Hermitian Schmidt Decomposition and Twin Observables of Bipartite Mixed States

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Abstract. Study of mixed-state quantum correlations in terms of opposite-subsystem observables the measurement of one of which amounts to the same as that of the other, of so-called twins, is continued. Twin events that imply biorthogonal mixing of states, called ”strong twin events”, are studied. It is shown that for each mixed state there exists a Schmidt (super state vector) decomposition in terms of Hermitian operators, and that it can be the continuation of the mentioned biorthogonal mixing due to strong twins. The case of weak twins and nonhermitian Schmidt decomposition is also investigated. For separable states a necessary and sufficient condition for the existence of nontrivial twins is derived. Utilization of the Hermitian Schmidt decomposition for finding all twins is illustrated in full detail in the case of the two spin-one-half-particle states with maximally disordered subsystems (mixtures of Bell states). It is shown that only rank two mixtures have nontrivial twins.

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1. Introduction

Nowadays one distinguishes sharply between separable bipartite mixtures, which are quasiclassically correlated, and nonseparable ones, endowed with entanglement, a purely quantum property. (A good example of the latter is the case of correlated pure states.) The term "quantum correlations" is used in the generic sense comprising both quasiclassical correlations and entanglement.

It was claimed in a recent investigation \[1\] that the study of quantum correlations through twin observables, or shortly twins, is expected to be important for quantum communication and quantum information theories because it is believed to reveal some basic properties of the correlations. Twin observables are opposite-subsystem observables such that the (subsystem) measurement of one of them amounts \textit{ipso facto} to a measurement also of the other. Equivalently put, the subsystem measurement of a twin gives rise, on account of the quantum correlations, to an orthogonal decomposition of the state of the opposite subsystem.

In bipartite mixed states it is easier to \textit{relate twins to quantum correlations} than to entanglement (though the latter is more important). A quantitative measure of the former is, what is called, von Neumann’s mutual information

\[ C(\rho_{12}) \equiv S(\rho_{12}|\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2) - S(\rho_{12}), \]

expressed in terms of the so-called relative (or conditional) entropy, and, alternatively, in terms of the von Neumann entropies of the reduced statistical operators \( \rho_i, \ i = 1, 2 \) (subsystem states) and the von Neumann entropy of the statistical operator (bipartite state) \( \rho_{12} \) itself. I denote it by "\( C \)" because it was thus designated and called "logarithmic correlation" by Lindblad \[2\]. He also made use of the classical discrete mutual information \( I(A,B|\rho_{12}) \) of two arbitrary opposite subsystem observables \( A \) and \( B \) (with purely discrete spectra) which were assumed to be simultaneously measured in a quantum state \( \rho_{12} \). Then, utilizing the spectral forms and the ensuing probabilities

\[ A = \sum_k a_k P_1^{(k)} , \ k \neq k' \Rightarrow a_k \neq a_{k'} ; \quad B = \sum_l b_l Q_2^{(l)} , \ l \neq l' \Rightarrow b_l \neq b_{l'} ; \]

\[ p(k,l) \equiv \text{Tr}[\rho_{12}(P_1^{(k)} \otimes Q_2^{(l)})] , \quad p_k = \sum_l p(k,l) , \quad p_l = \sum_k p(k,l) ; \]

one defines the mutual information

\[ H(A : B|\rho_{12}) \equiv H(p(k,l)|p_k p_l) = H(p_k) + H(p_l) - H(p(k,l)) , \]

where \( H(p_k) \) e. g. is the so-called Gibbs-Boltzmann-Shannon entropy \( H(p_k) \equiv -\sum_k p_k \log p_k \), etc. Finally, Lindblad defined

\[ I(A,B|\rho_{12}) \equiv \sup H(A : B|\rho_{12}) , \]

where the \textit{supremum} was taken over all possible choices of the observables.
Lindblad showed that
\[ I(A, B | \rho_{12}) \leq C(\rho_{12}), \]
and that
\[ C(\rho_{12}) > 0 \quad \Rightarrow \quad I(A, B | \rho_{12}) > 0 \]
are always valid.

Thus, in all correlated states, i.e., in states in which \( C(\rho_{12}) > 0 \), or equivalently, \( \rho_{12} \neq \rho_1 \otimes \rho_2 \), one can understand part of the quantum correlations in terms of simultaneous subsystem measurements and their maximal mutual information.

Now, twins take up a very special position among the subsystem observables, because, if \( A \) and \( B \) are twins then
\[ I(A, B | \rho_{12}) = H(p_k), \]

since \( H(p_k) = H(p_l) = H(p(k, l)) \) due to \( p(k, l) = p_k \delta_{l,f(k)} \), where \( f(k) \) is a fixed bijection of the values of \( k \) onto those of \( l \). This is the case of perfect correlations, called ”lossless and noiseless information channel” in information theory.

The investigation of twins began with pure states \([3],[4]\) \( \rho_{12} \equiv |\Phi \rangle \langle \Phi| \). Surprisingly, a necessary and sufficient condition for a subsystem observable \( A \) to have a nontrivial twin was found in terms of properties of \( \rho_1 \) alone (local properties, cf (2a)). The opposite subsystem observable \( B \) that is the twin of \( A \) was, naturally, expressed in terms of global properties of \( |\Phi \rangle \). These were in a simple way given in terms of an operator (called correlation operator cf (11)) mapping the range of \( \rho_1 \) onto that of \( \rho_2 \). It was defined by \( |\Phi \rangle \). This operator is most practically handled in terms of the so-called Schmidt decomposition \([5]\) because it is precisely the (antiunitary) operator determining which characteristic vector of \( \rho_2 \) should appear in the same term as that of \( \rho_1 \) in the mentioned decomposition \([3]\) (cf (11)).

When twins were investigated in the mixed-state case \([1]\), the mentioned condition (cf (2a)) was found to be only necessary. Actually, a sufficient condition for \( A \) to have a twin expressed as a property of \( \rho_1 \) alone (a local property) cannot exist because for every \( \rho_1 \) there is the uncorrelated state \( \rho_{12} \equiv \rho_1 \otimes \rho_2 \), which does not have nontrivial twins.

Thus, global properties inherent in \( \rho_{12} \) have to be made use of at the mentioned very first stage of investigation of twins in the mixed state case. It is not easy to ”extract” a minimal global property of \( \rho_{12} \) that does the job (as in the pure state case).

It is a striking fact that the Schmidt decomposition of state vectors can be generalized to all mixed states. It is the basic aim of this article to investigate the relevance of this decomposition to twins. It is proved that the Schmidt decomposition of any bipartite mixed state \( \rho_{12} \) need not be expressed in terms of some very general linear operators, it can be given exclusively in terms of Hermitian operators, which can, in principle, be physically interpreted as observables (cf Theorem 2 and Corollary 1).

The concept of strong twins, which are closely connected with biorthogonal decomposition of \( \rho_{12} \) (cf Theorem 1), is introduced as a step towards the mentioned...
Hermitian Schmidt decomposition of $\rho_{12}$. Also nonhermitian Schmidt decomposition of mixed states is studied (cf Theorem 3).

For mixtures of Bell states a Hermitian Schmidt decomposition is given in the literature (though not treated as such). On this simple example the problem of finding all twins is easily solved (cf Theorems 5 and 6) in order to illustrate the relevance of the Hermitian Schmidt decomposition to extracting the sought for global property inherent in $\rho_{12}$.

In the mentioned simple case it turns out that rank four mixtures do not allow nontrivial twins. This is not surprising because it was shown in the preceding study [1] that singularity of $\rho_{12}$ is a necessary condition. But, surprisingly, also rank three mixtures are shown to have no nontrivial twins. This suggests that perhaps a stronger necessary condition, some kind of "sufficient singularity", for the existence of nontrivial twins could be found in the general case. This will be followed up elsewhere.

