Cosmological anti-deSitter space-times and
time-dependent AdS/CFT correspondence

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Abstract

We study classes of five-dimensional cosmological solutions with negative curvature, which are obtained from static solutions by an exchange of a spatial and temporal coordinate, and in some cases by an analytic continuation. Such solutions provide a suitable laboratory to address the time-dependent AdS/CFT correspondence. For a specific example we address in detail the calculation of the boundary stress-energy and the Wilson line and find disagreement with the standard AdS/CFT correspondence. We trace these discrepancies to the time-dependent effects, such as particle creation, which we further study for specific backgrounds. We also identify specific time-dependent backgrounds that reproduce the correct conformal anomaly. For such backgrounds the calculation of the Wilson line in the adiabatic approximation indicates only a Coulomb repulsion.

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1 Introduction

The interest in time-dependent backgrounds in string theory is motivated by several reasons. The most important one is related to the attempts to understand the origin of the very early Universe, in particular within superstring theory and its M-theory unification, which are the most promising candidate for explaining the underlying unification of forces of nature. If string theory were able to shed light on the aspect of the early Universe, this would provide a strong incentive for its further investigation. From another point of view, the study of such time-dependent backgrounds may provide new insights into underlying symmetries that govern cosmological evolutions. For instance, AdS/CFT duality relates the properties of string theory on negative curvature D-dimensional space-times to the dual (D-1)-dimensional supersymmetric gauge theory. A time-dependent version of such a duality may in turn relate the properties of time-dependent gravitational evolutions (potentially describing the early Universe) to a dual description in terms of gauge theories in one dimension less.

It is known that quantum field theory (QFT) in curved time-dependent space-time (see e.g., [1] for introduction) is much more involved than the corresponding theory in a static space. In particular, there occurs particle creation caused by the gravitational field and the vacuum state changes with time. There are various non-zero quantum averages for "in-in", "out-in", "in-out" and "out-out" amplitudes. The standard QFT is well-developed to study only out-in amplitudes. At the same time, in quantum cosmology one needs in-in quantities as well. Moreover, we expect that a formulation of the time-dependent AdS/CFT correspondence should encode information for various vacuum states in time-dependent backgrounds.

The purpose of this paper is to discuss aspects of (conjectured) AdS$_5$/CFT$_4$ correspondence and its various predictions and manifestations when applied to the situation when the five-dimensional space-time is a cosmological AdS. We obtain such cosmological 5d AdS spaces from their AdS black hole cousins by exchanging the spatial and temporal coordinates, along with the corresponding analytical continuation when it is required. Examples of such cosmological AdS spaces include curved branes with deSitter space world-volume or a product of a sphere and a torus. For this class of backgrounds we focus on the evaluation of the surface stress tensor and the Wilson loop calculation on the gravitational side. In particular, for these
backgrounds one reproduces the holographic trace anomaly that agrees with
the prediction of the AdS/CFT correspondence. On the other hand, we also
demonstrate a discrepancy in the calculation of the energy momentum tensor
components on the gravity side with that of the AdS/CFT predictions can
be traced to the omitted non-local effects of particle creation in the bulk.

The paper is organized as follows. In the next section it is shown how
one can construct a cosmological AdS space from the 5d Schwarzschild-AdS
black hole by exchanging the role of temporal and radial coordinates. This
method can also be applied to BPS solutions which appear in 5d gauged
supergravity, and we demonstrate that on a specific example of bent BPS
domain walls with an asymptotic AdS space-time.

Section 3 is devoted to a general consideration, addressing which 5d cos-
mological AdS space may lead to the correct holographic conformal anomaly.
An explicit example of such a space is presented. As an additional check the
thermodynamical energy is found. We confirm that cosmological AdS back-
grounds, described in the previous section, belong to the class of backgrounds
with the correct holographic anomaly.

In section 4 we study the massive scalar propagator in a cosmological
AdS space and demonstrate that in these time-dependent backgrounds one
reproduces the conformal dimensions of these fields, in agreement with the
usual AdS/CFT predictions.

In section 5 we address particle creation in these time-dependent back-
grounds, which should provide non-local effects that account for the discrep-
ancy between the standard evaluation of the boundary stress energy on the
gravity side and the dual field theory prediction. In particular, we address
the number of particles created on deSitter brane in a time-dependent AdS
background.

Section 6 is devoted to the evaluation of an analog of Wilson loop in the
adiabatic approximation for the cosmological AdS spaces with the correct
holographic conformal anomaly. It is demonstrated that the potential has a
typically Coulomb-like form (as for the pure AdS space).

In the last section we summarize the results and discuss further directions.

In the Appendix we also summarize the construction of cosmological
space-times that are obtained from the AdS spaces by applying $T$-duality
(within the Neveu-Schwarz-Neveu-Schwarz sector of string theory). In this
case the obtained space-times are not asymptotically AdS, but correspond
to the so-called dilatonic vacua, and thus the standard AdS/CFT correspon-
dence is not applicable. Nevertheless there may be remnants of the standard AdS/CFT correspondence surviving due to the fact that these backgrounds are T-dual to the AdS ones. With this in mind, analogs of the surface stress-tensor and Wilson loop on the gravitational side are calculated for the T-dual backgrounds.

2 Cosmological AdS spaces

In this section we demonstrate on a number of examples how to obtain cosmological AdS space-times from the static ones via exchanges of spatial and temporal coordinates, and in some cases via further analytic continuation. We start with an example of the Schwarzschild-AdS (SAdS) black hole metric:

\[ ds^2_{\text{AdS-S}} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_3^2, \quad f(r) \equiv \frac{r^2}{l^2} + 1 - \frac{\mu}{r^2} \quad (1) \]

This space-time may correspond to a classical solution of a gauged supergravity theory that arises as a sphere compactification of an effective supergravity from string or M-theory. In (1), \( d\Omega_3^2 \) is the metric of the three dimensional sphere \( S^3 \) with unit radius. The horizon radius \( r_H \) is

\[ r_H^2 = \frac{l^2}{2} + \frac{1}{2} \sqrt{l^4 + 16\mu l^2} \quad (2) \]

and the Hawking temperature \( T_H \)

\[ T_H = \left| \frac{1}{4\pi} \frac{df(r)}{da} \right|_{r=r_H} = \frac{1}{2\pi r_H} + \frac{r_H}{\pi l^2}. \quad (3) \]

In the following we shall discuss how to obtain the cosmological AdS space from the above SAdS black hole. One approach was employed in ref.[2] (see also [3]) writing the metric \( d\Omega_3^2 \) in (1) as

\[ d\Omega_3^2 = d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (4) \]

and replacing

\[ t \to i\tilde{\phi}, \quad \chi \to \frac{\pi}{2} + i\tilde{t}, \quad (5) \]
the bubble solution has been constructed:

\[ ds^2 = -r^2 \tilde{d}t^2 + f(r)d\tilde{\phi}^2 + \frac{1}{f(r)}dr^2 + r^2 \cosh^2 \tilde{t} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) . \] (6)

If \( \tilde{\phi} \) is periodic, \( r \) is restricted by

\[ r \geq r_H \] (7)

and the singularity at \( r = r_H \) does not appear in (6).

