2}}{2\pi} \int d^2 \sigma \left\{ \left( \frac{n-1}{2} \right)(n-2) \phi (e^{2imy^+} - 1)^{n-3} \partial_a e^{2imy^+} \partial_a e^{2imy^+} \right. \]
\[ \left. + (n-1) \phi (e^{2imy^+} - 1)^{n-2} \partial_a \partial_a e^{2imy^+} \right. \]
\[ \left. + \frac{(n-1)(n-2)}{2} \phi (e^{2imy^+} - 1)^{n-2} \partial_a e^{2imy^+} \partial_a e^{2imy^+} \right. \]
\[ \left. + (n-1) \phi (e^{2imy^+} - 1)^{n-1} \partial_a \partial_a e^{2imy^+} \right\} \] (50)
where the first two terms are obtained by replacing one point of Fig.6 \( (e^{2imy^+} - 1) \) by \( \epsilon \phi \), and the last two terms by replacing one point of Fig.6 \( (e^{2imy^+} - 1) \) by \( (e^{2imy^+} - 1) \epsilon \phi \).

In deriving the quantum counterterm (45), the only interacting term is \( -\frac{1}{2} (e^{2imy^+} - 1) \partial_a y^i \partial_a y^i \). However, when calculating the \( \phi \)-dependent effective Lagrangian from n-gon, there are three interacting terms \( -\frac{1}{2} (e^{2imy^+} - 1) \partial_a y^i \partial_a y^i \), \( -\frac{1}{2} \partial_a y^i \partial_a y^i \epsilon \phi \) and \( -\frac{1}{2} \partial_a y^i \partial_a y^i \epsilon \phi \), and the interacting terms \( -\frac{1}{2} \partial_a y^i \partial_a y^i \epsilon \phi \) and \( -\frac{1}{2} (e^{2imy^+} - 1) \partial_a y^i \partial_a y^i \epsilon \phi \) only appear once respectively because of the logarithmic divergence.

The first part of the \( \phi \)-dependent effective action from \( y^i \) is
\[ L^\phi = \sum_{n=2}^{\infty} L^\phi_{(n)} = -\frac{m i}{\pi} \int d^2 \sigma \phi \partial_+ \partial_- y^+ \] (51)
which is proportional to the equation of motion $\partial_+ \partial_- y^+ = 0$, and can be ignored in the process of renormalization.

Similar to $B_3$ and $F_3$ the second part of the $\phi$-dependent effective action which can be obtained by the contraction between $A$ and $\partial_+ y^- \partial_- y^+ \epsilon \phi$ is

$$\Delta \mathcal{L}^\phi = \frac{-2m^2}{2\pi \alpha'} \int d^2 \sigma \, \sigma_1 \partial_+ y^+(\sigma_1) \partial_- y^+ (\sigma_1) \partial_+ y^-(\sigma_2) \partial_- y^-(\sigma_2) \epsilon \phi \quad \Rightarrow \quad \Delta \mathcal{L} = \frac{-2m^2}{\pi} \int d^2 \sigma \, \phi \, \partial_+ y^+ \partial_- y^+.$$  
(52)

The total logarithmically divergent term from $y^i$ and $y^{i'}$ is obtained by adding the same term with $m$ replaced by $-m$

$$\mathcal{L}_T^\phi = -\frac{mi}{\pi} \int d^2 \sigma \, \phi \, \partial_+ y^+ \partial_- y^+ + \frac{2m^2}{\pi} \int d^2 \sigma \, \phi \, \partial_+ y^+ \partial_- y^+$$

$$-\frac{-mi}{\pi} \int d^2 \sigma \, \phi \, \partial_+ y^- \partial_- y^- + \frac{2m^2}{\pi} \int d^2 \sigma \, \phi \, \partial_+ y^- \partial_- y^-$$

$$= \frac{4m^2}{\pi} \int d^2 \sigma \, \phi \, \partial_+ y^+ \partial_- y^-$$  
(53)

This cancels with the fermionic $\phi$-dependent effective action, i.e., $\mathcal{L}_T^\phi + F_1 + F_2 + F_3 = 0$.

Thus the pp-wave GS superstring in group coordinates is a conformal field theory.

V. SUMMARY AND CONCLUSION

We have studied the pp-wave GS superstring in the semi-light cone gauge $g_{ab} = e^{2\phi} \delta_{ab}$, $\hat{\gamma}^+ \theta = 0$. The original GS superstring action with $SO(8)$ spinors has been recast into a simple form with two $SU(4)$ spinors. For the $m = 0$ case, the conformal anomaly from $SU(4)$ spinors has a coefficient $+8$ while the $x^\mu$ contribute $\frac{10}{2}$ and reparametrization ghost contribute $-\frac{26}{2}$, thus the total conformal anomaly in (8) vanishes.

For the $m \neq 0$ case, we have calculated the $\phi$-dependent bosonic effective action in harmonic coordinates. When we compute the fermionic $\phi$-dependent effective action, we have found a new triangular one-loop Feynman diagram. We have shown that the bosonic $\phi$-dependent effective action cancels with the fermionic one, which indicates that the pp-wave GS superstring is a exact conformal field theory. The quartic interacting term in
pp-wave background is \( x^2 \partial_+ x^+ \partial_- x^+ \) and \( x^+ \) can only contract with \( x^- \), and there is no interacting term quadratic in \( x^- \), thus the higher-loop diagrams can be decomposed as the products of the one-loop diagrams, which is a crucial feature when we discuss the mixing of the vertex operators.

We have introduced the group coordinates preserving \( SO(4) \times SO(4) \) and conformal symmetry. And we found that in group coordinates there are logarithmic divergences from n-gons whose divergent structure is more complicated than that in harmonic coordinates. After summing over all contributions from n-gons, we have shown that in group coordinates, the GS superstring on pp-wave RR background is also a conformal field theory.

In the above, to prove the conformal invariance of the pp-wave GS superstring action, we have shown how the \( \phi \)-dependent effective action vanishes. The authors of [5, 6, 19], only commented on the UV finiteness of the string action. When we construct the vertex operators on pp-wave, we will calculate one-particle irreducible diagrams of an insertion of a vertex operator. In this case, the method for calculating \( \phi \)-dependent effective action is more useful than that for calculating UV finiteness of the string action. Actually, our work has paved the way to construct the vertex operators and discuss the mixing of the vertex operators on the pp-wave.

**Acknowledgments**

We thank N. Berkovits and P. Mathieu for their helpful correspondence. This work was supported in part by NSERC.