Relating twins to separability is fully clarified in this study in terms of a necessary and sufficient condition for the existence of nontrivial twins (cf Theorem 4). Relating twins to entanglement in the mixed-state case, and to the quantitative measures of entanglement like the so-called entanglement of creation and entanglement of distillation [6], or the quantum relative entropy [7] and others is an important open question that will be hopefully treated in further work.

The study of twins having been pursued in a number of mentioned articles is an ab ovo approach, which is already proved to be, in principle, relevant and perhaps even important to quantum information theory. It stands somewhat apart from the mainstream investigations. But it will be, hopefully, connected up with the latter as a result of further exploration.

2. Preliminary Relations

When a general, i.e., mixed or pure, bipartite state (statistical operator) $\rho_{12}$ is given, twins $(A_1, A_2)$ are algebraically defined as Hermitian (opposite subsystem) operators satisfying

$$A_1 \rho_{12} = A_2 \rho_{12}, \quad (1)$$

where $A_1$ is actually $(A_1 \otimes I_2)$, $I_2$ being the identity operator for the second subsystem, etc. It was shown [1] that (1) implies

$$[A_1, \rho_1] = 0, \quad [A_2, \rho_2] = 0 \quad (2a, b)$$

for the subsystem states (the reduced statistical operators). (The symbols $\text{Tr}_i$, $i = 1, 2$, denote the partial traces.) Relation (2a) is the mentioned local necessary condition on $A_1$ to have a twin.

If $P_1$ is a first-subsystem projector, one can decompose the statistical operator:

$$\rho_{12} = P_1 \rho_{12} + P_{1}^{\perp} \rho_{12}, \quad (3)$$
where $P_1^\perp$ is the orthocomplementary projector of $P_1$. Let $(P_1, P_2)$ be a pair of nontrivial twin events (twin projectors) for $\rho_{12}$. In general, the terms on the RHS are not even Hermitian. First, we are going to investigate the more important case when (3) is a mixture of states.

3. Strong twin projectors and biorthogonal mixtures

Let $(P_1, P_2)$ be a pair of nontrivial twin projectors for a composite-system statistical operator $\rho_{12}$.

Remark 1. Evidently, either both terms on the RHS of (3) are Hermitian or none of them is. They are Hermitian if and only if the projector $P_1$ (or equivalently, $P_1^\perp$) commutes with $\rho_{12}$:

$$[P_i, \rho_{12}] = 0, \quad i = 1, 2,$$

(4)

(any one of the equalities implies the other), as seen by adjoining the terms in (3).

Hermiticity of the terms in (3) implies that they are statistical operators (up to normalization constants), i.e., that (3) is a mixture. Namely, if (4) is valid, then idempotency leads to $P_1\rho_{12} = P_1\rho_{12}P_1$, which is evidently a positive operator. Since

$$\text{Tr}P_1\rho_{12}P_1 \leq \text{Tr}\rho_{12} = 1,$$

the operator has a finite trace.

Definition 1. Nontrivial twin events (projectors) we call either strong twin events (projectors), if they satisfy (4), or weak twin events (projectors) if (4) is not satisfied.

A strong twin event $P_1$ implies a mixture (3) of states that have a strong property called biorthogonality. To understand it, we first remind of (ordinary) orthogonality of states.

If $\rho'$ and $\rho''$ are statistical operators with $Q'$ and $Q''$ as their respective range projectors, then one has the known equivalences:

$$\rho'\rho'' = 0 \Leftrightarrow Q'Q'' = 0 \Leftrightarrow \mathcal{R}(\rho') \perp \mathcal{R}(\rho''),$$

(5)

where the last relation expresses orthogonality of the ranges.

Any of the three relations in (5) defines orthogonality of states.

Definition 2. If

$$\rho_{12} = w\rho_1' + (1 - w)\rho_2'', \quad 0 < w < 1,$$

(6)

is a mixture of states such that

$$\rho'_i\rho''_i = 0, \quad i = 1, 2,$$

(7)
where \( \rho_1' \equiv \text{Tr}_2 \rho_{12} \) etc. are the reduced statistical operators, then we say that (6) is a biorthogonal mixture.

To prove a close connection between strong twin events and biorthogonal mixtures, we need another known general property of composite-system statistical operators \( \rho_{12} \):

\[
\rho_{12} = Q_1 \rho_{12} = \rho_{12} Q_1 = Q_2 \rho_{12} = \rho_{12} Q_2,
\]

(8)

where \( Q_i \) is the range projector of the corresponding reduced statistical operator \( \rho_i \), \( i = 1, 2 \).

**Theorem 1.** If \( P_1 \) is a nontrivial twin event, (3) is a biorthogonal mixture if and only if \( P_1 \) is a strong twin event.

**Proof. Sufficiency.** If \( P_1 \) is a strong twin projector and (6) is obtained by rewriting (3), then \( \rho_{12}' = P_1 \rho_{12}' \) is valid, and this implies \( \rho_1' = P_1' \rho_1 \) for the reduced statistical operator, and, adjoining this, one arrives at \( \rho_1' = P_1' \rho_1 \). On the other hand, one has analogously \( \rho_{12}'' = P_1^\perp \rho_{12}'' \) implying \( \rho_1'' = P_1^\perp \rho_1'' \). Finally,

\[
\rho_1' \rho_1'' = (P_1' \rho_1')(P_1^\perp \rho_1'') = 0.
\]

The symmetrical argument holds for the second tensor factor.

**Necessity.** If (6) is a biorthogonal mixture, then we define \( P_i \equiv Q_i', i = 1, 2 \), i.e., we take the range projectors of the reduced statistical operators of \( \rho_{12}' \) as candidates for our twin projectors. On account of (8), we can write (6) as follows:

\[
\rho_{12} = wQ_1' Q_2' \rho_{12}' Q_1' Q_2' + (1 - w) Q_1'' Q_2'' \rho_{12}'' Q_1' Q_2' .
\]

Since in view of (5) biorthogonality (7) implies \( Q_i' Q_i'' = 0, i = 1, 2 \), it is now obvious that \( P_1 \) and \( P_2 \), multiplying from the left \( \rho_{12} \), give one and the same operator, i.e., that they are twins, and it is also obvious that they both give the same irrespectively if they multiply \( \rho_{12} \) from the left or from the right, i.e., that they are strong twin projectors.

\( \square \)

In view of (5), it is clear that biorthogonal decomposition of a statistical operator can be, in principle, continued: If, e.g., \( \rho_{12}' \) in a biorthogonal decomposition (6) is, in its turn, decomposed into biorthogonal statistical operators and replaced in (6), then any two of the new terms are biorthogonal etc.

An extreme case of a biorthogonal mixture is a separable one:

\[
\rho_{12} = \sum_k w_k \left( \rho_1^{(k)} \otimes \rho_2^{(k)} \right),
\]

(9)

where

\( \forall k: \ w_k > 0, \ \rho_i^{(k)} > 0, \ \text{Tr} \rho_i^{(k)} = 1, \ i = 1, 2, \ \sum_k w_k = 1 \)
"\(\rho > 0\)" denotes positivity of the operator. This decomposition cannot, of course, always be carried out, but examples are well known. For instance, if one performs ideal nonselective measurement of the \(z\)-component of spin of the first particle in a singlet two-particle state, one ends up with

\[
\rho_{12} \equiv (1/2)\left( |z+\rangle_1 \langle z+|_1 \otimes |z-\rangle_2 \langle z-|_2 + |z-\rangle_1 \langle z-|_1 \otimes |z+\rangle_2 \langle z+|_2 \right).
\]

This is obviously a biorthogonal separable mixture.

One wonders if, at the price of relaxing the requirement of statistical-operator terms as slightly as possible, there could exist a \textit{general} decomposition into \textit{uncorrelated} terms (like in (9)).