There is, however, more freedom in the choice of an analytic continuation. For example, instead of (6), if we consider,

\[ t \to i\tilde{\phi}, \quad \theta \to \frac{\pi}{2} + i\tilde{t}, \] (8)

one obtains

\[ ds^2 = -r^2 \sin^2 \chi \tilde{d}t^2 + f(r)d\tilde{\phi}^2 + \frac{1}{f(r)}dr^2 + r^2 \sin^2 \chi \cosh^2 i\tilde{d}\phi^2 . \] (9)

Alternatively, with

\[ t \to i\tilde{\phi}, \quad \chi \to i\tilde{t}, \quad \theta \to i\tilde{\theta}, \] (10)

we obtain

\[ ds^2 = -r^2 \tilde{d}t^2 + f(r)d\tilde{\phi}^2 + \frac{1}{f(r)}dr^2 + r^2 \sinh^2 \tilde{t} \left( d\tilde{\theta}^2 + \sinh^2 \tilde{\theta}d\phi^2 \right) . \] (11)

Furthermore, taking

\[ t \to i\tilde{\phi}, \quad \phi \to i\tilde{t}, \] (12)

we obtain the static space-time:

\[ ds^2 = -r^2 \sin^2 \chi \sin^2 \theta \tilde{d}t^2 + f(r)d\tilde{\phi}^2 + \frac{1}{f(r)}dr^2 + r^2 \left( d\tilde{\chi}^2 + \sin^2 \chi d\theta^2 \right) . \] (13)

For (9), (11), or (13), if \( \tilde{\phi} \) is periodic in (164), \( r \) is restricted by (7) and there is no singularity at \( r = r_H \).
Following [4], another cosmological model can be obtained from the SAdS black hole. Inside the horizon \( r < r_H \) in (1), if we rename \( r \) as \( t \) and \( t \) as \( r \) we get a metric describing a time-dependent background:

\[
ds^2 = -\frac{l^2 t^2}{(r_H^2 - t^2) (r_B^2 + t^2)} dt^2 + \frac{(r_H^2 - t^2) (r_B^2 + t^2)}{l^2 t^2} dr^2 + t^2 d\Omega_3^2 .
\]  

(14)

Here

\[
r_B^2 = \frac{l^2}{2} \left( 1 + \sqrt{1 + \frac{4\mu}{l^2}} \right). 
\]  

(15)

The appearance of the time-dependent background behind the horizon of SAdS black hole, where the physical role of time and radial coordinates was exchanged, is of course very analogous to the space-time properties of regular Schwarzschild black holes behind the horizon. Since \( t = 0 \) corresponds to the singularity of the black hole, there is a curvature singularity, where the square of the Riemann tensor diverges as

\[
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \sim \frac{72}{t^8} .
\]  

(16)

The singularity at \( t = 0 \) might be regarded as a big-bang singularity. The topology of the space is \( R_1 \times S_3 \), where \( R_1 \) corresponds to \( r \). The metric of \( R_1 \) vanishes at the horizon \( t = r_H \). The scalar curvature \( R \) is a negative constant

\[
R = -\frac{20}{l^2} .
\]  

(17)

as it should be for a typical AdS space-time.

If we change the coordinate by

\[
t^2 = \frac{r_H^2 + r_B^2}{2} \cos \frac{2\tau}{l} + \frac{r_H^2 - r_B^2}{2} = l^2 \sqrt{1 + \frac{4\mu}{l^2} \cos \frac{2\tau}{l}} - l^2 ,
\]  

(18)

one can rewrite the metric (14) in the following form:

\[
ds^2 = -d\tau^2 + \frac{\left( 1 + \frac{4\mu}{l^2} \right) \sin^2 \frac{2\tau}{l}}{\sqrt{1 + \frac{4\mu}{l^2} \cos \frac{2\tau}{l} - 1}} dr^2 \\
+ \left( l^2 \sqrt{1 + \frac{4\mu}{l^2} \cos \frac{2\tau}{l} - l^2} \right) d\Omega_3^2 .
\]  

(19)
In (19), \( \sin^2 \frac{2\tau}{l} = 0 \) corresponds to the horizon and \( \cos^2 \frac{2\tau}{l} = \frac{k}{2 \sqrt{1 + \frac{4\mu l^2}{l^2}}} \) corresponds to the curvature singularity.

Let us also point out another analytic continuation of the metric (19). First we write \( d\Omega_3^2 \) as in (4) and consider the following analytic continuation

\[
\tau \rightarrow i \tilde{\tau}, \quad r \rightarrow i \tilde{\phi}, \quad \phi \rightarrow i \tilde{\tau}.
\]

(20)

Then the metric (19) can be rewritten as a static one:

\[
ds^2 = d\tilde{r}^2 + \left(1 + \frac{4\mu}{l^2}\right) \frac{\sinh^2 \frac{2\tilde{r}}{l}}{\cosh^2 \frac{2\tilde{r}}{l} - 1} d\tilde{\phi}^2
\]

\[
+ \left(l^2 \sqrt{1 + \frac{4\mu}{l^2} \cosh \frac{2\tilde{r}}{l} - l^2}\right) \left(d\tilde{\chi}^2 + \sin^2 \tilde{\chi} \left(d\theta^2 - \sin^2 \theta d\tilde{\phi}^2\right)\right).
\]

(21)

As we will see in the next section, AdS/CFT is applied to such metric. This metric (21) is, however, another presentation of the metric (13), which are related by the coordinate transformation given by

\[
r^2 = l^2 \sqrt{1 + \frac{4\mu}{l^2} \cosh \frac{2\tilde{r}}{l} - l^2},
\]

(22)

When \(-C = \frac{4\mu}{l^2} < 0\), a non-trivial analytic continuation is given by

\[
r = \tilde{r}, \quad \tau = \frac{\pi l}{4} + i \tilde{\tau}, \quad \chi = i \tilde{\chi},
\]