To find an affirmative answer, we take resort to the known case of general (entangled or disentangled) composite-system \textit{state vectors} and their Schmidt decompositions. Let us sum up the sufficiently detailed relevant information on this [3].

The Schmidt decomposition of an arbitrary pure state vector \(|\Phi\rangle_{12}\) of a composite system is expressed in terms of its canonical entities. They are the following:

(i) The reduced statistical operators (subsystem states) \(\rho_1 \equiv \text{Tr}_2 |\Phi\rangle_{12} \langle \Phi|_{12}\) and \(\rho_2\) (defined symmetrically) are well known.

(ii) The spectral forms of the reduced statistical operators are

\[
\rho_1 = \sum_i r_i |i\rangle_1 \langle i|_1, \quad \rho_2 = \sum_i r_i |i\rangle_2 \langle i|_2, \quad \forall i: \quad r_i > 0.
\]

(Note that the positive spectra -multiplicities included - are always equal.)

(iii) Finally, the mentioned expansion utilizes the (antiunitary) \textit{correlation operator} \(U_a\), which maps the range \(\mathcal{R}(\rho_1)\) onto the range \(\mathcal{R}(\rho_2)\). (Note that they are always equally dimensional in the pure state case). The correlation operator is determined by \(|\Phi\rangle_{12}\), and, in turn, in conjunction with \(\rho_1\), it determines \(|\Phi\rangle_{12}\).

The Schmidt decomposition reads:

\[
|\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 \otimes (U_a |i\rangle_1)_2.
\]

The normalized characteristic vectors \(|i\rangle_2\) in (10b) may (and need not) be chosen to be equal to \((U_a |i\rangle_1)_2\).

In case of a state vector \(|\Phi\rangle_{12}\), the characteristic relation (1) for twins reduces to

\[
A_1 |\Phi\rangle_{12} = A_2 |\Phi\rangle_{12}.
\]

The corresponding twin \(A_2\) then satisfies

\[
A_2 = U_a A_1 U_a^{-1} Q_2 + A_2 Q_2^+, \quad (13)
\]

where \(Q_2\) is the range projector of \(\rho_2\), and \(Q_2^+\), its orthocomplementary projector, projects onto the null space of \(\rho_2\).

One should note that, on account of the commutation (2b), both the range and the null space of \(\rho_2\) are invariant for \(A_2\). Further, the second term on the RHS of (13), or rather the restriction of \(A_2\) to the null space, which corresponds to it, is completely arbitrary and immaterial for the twin property (12), because it acts as zero on \(|\Phi\rangle_{12}\).

(Naturally, the symmetric claim holds true for \(A_1\) and \(\rho_1\).)
4. Hermitian Schmidt decomposition of bipartite statistical operators

It is well known that linear Hilbert-Schmidt operators $A$ acting in a Hilbert space, i.e., those with a finite Hilbert-Schmidt norm $\left(\text{Tr} A^\dagger A\right)^{1/2}$, form a Hilbert space in their turn. Writing the operator $A$ as a (Hilbert-Schmidt) supervector $|A\rangle$, the scalar product is

$$\langle A|B \rangle \equiv \text{Tr} A^\dagger B.$$ 

Since for every statistical operator $\rho$, one has $\text{Tr}\rho^2 \leq 1$, it is a Hilbert-Schmidt operator. Therefore, every statistical operator has a Schmidt decomposition.

The trouble is that the operators that take the place of the state-vector tensor factors in the terms of (11), which are the sought for generalizations of the statistical operators $\rho_i^{(k)}$, $i = 1, 2$ in (9), are in general linear operators. This might be a too wide generalization. One wonders if one could be confined to Hermitian operators.

When we view the operators as supervectors, then we must view adjoining of operators as an antiunitary operator the square of which is the identity operator, i.e., which is an involution. Hence, we denote adjoining by $V_1^{(a)} \otimes V_2^{(a)}$ for a composite system. Hermitian are the operators that are invariant under the action of this antiunitary involution.

Fortunately, the Schmidt decomposition can always be expressed in terms of Hermitian operators. We put this in a more precise and a more detailed way. But it is simpler to return to the Hilbert space of state vectors to perform some elaboration.

**Theorem 2.** Let $V_1^{(a)} \otimes V_2^{(a)}$ be a given antiunitary involution acting on composite-system state vectors. One has the equivalence:

$$\left(V_1^{(a)} \otimes V_2^{(a)}\right) |\Phi\rangle_{12} = |\Phi\rangle_{12} \iff \left[\rho_i, V_1^{(a)}\right] = 0, \quad i = 1, 2; \quad V_2^{(a)} U_a V_1^{(a)} = U_a,$$

where $\rho_i$, $U_a$ are the above mentioned canonical entities of $|\Phi\rangle_{12}$. (Note that in the last relation we, actually, have the restriction of $V_i^{(a)}$ to $\mathcal{R}(\rho_i)$.)

**Proof.** Let $|\Phi\rangle_{12}$ be invariant under the action of the antiunitary involution. Then

$$V_1^{(a)} \rho_1 V_1^{(a)} = V_1^{(a)} \left(\text{Tr}_2 |\Phi\rangle_{12} \langle \Phi|_{12} \right) V_1^{(a)} =$$

$$\text{Tr}_2 \left( V_1^{(a)} |\Phi\rangle_{12} \langle \Phi|_{12} V_1^{(a)} \right) = \text{Tr}_2 \left( V_1^{(a)} \left[ \left( V_1^{(a)} \otimes V_2^{(a)} \right) |\Phi\rangle_{12} \langle \Phi|_{12} \left( V_1^{(a)} \otimes V_2^{(a)} \right) \right] V_1^{(a)} \right) =$$

$$\text{Tr}_2 \left( V_2^{(a)} |\Phi\rangle_{12} \langle \Phi|_{12} V_2^{(a)} \right) = \text{Tr}_2 |\Phi\rangle_{12} \langle \Phi|_{12} = \rho_1,$$

and symmetrically for $\rho_2$. One has to note that an antiunitary involution equals its inverse and its adjoint. Further, use has been made of some known basic properties of partial traces (which are analogous to the well known ones for ordinary traces).

Commutation of $\rho_1$ with the antiunitary involution $V_1^{(a)}$ allows one to choose the characteristic basis $\{|i\}_1 : \forall i\}$ of the former spanning its range consisting of vectors invariant under the action of $V_1^{(a)}$ (cf [8]).
Now, let us take the Schmidt decomposition (11) in terms of such an invariant basis. Then
\[(V_1^{(a)} \otimes V_2^{(a)}) |\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 \otimes V_2^{(a)}(U_a |i\rangle_1)_{2}.\]

Since $|\Phi\rangle_{12}$ is assumed to be invariant, it follows that also
\[|\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 \otimes V_2^{(a)}(U_a |i\rangle_1)_{2}.\]

The second tensor factor in each term is uniquely determined by the LHS and the corresponding first tensor factor (as a partial scalar product, cf [3]). Comparison with (11) then shows that
\[\forall i:\quad V_2^{(a)}U_a |i\rangle_1 = U_a |i\rangle_1.\]

Since $|i\rangle_1 = V_1^{(a)} |i\rangle_1$, we further have
\[V_2^{(a)}U_a V_1^{(a)} = U_a\]
as claimed.

Conversely, if the main canonical entities are in the relation to the antiunitary involutions as stated in (14), then we can expand $|\Phi\rangle_{12}$ in a characteristic basis of $\rho_1$ spanning its range that is invariant under the antilinear operator. Then (11) immediately reveals that, as a consequence, $|\Phi\rangle_{12}$ is invariant under $V_1^{(a)} \otimes V_2^{(a)}$. \(\square\)

**Corollary 1.** Every composite-system statistical operator $\rho_{12}$ has, after normalization, a Hermitian Schmidt decomposition.