(23)

which leads to

\[
ds^2 = d\tilde{r}^2 - \frac{M \cosh^2 \frac{2\tilde{r}}{l}}{\sqrt{M} \sinh \frac{2\tilde{r}}{l} + 1} d\tilde{\phi}^2
\]

\[
+l^2 \left(\sqrt{M} \sinh \frac{2\tilde{r}}{l} + 1\right) \left(d\tilde{\chi}^2 + \sinh^2 \tilde{\chi} \left(d\theta^2 + \sin^2 \theta d\tilde{\phi}^2\right)\right).
\]

(24)

In the above metric (24), there is a time-like singularity at \( \sqrt{M} \sinh \frac{2\tilde{r}}{l} = -1 \). If we put a boundary or a brane source at \( \tilde{r} = R > 0 \) (\( R \) can be a function of the “time” coordinate \( \tilde{\tau} \) and other coordinates, \( \tilde{\chi}, \theta, \) and \( \phi \)) and only consider the region where \( \tilde{r} > R \), then this procedure constrains the space-time to the non-singular region, only. Since the horizon is given by \( \tilde{r} = 0 \) and

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it is null, then for the brane with a velocity less than the speed of light, the brane does not cross the horizon and therefore shields the space-time from the singularity.

Note that the above suggested ways of obtaining cosmological solutions is not limited to a narrow class of cosmological models obtained from SAdS black holes but should also apply to other AdS black holes, such as charged AdS black holes [5] and rotating charged AdS black holes [6], and they deserve further study.

We conclude this section by deriving cosmological solutions from the BPS bent domain wall solutions in a class of \( N = 2 \) 5-dimensional gauged supergravity. Such domain walls were found in [7] (see also [8]). The world-volume of these domain walls corresponds to the AdS\(_4\) space-time. Examples of such configurations have the property that asymptotically \( (y \to \infty) \) the space-time is the boundary of AdS\(_5\) while in the interior \( (y \to y_0) \) there is a naked singularity. One such a configuration has the metric of the following form [7]:

\[
\begin{align*}
\text{ds}^2 &= e^{-2y} \tilde{g}_{mn} dx^m dx^n + \frac{4dy^2}{(e^{12y} - 1)^2 - 4\lambda^2 e^{2y}}, \\
&= e^{-2y} \tilde{g}_{mn} dx^m dx^n + \frac{4dy^2}{(e^{12y} - 1)^2 - 4\lambda^2 e^{2y}}.
\end{align*}
\]

and another is

\[
\begin{align*}
\text{ds}^2 &= e^{-2y} \tilde{g}_{mn} dx^m dx^n + \frac{16dy^2}{e^{12y} - 16\lambda^2 e^{2y}}.
\end{align*}
\]

Here \( \tilde{g}_{mn} \) is the metric of 4d AdS, whose Ricci tensor \( \tilde{R}_{mn} \) given by \( \tilde{g}_{mn} \) satisfies \( \tilde{R}_{mn} = -3\lambda^2 \tilde{g}_{mn} \). There are several ways of expressing the AdS metric, and one of them is

\[
\begin{align*}
\tilde{g}_{mn}(x) dx^m dx^n &= -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 \sum_{i,j=1}^{2} \gamma_{ij} dx^i dx^j, \\
f(r) &\equiv k + \lambda^2 r^2.
\end{align*}
\]

Here \( k = 0, \pm 1 \). Consider \( k = -1 \) case. Then \( \gamma_{ij} \) expresses the hyperboloid with unit length parameter. When \( \lambda^2 r^2 < 1 \), by exchanging \( t \) and \( r \), we obtain the following metric

\[
\begin{align*}
\tilde{g}_{mn}(x) dx^m dx^n &= -\frac{1}{f(t)} dt^2 + \hat{f}(t) dr^2 + t^2 \sum_{i,j=1}^{2} \gamma_{ij} dx^i dx^j, \\
\hat{f}(t) &\equiv 1 - \lambda^2 t^2.
\end{align*}
\]
which may be reinterpreted as a cosmological model.

In the Euclidean signature, we may write the 4-dimensional AdS (hyperboloid) as
\[
\tilde{g}_{mn}(x)dx^m dx^n = dy^2 + \frac{1}{\lambda^2} \cosh^2 (\lambda y) dH_3^2.
\]

Here \(dH_3^2\) is the metric of the 3-dimensional hyperboloid with a unit radius. Wick rotation \(y = it\) yields a cosmological model with the following metric:
\[
\tilde{g}_{mn}(x)dx^m dx^n = -dt^2 + \frac{1}{\lambda^2} \cos^2 (\lambda t) dH_3^2.
\]

In summary we presented a number of examples of cosmological AdS space-times. Since these space-times describe (time-dependent) AdS or Schwarzschild-AdS space-times, it might be natural to expect that some analog of AdS/CFT correspondence would be applicable. However, due to time dependence and the related effects, such as cosmological particle creation, the application of the AdS/CFT correspondence is not straightforward. In the following sections we shall explore to what extent the AdS/CFT correspondence can be applicable to this class of AdS cosmological backgrounds.

3 Holographic conformal anomaly

In this section we investigate constraints on the time-dependent AdS space-times that reproduce the correct holographic trace anomaly. The space-times with this feature do not reproduce the components of the boundary stress-tensor, and thus the standard AdS/CFT correspondence is not applicable. Nevertheless, the fact that the trace of the boundary stress-tensor can be reproduced correctly is an interesting feature of these backgrounds.

While we start with a general consideration of such space-times, we apply the explicit calculation of the trace anomaly to the time-dependent solutions obtained from the so-called topological AdS black holes \((k = -1)\) [18]. We also comment on the fact that the calculation of the conformal anomaly is applicable also to the time-dependent backgrounds derived in the previous section. Subsequently, we also calculate the thermodynamic energy for the time-dependent (topological) SAdS black hole background.
Using a general coordinate transformation, one can start with the following form of the metric:

\[ ds^2 = dy^2 + e^{2\rho y} \sum_{m,n=0}^{d-1} \tilde{g}_{mn}(y,x) dx^m dx^n. \]  

(31)

Defining a new coordinate \( \rho \):

\[ \rho = e^{-\frac{2\rho}{\lambda}}, \]  

(32)

the metric (31) can be rewritten as

\[ ds^2 = \frac{d\rho^2}{4\rho^2} + \rho^{-1} \sum_{m,n=0}^{d-1} \tilde{g}_{mn}(y,x) dx^m dx^n. \]  

(33)