**Proof.** Since every $\rho_{12}$, being Hermitian, is invariant under the antiunitary involution $V_1^{(a)} \otimes V_2^{(a)}$, Theorem 2 immediately implies that $\rho_{12}$, upon super vector normalization, has a Schmidt decomposition in terms of Hermitian operators. \(\square\)

Returning to a biorthogonal mixture, one wonders if one can continue such a decomposition by writing each term in a Hermitian Schmidt decomposition in order to obtain the latter decomposition for the entire statistical operator. The answer is affirmative on account of the following:

Going back to (5), we can add a fourth equivalent property.

**Proposition 1.** Two statistical operators $\rho'$ and $\rho''$ are orthogonal if and only if they are orthogonal as Hilbert-Schmidt supervectors.

**Proof.** It is obvious that orthogonality (in the sense of (5)) implies Hilbert-Schmidt orthogonality. To see the converse implication, we make use of the fact that every statistical operator has a purely discrete spectrum [9], and we decompose the statistical
operators in terms of characteristic vectors corresponding to positive characteristic values:
\[
\langle \rho' \mid \rho'' \rangle = \text{Tr} \rho' \rho'' = \text{Tr} \sum_k r_k |k\rangle \langle k| \sum_j \bar{r}_j |j\rangle \langle j| = \\
\sum_k \sum_j r_k \bar{r}_j \langle j \mid |k\rangle^2.
\]
Hence,
\[
\langle \rho' \mid \rho'' \rangle = 0 \Rightarrow \rho' \rho'' = 0
\]
(cf the third relation in (5)).

If \((A_1, A_2)\) is a pair of twin observables, then the detectable parts \(A'_i, i = 1, 2\), have a common purely discrete spectrum \(\{a_n : \forall n\}\) (with, in general, different multiplicities), and the corresponding (detectable) characteristic projectors \(\{P^{(n)}_i : i = 1, 2 \ \forall n\}\), are also pairs of twins [1].

**Definition 3.** If all mentioned characteristic projector pairs \(\{P^{(n)}_1, P^{(n)}_2\}\) are strong twin projectors, then \((A_1, A_2)\) is a pair of strong twin observables. If some of the detectable characteristic twin projectors are strong and some weak, we say that we have partially strong (or, synonymously, partially weak) twin observables. If all the mentioned twin projectors are weak, then we have a weak pair of twin observables.

A pair \((A_1, A_2)\) of nontrivial twin observables for \(\rho_{12}\) is a pair of strong ones if and only if
\[
[A_i, \rho_{12}] = 0, \quad i = 1, 2
\]
is valid. This is so because commutation with all characteristic projectors is equivalent to commutation with the Hermitian operator itself, and, if \(P_1\) e. g. is a nondetectable characteristic projector of \(A_1\), then one has commutation because
\[
P_1 \rho_{12} = (P_1 Q_1^+) \rho_{12} = 0 = \rho_{12} (Q_1^+ P_1) = \rho_{12} P_1
\]
on account of (8).

Strong twin observables, by means of their strong characteristic twin projectors, lead to a generalization of (3):
\[
\rho_{12} = \sum_n P^{(n)}_1 \rho_{12} = \sum_n w_n \rho^{(n)}_{12}, \quad (16a)
\]
where
\[
\forall n : \quad w_n \equiv \text{Tr} \rho_{12} P^{(n)}_1, \quad \rho^{(n)}_{12} \equiv (w_n)^{-1} P^{(n)}_1 \rho_{12}. \quad (16b)
\]
Naturally, if \(P^{(n)}_1 \rho_{12} = 0\), then \(\rho^{(n)}_{12}\) is not defined. Any two terms in (16a) are biorthogonal.

Note that we utilize the entire characteristic projectors, which are the orthogonal sums of the detectable and the nondetectable parts: \(P^{(n)}_1 = (P'^{(n)}_1) \oplus (P'')^{(n)}_1\) paralleling
\[ H_1 = \mathcal{R}(\rho_1) \oplus \mathcal{R}^\perp(\rho_1) \] because \((P'_1)^{(n)}\rho_{12} = P'_1 \rho_{12}\).

**Proposition 2.** If 
\[ \rho^{(n)}_1 \equiv \text{Tr}_2 \rho^{(n)}_{12}, \]
and symmetrically for \(\rho^{(n)}_2\), are the reduced statistical operators of the terms in a biorthogonal mixture (16a), then
\[ P^{(n)}_i \rho^{(n)}_i = \rho^{(n)}_i, \quad i = 1, 2, \tag{17a} \]
or equivalently,
\[ \mathcal{R}(\rho^{(n)}_i) \subseteq \mathcal{R}(P^{(n)}_i), \quad i = 1, 2. \tag{17b} \]

**Proof.** On account of the definition of (16a), one has \(P^{(n)}_i \rho^{(n)}_{12} = \rho^{(n)}_{12}\). Taking the opposite-subsystem partial trace, one obtains \(P^{(n)}_i \rho^{(n)}_i = \rho^{(n)}_i\) \(i = 1, 2\). \(\square\)

**Corollary 2.** If the detectable part \(A'_1\) of a twin observable \(A_1\) has a nondegenerate characteristic value \(a_n\) corresponding to a strong characteristic twin projector \((P'_1)^{(n)} = |\psi^{(n)}_1\rangle \langle \psi^{(n)}_1|_1, \quad |\psi^{(n)}_1\rangle_1 \in \mathcal{R}(\rho_1)\), then the term in the biorthogonal mixture (16a) that corresponds to it has the form
\[ w_n |\psi^{(n)}_1\rangle \langle \psi^{(n)}_1|_1 \otimes \rho^{(n)}_2, \tag{18} \]
where \(\rho^{(n)}_2\) is a (second-subsystem) statistical operator and (18) is a term in a final Hermitian Schmidt decomposition of \(\rho_{12}\).

Any biorthogonal decomposition of a composite-system statistical operator \(\rho_{12}\) (into two or more terms) can be continued in each term separately into a Schmidt decomposition of \(\rho_{12}\) in terms of Hermitian operators.

The biorthogonal decomposition is an intermediate step. This is similar to the case when we can partially diagonalize the Hamiltonian of a quantum system (due to some symmetry e. g.). The diagonalization is then continued separately with each submatrix on the diagonal of the Hamiltonian.

The continuation from a biorthogonal mixture to a Hermitian Schmidt decomposition can always be performed, in principle, "by brute force": diagonalizing the reduced statistical superoperator \(\hat{\rho}_1\) of the normalized supervector \(|\rho_{12}\rangle\) (analogously as it is done for an ordinary state vector), and by finding an invariant basis for \(V_1^{(a)}\) in each characteristic subspace thus obtained [8].

### 5. Weak twins and nonhermitian Schmidt decomposition

For the sake of completeness it is desirable to investigate decomposition (3) also for a weak nontrivial twin projector \(P_1\). First, we take an analytical view of Theorem 1 to realize that the biorthogonality of the two terms in (3) is connected with the twin property (strong or weak), and the strong twin property corresponds to the hermiticity
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Remark 2. A decomposition
\[ \rho_{12} = A_{12} + B_{12} \]
of a composite-system statistical operator \( \rho_{12} \) into two linear operators is biorthogonal if there exist two opposite-subsystem projectors \((P_1, P_2)\) such that
\[ A_{12} = P_1 A_{12} = P_2 A_{12}, \quad 0 = P_1 B_{12} = P_2 B_{12}; \]
\[ 0 = P_1^{\perp} A_{12} = P_2^{\perp} A_{12}, \quad B_{12} = P_1^{\perp} B_{12} = P_2^{\perp} B_{12}. \]

It is clear from Theorem 1 that any biorthogonal mixture (of states) (6) satisfies the condition given in Remark 2. Having in mind (3), it is also evident that biorthogonality is equivalent to the existence of a pair of twin projectors (weak or strong). Finally, the strongness property of the twins is equivalent to the hermiticity of the terms in (3), which results in having statistical operator terms (and a mixture).