The above metric is invariant under the following transformation with a constant parameter \( \lambda \) of the transformation:

\[ \rho \rightarrow \lambda \rho, \quad \tilde{g}_{mn} \rightarrow \lambda \tilde{g}_{mn}, \]  

(34)

which can be identified with a scale transformation for the metric \( \tilde{g}_{mn} \). First we consider the case that \( \tilde{g}_{mn} \) and the Lagrangian density \( \mathcal{L} \) can be expanded as the power series on \( \rho \):

\[
\tilde{g}_{mn}(\rho, x) = \tilde{g}_{mn}^{(0)}(x) + \rho \tilde{g}_{mn}^{(1)}(x) + \rho^2 \tilde{g}_{mn}^{(2)}(x) + \cdots,
\]

\[
\mathcal{L}(\rho, x) = \mathcal{L}^{(0)}(x) + \rho \mathcal{L}^{(1)}(x) + \rho^2 \mathcal{L}^{(2)}(x) + \cdots.
\]  

(35)

Then the action has the following form:

\[
S = \int d^{d+1}x \sqrt{-g} \mathcal{L}
= \frac{l}{2} \int d^{d}x d\rho \rho^{-1 - \frac{d}{2}} \left( \mathcal{L}^{(0)}(x) + \rho \mathcal{L}^{(1)}(x) + \rho^2 \mathcal{L}^{(2)}(x) + \cdots \right).
\]  

(36)

The \( \rho \) integration of the terms containing \( \mathcal{L}(n) \) with \( n \leq \frac{d}{2} \) leads to a divergence. If \( n < \frac{d}{2} \), the divergence can be subtracted by adding the counter-terms. The subtraction does not break the scale transformation in (34). On the other hand, the subtraction of the term with \( n = \frac{d}{2} \) breaks the scale invariance in (34) and therefore gives the holographic trace anomaly [16, 17]

\[
T = -l \mathcal{L}^{\left( \frac{d}{2} \right)}.
\]  

(37)
Now let us assume that the metric has the following, more general form than that in (33):

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \rho^\alpha \sum_{m,n=0}^{d-1} \tilde{g}_{mn}(\rho, x) dx^m dx^n , \quad (38)$$

One should note that the metric (38) still has an invariance under a kind of scale transformation, which is analogous to (34):

$$\rho \rightarrow \lambda^{\frac{\alpha}{d}} \rho , \quad \tilde{g}_{mn} \rightarrow \lambda \tilde{g}_{mn} . \quad (39)$$

We now also assume $\tilde{g}_{mn}(\rho, x)$ (38) can be expanded as a power series in $\rho$ as in (35). Then the scalar curvature $R$ has the following form:

$$R = -\alpha^2 d(d + 1) + \rho^{-2\alpha} \tilde{R} . \quad (40)$$

Here $\tilde{R}$ is the scalar curvature constructed with $\tilde{g}_{mn}$. Eq.(40) implies that the behavior of the Lagrangian density $\mathcal{L}$ for small $\rho$ may be changed by the sign of $\alpha$. We now assume that the Lagrangian density behaves as $\mathcal{L} \sim R$. Then in the case of $\alpha < 0$, one can expand the Lagrangian as

$$\mathcal{L}(\rho, x) = \mathcal{L}^{(0)}(x) + \rho \mathcal{L}^{(1)}(x) + \rho^2 \mathcal{L}^{(2)}(x) + \cdots \quad (41)$$

and the action has the following form:

$$S = \frac{l}{2} \int d^4 x d\rho \rho^{-1 + \frac{d\alpha}{2}} \left( \mathcal{L}^{(0)}(x) + \rho \mathcal{L}^{(1)}(x) + \rho^2 \mathcal{L}^{(2)}(x) + \cdots \right) . \quad (42)$$

If $\frac{d\alpha}{2}$ is a non-positive integer, the term including $\mathcal{L}^{(-\frac{d\alpha}{2})}$ diverges logarithmically. If one subtracts this term, the scale invariance (39) breaks down. Then the term including $\mathcal{L}^{(-\frac{d\alpha}{2})}$ may give the holographic trace anomaly:

$$T = -l \mathcal{L}^{(-\frac{d\alpha}{2})} . \quad (43)$$

On the other hand, in the case of $\alpha > 0$, (40) indicates the following expansion of the Lagrangian:

$$\mathcal{L}(\rho, x) = \rho^{-2\alpha} \left( \mathcal{L}^{(0)}(x) + \rho \mathcal{L}^{(1)}(x) + \rho^2 \mathcal{L}^{(2)}(x) + \cdots \right) , \quad (44)$$
and the action has the following form:

\[
S = \frac{l}{2} \int d^d x d\rho \rho^{-1+\left(\frac{d}{2}-2\right)\alpha} \left( \mathcal{L}(0)(x) + \rho \mathcal{L}(1)(x) + \rho^2 \mathcal{L}(2)(x) + \cdots \right).
\] (45)

Then if \(\left(\frac{d}{2}-2\right)\alpha\) is non-positive integer, which requires \(d \leq 4\), the term including \(\mathcal{L}(\pm\frac{d}{2})\) diverges logarithmically and may give the holographic trace anomaly again:

\[
T = -l \mathcal{L}(\frac{d}{2} - 2\alpha).
\] (46)

For the special case: \(d = 4\), we have \(\left(\frac{d}{2} - 2\right)\alpha = 0\). Then (37) has the form

\[
T = -l \mathcal{L}(0).
\] (47)

In the case of the metric (156), one may identify \(\rho\) in (38) as \(\frac{1}{r^2}\) in (156). When \(r\) is large (\(\rho\) is small), the metric behaves as in (165), i.e., \((t, t)\)-component of the metric is \(O(r^2)\) and \((\xi^i, \xi^j)\)-components is \(O(r^{-2})\). Then we should choose \(\alpha = -1\) in (38) from the \((t, t)\)-component of the metric. Then \((\xi^i, \xi^j)\)-components of the metric corresponding to \(\tilde{g}^{(0)}_{mn}\) (38) vanish. Therefore the corresponding metric \(\tilde{g}^{(0)}_{mn}\) (38) cannot be defined or becomes singular. On the other hand, the metric (183) for large \(r\) behaves as

\[
ds^2 \to \frac{l^2}{r^2} dr^2 + \frac{1}{r^2} \left(-l^2 dt^2 + 2d\xi^2\right).
\] (48)

Defining

\[
\rho = \frac{1}{r^2},
\] (49)

one gets

\[
ds^2 \to \frac{l^2}{4\rho^2} d\rho^2 + \rho \left(-l^2 dt^2 + 2d\xi^2\right).
\] (50)

Since \(\alpha = 1 > 0\) and \(d = 4\), the quantity which we identify with the holographic trace anomaly is given by (47).

Let us now consider another metric:

\[
ds^2 = dy^2 + l^2 \sinh^2 \frac{y}{l} \sum_{m,n=0}^{d-1} \tilde{g}_{mn}(x) dx^m dx^n.
\] (51)
Redefining the radial coordinate $y$ as in (32), and taking the limit $\rho \to 0$, we have (38):

$$ds^2 \to \frac{dp^2}{4\rho^2} + \rho^{-1} \sum_{m,n=0}^{d-1} \tilde{g}_{mn}(x)dx^m dx^n.$$  

(52)

The $(d+1)$-dimensional vacuum Einstein equation with negative cosmological constant requires that

$$\tilde{R}_{mn} = (d-1)\tilde{g}_{mn}.$$  

(53)

We shall take as an explicit example the Schwarzschild-anti-deSitter black hole which is a solution of (52), i.e.,

$$\tilde{g}_{mn}(x)dx^m dx^n = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^{d-2} \sum_{i,j=1}^{d-2} \gamma_{ij}dx^i dx^j ,$$  

$$f(r) \equiv \frac{k}{d-3} - \frac{\mu}{r^{d-3}} - r^2.$$  

(54)

The metric $\gamma_{ij}$ satisfies the condition $R^{(\gamma)}_{ij} = k\gamma_{ij}$. Here $R^{(\gamma)}_{ij}$ is the Ricci tensor given by $\gamma_{ij}$. The metric has a curvature singularity at $r = 0$. In fact,

$$\tilde{R}_{klmn} \tilde{R}_{klmn} \sim \frac{(d-1)(d-2)(d-3)}{r^{2(d-1)}}.$$  

(55)

In order for space-time (54) to describe a black hole with a horizon, we should choose $k$ to be positive. When $k$ is not positive, the curvature singularity in (54) becomes naked. However, the interchange of the radial coordinate $r$ and the time coordinate $t$ yields a time-dependent (cosmological) solution, which may be referred to as a topological Schwarzschild-anti-deSitter solution [18]:

$$\tilde{g}_{mn}(x)dx^m dx^n = -\frac{dt^2}{\tilde{k}_{d-3} + \frac{\mu}{t^{d-3}}} + \left( \frac{\tilde{k}}{d-3} + \frac{\mu}{t^{d-3}} + t^2 \right) dr^2 + t^{d-2} \sum_{i,j=1}^{d-2} \gamma_{ij}dx^i dx^j.$$  

(56)

Here $\tilde{k} \equiv -k$. This $d$-dimensional space-time is embedded in $d+1$-dimensional space-time as in (51) or (52). In this case the metric of the
total $d + 1$ dimensional space-time is given by
\[
 ds^2 = dy^2 + l^2 \sinh^2 \frac{y}{l} \sum_{m,n=0}^{d-1} \tilde{g}_{mn}(x) dx^m dx^n \\
= dy^2 + l^2 \sinh^2 \frac{y}{l} \left( -\frac{k}{d-3} + \frac{\mu}{t^{d-3}} + t^2 \right)
\]
\[
+ \left( \frac{k}{d-3} + \frac{\mu}{t^{d-3}} + t^2 \right) dr^2 + t^2 \sum_{i,j=1}^{d-2} \gamma_{ij} dx^i dx^j \right) . \tag{57}
\]

or
\[
 ds^2 = \frac{d\rho^2}{4\rho^2} + \rho^{-1} \left( -\frac{k}{d-3} + \frac{\mu}{t^{d-3}} + t^2 \right) dr^2 + t^2 \sum_{i,j=1}^{d-2} \gamma_{ij} dx^i dx^j \right) . \tag{58}
\]

One can put a probe brane at $y = y_0$. The structure of the metric (51) is similar to the case of deSitter brane in the AdS bulk space (see, for instance, Ref.[19]). The stability of the brane may be achieved by the quantum effects produced via the trace anomaly [20, 21, 22].

Using the standard procedure[16, 17], the general form of the holographic trace anomaly is:
\[
 T = \frac{l^3}{8\pi G} \left( \frac{1}{8} \tilde{R}_{mn}\tilde{R}^{mn} - \frac{1}{24} \tilde{R}^2 \right) . \tag{59}
\]

Here $G$ is the $(d + 1)$-dimensional Newton constant ($16\pi G = \kappa^2$). For $d = 4$ Eq.(53) gives
\[
 T = -\frac{3}{16\pi G l} . \tag{60}
\]

One should note that the obtained holographic trace anomaly exactly coincides with the trace anomaly calculated on the CFT side. Therefore even for the time-dependent backgrounds of the type (57) or (58), the holographic trace anomaly is a constant.

We now consider the case that the space-time metric is given by (21). Rewriting the metric with
\[
 \tilde{r} = \frac{l}{2} \ln \rho , \tag{61}
\]
as
\[
\begin{align*}
ds^2 &= \frac{l^2 \rho^2}{\rho^2} \left( 1 + \frac{4\mu}{l^2} \left( \frac{\rho - \rho^{-1}}{2} \right)^2 \right) \frac{d\bar{\phi}^2}{\sqrt{1 + \frac{4\mu \rho + \rho^{-1}}{2} - 1}} \\
&\quad + \left( l^2 \sqrt{1 + \frac{4\mu \rho + \rho^{-1}}{2} - l^2} \right) \left( 1 + \frac{4\mu}{l^2} \left( \frac{\rho - \rho^{-1}}{2} \right)^2 \right) \left( d\chi^2 + \sin^2 \chi \left( d\theta^2 - \sin^2 \theta d\bar{\tau}^2 \right) \right)
\end{align*}
\] (62)
in the limit of \( \rho \to 0 \), one gets
\[
\begin{align*}
ds^2 &= \frac{l^2 \rho^2}{\rho^2} \left( 1 + \frac{4\mu}{l^2} \left( \frac{\rho - \rho^{-1}}{2} \right)^2 \right) \frac{d\bar{\phi}^2}{\sqrt{1 + \frac{4\mu \rho + \rho^{-1}}{2} - 1}} \\
&\quad + \left( l^2 \sqrt{1 + \frac{4\mu \rho + \rho^{-1}}{2} - l^2} \right) \left( d\chi^2 + \sin^2 \chi \left( d\theta^2 - \sin^2 \theta d\bar{\tau}^2 \right) \right)
\end{align*}
\] (62)
in the limit of \( \rho \to 0 \), one gets
\[
\begin{align*}
ds^2 &= \frac{l^2 \rho^2}{\rho^2} + \frac{1}{\rho} \sum_{m,n=0}^{3} \tilde{g}_{mn}(x) dx^m dx^n , \\
\sum_{m,n=0}^{3} \tilde{g}_{mn}(x) &= \sqrt{1 + \frac{4\mu}{l^2} \left( \frac{1}{2} d\bar{\phi}^2 \right) + l^2 \left( 1 + \frac{4\mu}{l^2} \left( \frac{\rho - \rho^{-1}}{2} \right)^2 \right) \left( d\chi^2 + \sin^2 \chi \left( d\theta^2 - \sin^2 \theta d\bar{\tau}^2 \right) \right) } .
\end{align*}
\] (63)