Theorem 3. If \((P_1, P_2)\) is a pair of weak twin projectors for a composite-system statistical operator \( \rho_{12} \), then the terms in (3) are super vectors, and replacing each by a (nonhermitian) Schmidt decomposition, one obtains a decomposition of the same kind for the entire statistical operator.

Proof. Since in
\[ 1 \geq \text{Tr} \rho_{12}^2 = \text{Tr} \rho_{12} P_1 \rho_{12} + \text{Tr} \rho_{12} P_1^{\perp} \rho_{12} \]
the terms are nonnegative (as traces of positive operators), the terms in (3) are Hilbert-Schmidt operators, i.e., super vectors. Suppose we have decomposed the first term in (3) in the Schmidt way:
\[ P_1 \rho_{12} = c \sum_i r_i^{1/2} A_1^{(i)} \otimes B_2^{(i)}, \]
where \( c \) is a normalization constant (because the statistical operator is not a super state vector unless it is a pure state). Since the LHS is invariant under \( P_1 \), so is each first-subsystem linear operator \( A_1^{(i)} \), because the second factors in the expansion have unique corresponding first factors. If we decompose also the second term in (3) in the Schmidt way
\[ P_1^{\perp} \rho_{12} = c' \sum_j r_j^{1/2} C_1^{(j)} \otimes D_2^{(j)}, \]
then, analogously, invariance of each factor \( C_1^{(j)} \) under \( P_1^{\perp} \) follows. This results in super vector orthogonality:
\[ \forall i, j : \quad \text{Tr} \left[ (A_1^{(i)})^\dagger C_1^{(j)} \right] = \text{Tr} \left[ (A_1^{(i)})^\dagger P_1 P_1^{\perp} C_1^{(j)} \right] = 0. \]
The symmetrical argument goes for the second factors and \( P_2 \). Thus, replacing both terms in (3) by their nonhermitian Schmidt decompositions, we have biorthogonality.
between any term of the first decomposition and any term of the second one. Therefore, we have a decomposition of the same kind of the entire $\rho_{12}$.

It is now clear that also in the case of weak twin projectors the decomposition (3) can be continued, but this time to a nonhermitian Schmidt decomposition.

A nonhermitian Schmidt decomposition need not be wild and far fetched from the physical point of view. Let me illustrate this by the obvious fact that a Schmidt decomposition of a state vector $|\Phi\rangle_{12}$

$$|\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 |i\rangle_2, \quad \forall i \neq i': \quad \langle i|_p |i'\rangle_p = 0, \quad p = 1, 2$$

immediately results in a nonhermitian Schmidt decomposition of the statistical operator $|\Phi\rangle_{12}\langle\Phi|_{12}$:

$$|\Phi\rangle_{12}\langle\Phi|_{12} = \sum_i \sum_{i'} r_i^{1/2} r_{{i'}}^{1/2} |i\rangle_1 \langle i|_1 \otimes |i'\rangle_2 \langle i'|_2.$$  

Finally, let us return to separable mixtures.

6. Nontrivial twin projectors for separable mixtures

Let (9) be a general separable mixture. Let us clarify under what conditions it has nontrivial twin events.

**Theorem 4.** A general separable mixture (9) has a nontrivial twin projector $P_1$ if and only if the set of all values of the index ”$k$” is the union of two nonoverlapping subsets, say, consisting of ”$k''$” values and of ”$k'''$” values respectively, and, when (9) is rewritten accordingly:

$$\rho_{12} = \sum_{k'} w_{k'} \rho^{(k')}_1 \otimes \rho^{(k')}_2 + \sum_{k''} w_{k''} \rho^{(k'')}_1 \otimes \rho^{(k'')}_2,$$

then one has biorthogonality between the two groups of terms:

$$\forall k', \forall k'': \quad \rho^{(k')}_i \rho^{(k'')}_i = 0, \quad i = 1, 2.$$  

Before we prove the theorem, we first prove subsidiary results.

**Lemma 1.** Let

$$\rho_{12} = \sum_m w_m |\Psi^{(m)}\rangle_{12}\langle\Psi^{(m)}|_{12}$$

be an arbitrary pure-state mixture. Then, a pair of subsystem observables $(A_1, A_2)$ are twins for $\rho_{12}$ if and only if they are twins for all pure-state terms.

**Proof.** Necessity follows from the general result that all twins of $\rho_{12}$ are also twins of all state vectors from the topological closure $\mathcal{R}(\rho_{12})$ of the range of $\rho_{12}$ (cf section 3, C1 in [1]). As well known, the vectors $\{|\Psi^{(m)}\rangle_{12}: \forall m\}$ span the mentioned subspace.
Sufficiency is obvious. □

\textit{Lemma 2.} Let 
\[ \rho_{12} = \sum_k w_k \rho_{12}^{(k)} \]
be an arbitrary mixture. The pair \((A_1, A_2)\) are twin observables for \(\rho_{12}\) if and only if they are twin observables for \(\) all term states \(\rho_{12}^{(k)}\).

\textit{Proof} is immediately obtained from Lemma 1 if one rewrites each term state as a pure-state mixture. □

\textit{Lemma 3.} An uncorrelated state \(\rho_1 \otimes \rho_2\) has only trivial twins.

\textit{Proof} is an immediate consequence of the fact that the tensor factors of a nonzero uncorrelated vector, say \(a \otimes b\), are unique up to an arbitrary nonzero complex number \(\alpha\), but if \(a\) is replaced by \(\alpha a\), \(b\) must be replaced by \((1/\alpha)b\).

If two observables are twins for an uncorrelated state, then
\[ A_1 \rho_1 \otimes \rho_2 = \rho_1 \otimes A_2 \rho_2. \]
If \(A_1 \rho_1 = \alpha \rho_1\), then, applying the above remark to supervectors, one has \(\rho_2 = (1/\alpha) A_2 \rho_2\). □

\textit{Proof of Theorem 3} now immediately follows from Lemma 2 and Lemma 3. Namely, the two groups of terms stated in the Theorem, make up the two terms in (3). □

\textit{Corollary 3.} Nontrivial twin events of a separable mixture (9) are \textit{necessarily strong twin events}.

\textit{Corollary 4.} If \((A_1, A_2)\) are nontrivial twin observables for a separable mixture (9), they are strong twin observables (cf Definition 3), and the mixture terms can be grouped into as many biorthogonal groups of terms as there are distinct characteristic values of \(A_1\) in \(\mathcal{R}(\rho_1)\) (generalization of (19a,b)).

It is known that if a statistical operator and a Hermitian operator commute, then the corresponding state can be written as a mixture so that each term-state has a definite value of the corresponding observable [10]. But, for the same statistical operator, there are also mixtures violating this.

To take an example, let us think of an unpolarized mixture of spin-one-half states: \(\rho = (1/2)I\) (in the two-dimensional spin factor space). This statistical operator commutes with \(s_z\), nevertheless one can write down the mixture
\[ \rho = (1/2) \left( |x, +\rangle\langle x, +| + |x, -\rangle\langle x, -| \right) = (1/2)I, \]
in which the term-states do not have a definite value of the z-component.

It is interesting that in the case of a separable mixture with a nontrivial twin observable it is necessarily its term-states that have the sharp detectable values of the corresponding observable.