Here \( (x^m) = (\bar{\tau}, \bar{\phi}, \chi, \theta) \). The topology of the surface with constant \( \rho \) is \( R \times dS^3 \) and the radius of \( dS^3 \) is \( l \sqrt{1 + \frac{4\mu}{l^2}} \). Since the metric (63) has a general form (52), we can use the formula (59). Since now
\[
\tilde{R}_{\bar{\phi}\bar{\phi}} = 0 , \quad \tilde{R}_{ij} = \frac{2}{l^2 + 4\mu} \tilde{g}_{ij} ,
\] (64)
(Here \( i, j \) corresponds to \( (\bar{\tau}, \chi, \theta) \)), one finds the conformal anomaly vanishes
\[
T = 0 ,
\] (65)
which is consistent with the field theory result.

Note that we can also apply the calculation of the trace anomaly to the time-dependent backgrounds considered in the previous section. In particular, for the time-dependent metric (19), which was obtained from the static one (21) by employing the analytic continuation (20), the metric can again be cast in the form (63). Thus the conformal anomaly can again be calculated along the same lines, and it agrees with the field theory result. Note that the surface with constant \( \rho \) corresponds to the space-like surface with constant \( \bar{\tau} \), and thus the boundary field theory lives on a space-like surface.
We now turn to the calculation of the thermodynamic energy for the time-dependent background (57). Note that Eq.(55) (with the exchanged $r$ and $t$ coordinates) implies that there is a curvature singularity at $t = 0$. As we are considering the case with negative $k$, there is no horizon for positive $\mu$ and the singularity is naked and corresponds to the big-bang (or possibly big-crunch) singularity. In the 5d metric (57), the singularity is $y$-dependent and the shape of the singularity is the line.

Since these metrics have naked singularities for positive $\mu$, it is not straightforward to define thermodynamical quantities. If we choose, however, $\mu$ to be negative, there appears a cosmological horizon. For $d = 4$, from Eq.(56), we find the horizon exists at $t = t_c$, which is defined by

$$0 = \tilde{k} - \frac{\tilde{\mu}}{t_c} + t_c^2, \quad \bar{\mu} \equiv -\mu$$

(66)

and one can define the Hawking temperature $T_H$:

$$T_H \equiv \frac{1}{4\pi} \left. \frac{df(t)}{dt} \right|_{t=t_c} = \frac{1}{4\pi} \left( \frac{\bar{\mu}}{t_c^2} + 2t_c \right) = \frac{1}{4\pi} \left( \frac{\tilde{k}}{t_c^2} + 3t_c \right).$$

(67)

Assuming the entropy $S$ is given by the horizon area, one finds

$$S = \frac{r_c^2 V_2}{4G_4}, \quad G_4 \equiv \frac{2G}{l}. \quad (68)$$

Here $V_2$ is the volume of the 2d manifold corresponding to $k = -\tilde{k} : V_2 \equiv \int d^2 x \sqrt{\gamma}$. When $V_2$ is infinite, the density of the entropy may be considered. Using the method of [23], the thermodynamical energy $E$ for $d = 4$ is nothing but $\bar{\mu}$ [18]:

$$E = \frac{3\bar{\mu} V_2}{16\pi G} = -\frac{3\mu V_2}{16\pi G}.$$

(69)

In [24], the thermodynamical energy is related with the entropy by modifying the Cardy-Verlinde formula [25, 26]. As the (topological) black hole solution exists at the constant $y$ surface, the entropy and the energy should correspond to those on the CFT side.

We can also calculate the analog of the surface energy momentum tensor (170) for the space-time (57) with $d = 4$. Now

$$T_1^{mn} = \frac{6}{\kappa^2 l^3 \sinh^3 \frac{y}{l}} g^{mn}, \quad T_2^{mn} = \left( -\frac{\eta_1}{l^2 \sin^2 \frac{y}{l}} - \frac{6\eta_2}{l^4 \sinh^4 \frac{y}{l}} \right) \tilde{g}^{mn}.$$

(70)
Choosing

\[ \eta_1 = \frac{6}{\kappa^2 l}, \quad \eta_2 = \frac{l}{2\kappa^2}, \]  

which is a standard choice in AdS/CFT correspondence [13], the stress tensor in the large \( y \) limit takes the form:

\[ \hat{T}^{mn} = T_1^{mn} + T_2^{mn} \rightarrow -\frac{48}{16\pi G l^3} e^{-\frac{6\eta}{l}} \tilde{g}^{mn}. \]  

(72)

Then the trace of \( \hat{T}^{mn} \) is given by

\[ \hat{T}^m_m = g_{mn} \hat{T}^{mn} = l^2 \sinh^2 \frac{y}{l} \tilde{g}_{mn} \hat{T}^{mn} \rightarrow -\frac{48}{16\pi G l^3} e^{-\frac{6\eta}{l}}, \]  

(73)

which is related to the holographic trace anomaly \( T \) (60) by

\[ T = \frac{\sqrt{-g(4)}}{\sqrt{-g}} \hat{T}^m_m \rightarrow -\frac{3l^3}{16\pi G}. \]  

(74)

Here \( g_{(4)mn} \) is the 4-dimensional part \( (d = 4) \) of the metric (52): \( g_{(4)mn} = g_{mn} = l^2 \sinh^2 \frac{t}{l} \tilde{g}_{(4)} \).