7. States with Maximally Disordered Subsystems

Now we turn to the example that is, for illustrative purposes, investigated in this study, i.e., to states (statistical operators) \( \rho \) in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \). We say that \( \rho \) is an MDS state (one with maximally disordered subsystems or rather subsystem states) if \( \rho_1 = (1/2)I_1 \) and \( \rho_2 = (1/2)I_2 \). R. and M. Horodecki have shown [11] that for every MDS state there exist unitary subsystem operators \( U_1 \) and \( U_2 \) such that

\[
(U_1 \otimes U_2) \rho (U_1^\dagger \otimes U_2^\dagger) = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i \right) \equiv T,
\]

where \( \sigma_i, i = 1, 2, 3 \), are the well known Pauli matrices \( \sigma_x, \sigma_y \) and \( \sigma_z \); and it is seen from their place in the expression if they are meant for the first or for the second spin-one-half particle.

Further, they have shown that the operator \( T \) is a statistical operator (a quantum state) if and only if the vector \( \vec{t} \) from \( \mathbb{R}^3 \) the components of which appear in \( (20) \) is not outside the tetrahedron determined as the set of all mixtures of the four pure Bell states:

\[
| \psi_{12}^s \rangle \equiv (1/2)^{1/2} \left( |+\rangle |+\rangle + |-\rangle |-\rangle \right), \quad | \psi_{0}^3 \rangle \equiv (1/2)^{1/2} \left( |+\rangle |-\rangle + |-\rangle |+\rangle \right), \quad (21)
\]

where \( |+\rangle \) and the spin-up and the spin-down state vectors respectively.

It is straightforward to see that the three nonsinglet Bell states \( | \psi_s \rangle \), \( s = 1, 2, 3 \), when written in the form (20), are given by \( t_s = -1 \), and the other two components of \( \vec{t} \) equal to +1. The singlet state \( | \psi_0 \rangle \) is in the form (20) determined by all three components of \( \vec{t} \) being equal to \( -1 \).

It is also easy to see that for all mixtures one has

\[
-1 \leq t_i \leq +1 \quad i = 1, 2, 3.
\]

This is a necessary, but not a sufficient condition for \( T \) being a state. In other words, the tetrahedron is embedded in a cube, in which there are also nonphysical \( \vec{t} \). In view of the LHS of (20), we call \( T \) that belong to the tetrahedron: generating MDS states.

What we want to find out is: Which of the MDS states have nontrivial twins? For those that do have, we want to find the set of all nontrivial pairs of twins.

It is sufficient to find the generating MDS states \( T \) with nontrivial twins, because the validity of

\[
A_1 T = A_2 T
\]
obviously implies
\[
( U_1 A_1 U_1^\dagger ) ( U_1 U_2 T U_1^\dagger U_2^\dagger ) = ( U_2 A_2 U_2^\dagger ) ( U_1 U_2 T U_1^\dagger U_2^\dagger ),
\]
i. e., if the generating MDS states have nontrivial twins, then also the generated MDS states do have nontrivial twins, and they are immediately obtained.

As far as the pure generating MDS states (the Bell states) are concerned, the first-particle reduced statistical operator \( \rho_1 \) is equal to \((1/2)I_1 \), all nontrivial Hermitian operators \( A_1 \) commute with it, hence [3], they are twins. To evaluate the corresponding twin \( A_2 \), one has to read off the antilinear correlation operator \( A_1 \) from (21) having in mind (11), and then utilize (13). For the best known Bell state, the singlet state \( |\psi_0\rangle \), e. g., \( U_a \) takes \(|+\rangle\) into \(|-\rangle\), and \(|-\rangle\) into \((-|+\rangle) \) (cf (21)). If
\[
A_1 = \alpha_{++} |+\rangle\langle+| + \alpha_{--} |-\rangle\langle-| + \alpha_{+-} |+\rangle\langle-| + (\alpha_{+-})^* |-(\rangle\langle+|,
\]
then the twin \( A_2 \) has the form:
\[
A_2 = \alpha_{--} |+\rangle\langle+| + \alpha_{+-} |-\rangle\langle-| - \alpha_{+-} |+\rangle\langle-| - (\alpha_{+-})^* |-(\rangle\langle+|.\]

Now we turn to the mixtures of Bell states in our search for nontrivial twins.

8. Mixtures of Bell states

Viewing statistical operators as super vectors, and utilizing (redundantly, but for the sake of better overview) the ket notation for super state vectors (i. e., Hilbert-Schmidt operators as normalized super vectors), one can rewrite the generating vectors \( T \) given by (20) as a biorthogonal expansion with positive expansion coefficients:

\[
|T\rangle|T\rangle^{-1}_{12} = (1 + \sum_{i=1}^{3} t_i^2)^{-1/2} \left( |(1/2)^{1/2}I_1\rangle \otimes |(1/2)^{1/2}I_2\rangle + \sum_{i=1}^{3} |t_i| |(1/2)^{1/2}\sigma_i\rangle_1 \otimes |sg(t_i)(1/2)^{1/2}\sigma_i\rangle_2 \right) \tag{22}
\]

("sg" denotes the sign), i. e., as a (super state vector) Hermitian Schmidt decomposition.

One can read off (22) the following canonical entities of the super state vector \( |T\rangle|T\rangle^{-1}_{12} \) (cf (10a), (10b) and (11)):

The first-subsystem reduced statistical super operator \( \hat{\rho}_1 \) has the characteristic super state vectors \( \{|(1/2)^{1/2}I_1\rangle_1, |(1/2)^{1/2}\sigma_i\rangle_1 : i = 1, 2, 3\} \); the second subsystem reduced statistical super operator \( \hat{\rho}_2 \) has the characteristic state vectors \( \{|(1/2)^{1/2}I_2\rangle_2, |sg(t_i)(1/2)^{1/2}\sigma_i\rangle_2 : i = 1, 2, 3\} \); and the common spectrum of \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) is \( \{R_0 \equiv 1 + \sum_{i=1}^{3} t_i^2\}^{-1}, R_i \equiv R_0 t_i^2 : i = 1, 2, 3\} \). Finally, the antiunitary correlation super operator \( U_a \) maps the enumerated characteristic state vectors of \( \hat{\rho}_1 \) into the correspondingly ordered ones of \( \hat{\rho}_2 \).
9. Nontrivial MDS twins

Every super operator \( \hat{A}_1 \) that commutes with \( \hat{\rho}_1 \), i.e., for which every characteristic subspace of the latter is invariant (and no other super operator), has a twin super operator \( \hat{A}_2 \) \([3]\). But we are interested only in those pairs \((\hat{A}_1, \hat{A}_2)\) in which both super operators are, what may be called, multiplicative ones, i.e., which have the form

\[
\hat{A}_1 \rho_{12} = A_1 \rho_{12}, \quad \hat{A}_2 \rho_{12} = A_2 \rho_{12},
\]

where \( A_p, p = 1, 2, \) are ordinary (subsystem) operators. It is easy to see that a multiplicative super operator is Hermitian (in the Hilbert-Schmidt space of supervectors) if so is the ordinary operator (in the usual sense) that determines it.

The basic result of the expounded illustration is given in the following two theorems:

**Theorem 5.** Mixed generating MDS states have nontrivial twins if and only if they are mixtures of two Bell states (binary mixtures).

**Theorem 6.**

A) Let us take a binary mixture of two Bell states both distinct from the singlet one, and let \( T_i \equiv |\psi_i\rangle\langle\psi_i| \) (cf (21)) be the nonsinglet Bell state that does not participate in the mixture. Then the nontrivial twins are:

\[
A_1 \equiv \alpha I_1 + \beta \sigma_i^{(1)}, \quad A_2 \equiv \alpha I_2 + \beta \sigma_i^{(2)}, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \neq 0,
\]

where the suffix on \( \sigma_i \) refers to the corresponding tensor factor space.

B) In case of a binary mixture of the singlet state with another Bell state, say \( T_i \equiv |\psi_i\rangle\langle\psi_i| \) (cf (21)), the twins are:

\[
A_1 \equiv \alpha I_1 + \beta \sigma_i^{(1)}, \quad A_2 \equiv \alpha I_2 - \beta \sigma_i^{(2)}, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \neq 0.
\]

Proof of the two theorems and of some subsidiary results is given in the Appendix. The proof of Theorem 6 that is first given in the Appendix is only of methodological significance: it illustrates a method how to evaluate nontrivial twins. In our case of binary mixtures \( T^{(2)} \), another method gives a simpler evaluation. It is given at the end of the Appendix.

It is known that any Bell state can be converted into any other one by local unitary transformation \([6], [12]\). Hence, for proving the existence of nontrivial twins it would have sufficed to take mixtures of one pair of Bell states. Theorem 6 is, nevertheless, more elaborate because the explicit form of the twins depends on which Bell states are involved.

It is known that all binary mixtures of Bell states are nonseparable except those with equal weights. The latter, as easily seen, are examples of Theorem 4, e.g., as one can easily ascertain making use of (21), one has:

\[
(1/2)\left((|+\rangle\langle+| \otimes |+\rangle\langle+|) + (|-\rangle\langle-| \otimes |-\rangle\langle-|)\right) = \ldots
\]
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\[ (1/2) \left( |\psi_1\rangle\langle \psi_1| + |\psi_2\rangle\langle \psi_2| \right). \]  

(25)

The nonseparable binary Bell state mixtures are distillable even in the single copy case [13]. Unfortunately, there is no simple relation between the (investigated) existence of nontrivial twins and distillability as seen from the fact that also rank four mixtures are distillable (if and only if one of the weights is larger than 1/2), and they do not have nontrivial twins.

In conclusion, I would like to point out that the entangled pure state case [3], [4] is a well explored illustration for the fact that nontrivial twins can exist on account of entanglement. The nonseparable binary Bell state mixtures are another simple illustration for this fact. One should have in mind that, as it was seen in Lemma 3, uncorrelated bipartite states do not have nontrivial twins. Separable states can have nontrivial twins if and only if biorthogonal grouping of the terms is possible (cf Theorem 4). Unfortunately, for the time being, we do not have a necessary and sufficient condition for the existence of nontrivial twins on account of entanglement (except in the pure state case), let alone a way of generating all of them for a given composite-system mixed state (except in the pure state case).

At last, but not at least, a relation between the reported twin investigation ([3], [4], and [1] besides this article) and the mainstream research on entanglement (take the cited Bennett et al. articles, the article of Vedral et al., and the cited Horodecki family articles as examples) is still lacking. But I believe that there exists a connection. Further research will, hopefully, uncover it.

10. Appendix

Since we are going to prove the theorems making use of (22), first we must be able to recognize the binary mixtures \( T^{(2)} \) on the Horodecki tetrahedron.

**Proposition A.1.** One has a binary mixture \( T^{(2)} \) if and only if precisely one of the three \( |t_i| \) values in (22) equals 1.

A) If \( t_i = +1 \), \( |t_{i+1}|, |t_{i+2}| < 1 \) (where the three values \( \{1, 2, 3\} \) of \( i \) are meant cyclically), then the mixture is of two Bell states both distinct from the singlet state. If \( T_i \) is the nonsinglet Bell state that does not participate in the mixture, one has \( t_{i+2} = -t_{i+1} \). Finally, the binary mixture \( T^{(2)} \) in question is

\[ T^{(2)} = \left[ (1 - t_{i+1})/2 \right] T_{i+1} + \left[ (1 - t_{i+2})/2 \right] T_{i+2}. \]  

(A1)

B) If \( t_i = -1 \), \( |t_{i+1}|, |t_{i+2}| < 1 \) (in the cyclic sense), then one deals with a mixture of two states: the singlet state and another Bell state \( T_i \). One has \( t_{i+1} = t_{i+2} \), and the binary mixture \( T^{(2)} \) in question is

\[ T^{(2)} = \left[ (1 + t_{i+1})/2 \right] T_i + \left[ (1 - t_{i+1})/2 \right] T_0. \]  

(A2)

Both in the cases (A) and (B), \( t_{i+1} \) can be any number in the interval \(-1 \leq t_{i+1} \leq +1\); equivalently, one can have any point on the corresponding border of the Horodecki...
tetrahedron (the vertices excluded).

For proof a few subsidiary results are required.

**Lemma A.1.** If among the four numbers \{1, |t_i| : i = 1, 2, 3\} appearing in the form (22) of the generating MDS state \( T \) there is one distinct from the rest, then \( T \) has no nontrivial twins.

**Proof.** As clearly follows from the above stated spectrum of \( \hat{\rho}_1 \), the mentioned "one number distinct from the rest" corresponds to a nondegenerate characteristic value. Assuming that \( A_1 \) is a twin, it is a multiplicative superoperator reducing in each characteristic subspace of \( \hat{\rho}_1 \). (This is equivalent to commutation with \( \hat{\rho}_1 \).)

a) Let us take the case when \(|t_i| < 1, \ i = 1, 2, 3\). Then the first characteristic value of \( \hat{\rho}_1 \) is nondegenerate, and the corresponding characteristic super state vector has to be invariant (up to a constant):

\[
A_1(1/2)^{1/2}I_1 = \alpha(1/2)^{1/2}I_1,
\]

i. e., \( A_1 = \alpha \), and the twin is trivial.

b) Let \(|t_i| \) for some value of \( i \) be distinct from the other three numbers. Then the corresponding characteristic super state vector \( \sigma_i(1/2)^{1/2} \) must be invariant (up to a constant):

\[
A_1 \sigma_i(1/2)^{1/2} = \alpha \sigma_i(1/2)^{1/2},
\]

which, upon multiplication with \( \sigma_i \) from the right, implies \( A_1 = \alpha \) again. \( \square \)

**Corollary A.1.** If a generating MDS state \( T \) has nontrivial twins, then for at least one value of \( i \): \(|t_i| = 1\).

**Proof** is obvious from Lemma A.1. \( \square \)

**Lemma A.2.** Expressing a generating MDS state \( T \) written in the form (22) in terms of the statistical weights with respect to the Bell states \( \{T_k \equiv |\psi_k\rangle \langle \psi_k| : k = 0, 1, \ldots, 3\} \) (cf (6)), one has:

\[
T = \sum_{k=0}^{3} w_k T_k = (1/4) \left[ I \otimes I + (-w_1 + w_2 + w_3 - w_0)\sigma_1 \otimes \sigma_1 + (w_1 - w_2 + w_3 - w_0)\sigma_2 \otimes \sigma_2 + (w_1 + w_2 - w_3 - w_0)\sigma_3 \otimes \sigma_3 \right],
\]

(A3)

where

\[
\forall k: \ w_k \in [0, 1], \ k = 0, 1, 2, 3; \ \sum_{k=0}^{3} w_k = 1.
\]

**Proof** is straightforward substituting the Bell states in (22) (cf (21) and beneath it). \( \square \)
Lemma A.3. If one has \(|t_i| = 1, \ i = 1, 2, 3,\) for a generating MDS state \(T\) in the form (22), then it is a Bell state.