The thermodynamical energy \( \hat{E} \) can be evaluated from the bulk side by using the mass formula (175). Now \( q_t = \sqrt{-g_{tt}} = l \sinh \frac{t}{l} \sqrt{-g_{tt}} \) and other components of \( q_{\mu} \) vanish. The expression for \( \hat{E} \) follows:

\[ \hat{E} = M = \frac{l^3}{16\pi G} \frac{3}{4} \int d^3x \sqrt{\gamma} t^2. \]  

(75)

Here the integration \( \int d^3x \cdots \) is the integration with respect to the spatial part of the 4d surface. We shall first consider the case with \( t > t_c \). In order to avoid the orbifold singularity, when Wick-rotating the time coordinate, the radial coordinate \( r \) has to be periodic with the periodicity of the inverse of the Hawking temperature \( T_H \) (67): \( r \sim r + \frac{1}{T_H} \). As a result

\[ \hat{E} = \frac{l^3}{16\pi G} \frac{3 V_2}{4 T_H}. \]  

(76)

Note that the energy is time dependent. When considering the case with \( t < t_c \), the role of \( t \) and \( r \) is interchanged. With \( r \) now as the time coordinate,
one can define the integration\(^4\) \(\int d^3 x \sqrt{\gamma^2} t^2\) as \(\int d^3 x \sqrt{\gamma^2} t^2 = V_2 \int_0^{t_c} dt t^2\) and the energy as

\[
\hat{E} = \frac{l^3}{16\pi G} \frac{t_c^3 V_2}{4} .
\]  

(77)

In the case \(\tilde{k} = -k = 0\), Eq.(66) implies \(\mu = t_c^3\) and Eq.(77) can be rewritten as

\[
\hat{E} = \frac{l^3}{16\pi G} \tilde{\mu} V_2 .
\]  

(78)

This expression differs from (69) by a factor \(\frac{1}{12}\). This discrepancy could be due to the fact that in the cosmological models the definition of the thermodynamical quantities could be modified. Another reason could be that definition of energy inside and outside of \(t_c\) is different. It is however interesting that there is an analogous phenomenon within standard AdS/CFT correspondence. Namely, in the calculation of the free energy for the 5d SAdS black hole on the gravity side and on the dual field theory side there is a numerical discrepancy. This discrepancy is attributed to the fact that the gravity side the calculation corresponds to the result for a non-perturbative, strongly coupled field theory, while the calculation on the field theory side is that of the perturbative (one-loop) CFT.

In conclusion, in this section we demonstrated for several time-dependent explicit examples that some aspects of standard AdS/CFT correspondence are applicable. In particular, we spelled out the calculation of the quantity which plays a role of the holographic anomaly on the gravity side and found agreement with that on the CFT side.

4 Scalar propagator

In this section we shall calculate the propagator of the scalar field \(\phi\) with mass \(m\) in the time-dependent background (57). The aim is to determine on the gravity side the scaling dimensions of these fields and relate the results to the possible dual field theory interpretation.

For simplicity, the space with \(k = -\tilde{k} = 0\) is taken. We also specialize to the region near the horizon. Note that the region far from the horizon

\(^4\)There is a curvature singularity at \(t = 0\). Then there might be subtlety in the integration. Since the integrand is not singular at \(t = 0\), we now proceed by assuming that we can integrate from \(t = 0\).
yields the results that are close to those in the pure AdS space-time. Even for \( k \neq 0 \) case, the qualitative structure does not change.

As a starting point we redefine the coordinate \( t \) as

\[
t = t_c + s ,
\]

and assume \( 0 \leq s \ll t_c \). In this case the Klein-Gordon equation has the following form:

\[
0 = \left( \Box - m^2 \right) \phi
= \frac{1}{\sinh^4 \frac{y}{t}} \partial_y \left( \sinh^4 \frac{y}{t} \partial_y \phi \right) + \frac{1}{l^2 \sinh^2 \frac{y}{t}} \left( -\frac{3}{t^3} \partial_s \left( s \partial_s \phi \right) + \frac{1}{3t^3 s} \partial_s^2 \phi \right)
+ \frac{1}{t^2 l^2 \sinh^2 \frac{y}{t}} \sum_{i=1,2} \frac{\partial^2 \phi}{\partial x_i^2} - m^2 \phi .
\]

Introducing new coordinates \( \xi \) and \( \eta \) by

\[
\xi = s^{\frac{1}{2}} \cosh \left( \frac{3t_c}{2} r \right) , \quad \eta = s^{\frac{1}{2}} \sinh \left( \frac{3t_c}{2} r \right) ,
\]

one finds

\[
-\frac{3}{t^3} \partial_s \left( s \partial_s \phi \right) + \frac{1}{3t^3 s} \partial_s^2 \phi = \frac{3}{4t^3} \left( -\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \phi ,
\]

which indicates that in this case the contribution associated with coordinates \( s \) and \( r \) is that of the flat space-time. In the momentum space one replaces \( \frac{1}{t^2} \partial^2_{x_i} \) by \( -k^2 \) and \( \frac{3}{4t^3} \left( -\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \) by \( \omega^2 \). Then Eq.(80) can be rewritten as

\[
0 = \left( \Box_y - m^2 \right) \phi
\]

\[
\Box_y \equiv \frac{1}{\sinh^4 \frac{y}{t}} \partial_y \left( \sinh^4 \frac{y}{t} \partial_y \phi \right) + \frac{\omega^2 - k^2}{l^2 \sinh^2 \frac{y}{t}} \phi .
\]

If we define the parameters \( \mu \) and \( \nu \) by

\[
\mu^2 = \frac{9}{4} - \omega^2 + k^2 , \quad \nu (\nu + 1) = \frac{15}{4} + l^2 m^2 ,
\]

or

\[
\mu = \sqrt{\frac{9}{4} - \omega^2 + k^2} , \quad \nu = -\frac{1}{2} + \sqrt{4 + l^2 m^2} ,
\]

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the solutions of Eq.(83) are given by the associated Legendre functions $P^\mu_\nu$ and $Q^\mu_\nu$:

$$\phi(y) = \phi_P(y), \phi_Q(y),$$

$$\phi_P(y) \equiv \left( \sinh \frac{y}{l} \right)^{\frac{3}{2}} P^\mu_\nu \left( \cosh \frac{y}{l} \right),$$

$$\phi_Q(y) \equiv \left( \sinh \frac{y}{l} \right)^{\frac{3}{2}} Q^\mu_\nu \left( \cosh \frac{y}{l} \right). \quad (86)$$

Since $P^\mu_\nu(x) \rightarrow \text{constant}, Q^\mu_\nu(x) \rightarrow \infty$ when $x \rightarrow 0$ and $P^\mu_\nu(x) \rightarrow \infty, Q^\mu_\nu(x) \rightarrow 0$ when $x \rightarrow +\infty$, we may define a propagator $G(y_1, y_2; \omega, k)$ by

$$G(y_1, y_2; \omega, k) = C (\phi_P(y_1) \phi_Q(y_2) \theta(y_2 - y_1) + \phi_P(y_2) \phi_Q(y_1) \theta(y_1 - y_2)). \quad (87)$$