**Proof.** Each \(t_i\) has two sign possibilities; altogether there are \(2^3 = 8\) possibilities. A straightforward analysis of each of these, taking into account Lemma A.2 and \(\sum_{k=0}^{3} w_k = 1,\) shows that 4 possibilities do not give states. These are: \(\{sg(t_i) = + : i = 1, 2, 3\}, \{+ - -\}, \{- + -\}, \text{and} \{- - +\}.\) The remaining four sign possibilities give the four Bell states:
\[
\begin{align*}
&\{- + +\} : T_1; \quad \{+ - +\} : T_2; \quad \{+ + -\} : T_3; \quad \{- - -\} : T_0.
\end{align*}
\]

\(\square\)

**Proof of claim (A) in Proposition A.1.** Since it is clear from (A3) that the \(t_i\) as functions of \(w_k\) are symmetric (in the sense of the cycle \(\{1, 2, 3\}\)), it is sufficient to take \(i = 1.\) Then
\[
-w_1 + w_2 + w_3 - w_0 = 1, \quad \text{and} \quad \sum_{k=0}^{3} w_k = 1.
\]
This gives \(w_2 + w_3 = 1, \ w_1 = w_0 = 0, \) and \(t_2 = w_3 - w_2 = -t_3.\) Hence, \(w_2 = (1 - t_2)/2\) and \(w_3 = (1 + t_2)/2\) as claimed. Since \(0 < w_1, w_0 < 1,\) the claimed intervals for \(t_2\) and \(t_3\) follow. \(\square\)

**Proof of claim (B) of the Proposition.** It runs in full analogy with the proof for case (A). \(\square\)

**Proof of the main claim of the Proposition.** It is easy to see that the proofs of claims (A) and (B) of the Proposition go through also for the case when \(|t_{i+1}|\) or \(|t_{i+2}|\) equals one. Hence, one cannot have \(|t_i| = 1\) for precisely two values of \(i.\) If it is so for one value, then either it is so for all three values, and one has a pure Bell state, or it is so for precisely one value of \(i,\) then we have a binary mixture. \(\square\)

**Proof of Theorem 6.** We now assume that for one value of \(i, \ |t_i| = 1,\) and that the other two components of \(\vec{t}\) in (22) are by modulus less than one. Then it is sufficient and necessary for an observable \(A_1\) that defines a superoperator \(\hat{A}_1\) by multiplication (we write this as \(\hat{A}_1 \equiv (A_1 \bullet)\)) to have a superoperator twin \(A_2\) (that is not necessarily multiplicative as \(\hat{A}_1\)) that it reduces in the two-dimensional supervector subspace spanned by \(I_1\) and \(\sigma_i^{(1)}\). If we write \(A_1 = \alpha I_1 + \sum_{j=1}^{3} \beta_j \sigma_j^{(1)}\) \((\alpha, \beta_j \in \mathbb{R})\), and multiply with this from the left \(\sigma_i^{(1)},\) it turns out that the condition amounts to \(\beta_j = 0, \ j \neq i.\) The symmetrical argument gives the symmetrical result. Thus the multiplicative superoperators defined by \(A_1\) and, separately, by \(A_2\) do have superoperator twins if and only if they are of the form
\[
A_1 = \alpha I_1 + \beta \sigma_i^{(1)}, \quad A_2 = \gamma I_2 + \delta \sigma_i^{(2)}, \quad (A4)
\]
where \(\alpha, \beta, \gamma, \delta \in \mathbb{R}.\)
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The mentioned operators are twins of each other if and only if

\[(A_2 \bullet) = \hat{U}_a (A_1 \bullet) \hat{U}_a^{-1}.\]  \hspace{1cm} (A5)

Now we find out the necessary and sufficient conditions when (A5) is valid for the operators given by (A4). Since both sides of (A5) are linear operators, we apply them to the basis of supervectors \(\{I_2, \sigma^{(2)}_i : i = 1, 2, 3\}:

\[(A_2 \bullet) I_2 = \gamma I_2 + \delta \sigma^{(2)}_i;\]

\[\left(\hat{U}_a (A_1 \bullet) \hat{U}_a^{-1}\right) I_2 = \hat{U}_a (\alpha I_1 + \beta \sigma^{(1)}_i) = \alpha I_2 + sg(t_i) \beta \sigma^{(2)}_i.\]

Thus, we obtain the condition

\[\gamma = \alpha, \quad \delta = sg(t_i) \beta.\]

Utilizing the well known relation

\[\sigma_i \sigma_j = \delta_{ij} I + \sum_{m=1}^{3} i\epsilon_{ijm} \sigma_m,\]

we, further, have

\[(A_2 \bullet) \sigma^{(2)}_j = (\gamma I_2 + \delta \sigma^{(2)}_i) \sigma^{(2)}_j = \gamma \sigma^{(2)}_j + \delta (\delta_{ij} I_2 + \sum_{m=1}^{3} i\epsilon_{ijm} \sigma^{(2)}_m);\]

\[\left(\hat{U}_a (A_1 \bullet) \hat{U}_a^{-1}\right) \sigma^{(2)}_j = sg(t_j) \hat{U}_a (\alpha I_1 + \beta \sigma^{(1)}_i) \sigma^{(1)}_j =\]

\[sg(t_j) \hat{U}_a (\alpha \sigma_j^{(1)} + \beta (\delta_{ij} I_1 + \sum_{m=1}^{3} i\epsilon_{ijm} \sigma^{(1)}_m)) =\]

\[sg(t_j) (\alpha sg(t_j) \sigma^{(2)}_j + \beta (\delta_{ij} I_2 - \sum_{m=1}^{3} i\epsilon_{ijm} sg(t_m) \sigma^{(2)}_m)).\]

For \(i = j\) we obtain the condition \(\gamma = \alpha\), and \(\delta = sg(t_i) \beta\), and, for \(j \neq i\), in addition: \(\delta = -sg(t_j) sg(t_m) \beta\). Since \(i \neq m \neq j\), we know from the Proposition that, irrespective of \(sg(t_i)\), one has \(-sg(t_j) sg(t_m) = sg(t_i)\). Hence, we actually obtain the condition expressed by (23) and (24).

The claim in Theorem 5 that binary mixtures \(T^{(2)}\) do have nontrivial twins is an immediate consequence of Theorem 6.

The mentioned second, simpler, proof goes as follows: According to Lemma 1, a pair of opposite-subsystem observables \((A_1, A_2)\) are twins for a composite-system mixture if and only if they are simultaneously twins for each of the pure term states.

Utilizing (13), it is straightforward to evaluate the twins in the operator basis consisting of the four supervectors \(|\pm\rangle \langle \pm|\). But for comparison with the results (23) and (24) obtained by the Hermitian Schmidt decomposition method, we do this in a little bit more difficult way using the form (22) for the Bell states (see their description beneath (22)).
We can read off the antiunitary correlation superoperator $\hat{U}_a$ from the mentioned form (22) of the Bell state. As it was stated before, every first-subsystem observable $A_1 \equiv \alpha I_1 + \sum_{i=1}^{3} \beta_i \sigma_i^{(1)}$, $(\alpha, \beta_i \in \mathbb{R}, \ i = 1,2,3)$, is a twin. The corresponding second-subsystem twins for the Bell states are:

$$T_1 : \quad A_2 \equiv \alpha I_2 - \beta_1 \sigma_1^{(2)} + \beta_2 \sigma_2^{(2)} + \beta_3 \sigma_3^{(2)};$$

$$T_2 : \quad A_2 \equiv \alpha I_2 + \beta_1 \sigma_1^{(2)} - \beta_2 \sigma_2^{(2)} + \beta_3 \sigma_3^{(2)};$$

$$T_3 : \quad A_2 \equiv \alpha I_2 + \beta_1 \sigma_1^{(2)} + \beta_2 \sigma_2^{(2)} - \beta_3 \sigma_3^{(2)};$$

$$T_0 : \quad A_2 \equiv \alpha I_2 - \beta_1 \sigma_1^{(2)} - \beta_2 \sigma_2^{(2)} - \beta_3 \sigma_3^{(2)}.$$ 

Now, in view of the position of the minus sign in $A_2$, evidently, utilizing $m \neq i \neq j \neq m \ i, j, m \in \{1,2,3\}$, and $0 < w < 1$, the simultaneous twins are:

$$wT_j + (1-w)T_m : \quad A_2 \equiv \alpha + \beta_i \sigma_i;$$

$$wT_0 + (1-w)T_i : \quad A_2 \equiv \alpha - \beta_i \sigma_i;$$

and $A_2$ is, of course, the twin of $A_1 \equiv \alpha + \beta_i \sigma_i$.

In this way proof of (23) and (24) is obtained.

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