Here $C$ is a normalization constant and $\theta(x)$ is a step function defined by

$$\theta(x) \equiv \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}. \quad (88)$$

In this case

$$\Box_{\gamma_1} G(y_1, y_2; \omega, k)$$

$$= \frac{C}{l \sinh^2 \frac{y_1}{l}} \left( -P^\mu_\nu \left( \cosh \frac{y_1}{l} \right) Q^\mu_\nu \left( \cosh \frac{y_1}{l} \right) \\ + P^\mu_\nu \left( \cosh \frac{y_1}{l} \right) Q^\mu_\nu \left( \cosh \frac{y_1}{l} \right) \right) \delta(y_1 - y_2)$$

$$= -\frac{C}{l \sinh^4 \frac{y_1}{l}} \left( \frac{2^{2\mu}e^{2\mu\pi i}}{\Gamma \left( \frac{\nu + \mu + 1}{2} \right) \Gamma \left( \frac{\nu - \mu + 1}{2} \right)} \right). \quad (89)$$

Here the following formula for the associated Legendre functions was used:

$$-\frac{dP^\mu_\nu(x)}{dx} Q^\mu_\nu(x) + P^\mu_\nu(x) \frac{dQ^\mu_\nu(x)}{dx} = \frac{2^{2\mu}e^{2\mu\pi i}}{(1 - x^2)^\frac{\nu + \mu + 1}{2}} \Gamma \left( \frac{\nu + \mu + 1}{2} \right) \Gamma \left( \frac{\nu - \mu + 1}{2} \right). \quad (90)$$
Since $\sqrt{-g} \propto l^4 \sinh^4 \frac{y}{l}$, the normalization constant $C$ can be chosen to be:

$$\frac{1}{C} = -\frac{l^3 2^{2\mu} e^{2\mu \pi i} \Gamma \left( \frac{\nu+\mu+1}{2} \right) \Gamma \left( \frac{\nu-\mu}{2} + 1 \right)}{\Gamma \left( \frac{\nu-\mu+1}{2} \right) \Gamma \left( \frac{\nu-\mu}{2} + 1 \right)}.$$  

(91)

In the limit $x \to +\infty$, $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ behave as:

$$P_\nu^\mu(x) \sim \frac{\Gamma \left( \nu + \frac{\mu}{2} \right) (2x)^\nu}{\sqrt{\pi} \Gamma (\nu - \mu + 1)} , \quad Q_\nu^\mu(x) \sim \frac{e^{\mu \pi i} \sqrt{\pi} \Gamma (\nu + \mu + 1)}{\Gamma \left( \nu + \frac{\mu}{2} \right) (2x)^{\nu+1}}.$$  

(92)

For large $y$

$$\phi_P(y) \sim e^{-\left( \frac{\nu}{2} - \nu \right) y} = e^{-\left( 2 - \sqrt{4 + l^2 m^2} \right) y},$$
$$\phi_Q(y) \sim e^{-\left( \frac{\nu}{2} + \nu \right) y} = e^{-\left( 2 + \sqrt{4 + l^2 m^2} \right) y}.$$  

(93)

Therefore the scalar field $\phi$ with mass $m$ corresponds to the conformal field with the conformal weight $h$ given by

$$2h = 2 \pm \sqrt{4 + l^2 m^2},$$  

(94)

Interesting, despite the fact that we started from the cosmological AdS space, the above result is in agreement with the standard $\text{AdS}_5/\text{CFT}_4$ correspondence. The metric (57) has the form of (63) in the limit of $\rho \to 0$. Then ss clear from the arguments in section 3, the system has an invariance under the scale transformation (34). Then there might exist a corresponding CFT even for these specific time-dependent backgrounds.

In the limit where $y_1 = y_2 = y \to \infty$, $G(y_1, y_2; \omega, k)$ behaves as

$$G(y_1, y_2; \omega, k) \sim 8Ce^{-4\pi \omega} e^{\mu \pi i} \frac{\Gamma (\nu + \mu + 1)}{(\nu + \frac{1}{2}) \Gamma (\nu - \mu + 1)}$$
$$= -\frac{e^{-4\pi \omega} \Gamma \left( \frac{\nu-\mu+1}{2} \right) \Gamma \left( \frac{\nu+\mu}{2} + 1 \right)}{l^3 2^{2\mu} e^{2\mu \pi i} \Gamma \left( \frac{\nu+\mu+1}{2} \right) \Gamma \left( \frac{\nu-\mu}{2} + 1 \right) \Gamma \left( \nu + \frac{1}{2} \right) \Gamma (\nu - \mu + 1)},$$  

(95)

which has poles at

$$\nu - \mu + 1 = -2N, \quad N = 0, 1, 2, 3, \cdots.$$  

(96)
Employing (84) or (86), we obtain the following relationship:

\[ \omega^2 - k^2 = m_\omega^2 = \frac{9}{4} + \left\{ \frac{1}{2} + 2N + \sqrt{4 + l^2 m_\omega^2} \right\}. \]  

(97)

Eq.(97) gives an effective mass in the 4d space-time associated with \((t, r, x^i)\) coordinates. Using (81), the time evolution of the scalar field \(\phi\) is of the form

\[ \phi \propto e^{i(-\omega_\xi t + \omega_\eta \xi)} = e^{-i\omega_\xi T \cosh \left( \frac{3}{2} r - \theta_0 \right)}. \]  

(98)

Here the components of \(\omega\) are written as

\[ \omega_\xi = \omega \cosh \theta_0, \quad \omega_\eta = \omega \sinh \theta_0. \]  

(99)

A massless scalar field \(\phi\) depends only on the coordinate \(t\) or \(s\). Then Eq.(80) reduces as

\[ 0 = \partial_s (s \partial_s \phi), \]  

(100)

whose solution is given by

\[ \phi = \alpha_0 \ln \frac{s}{s_0}. \]  

(101)

Here \(\alpha_0\) and \(s_0\) are constants of the integration. The solution (101) diverges at the horizon \(s = 0\). In fact, Eq.(101) has the form of the scalar propagator in 2d space-time, however with the non-trivial time dependence.

The above results indicate again that the scalar propagator in the time-dependent AdS backgrounds has the structure that indicates the scaling dimensions of the fields in agreement with the standard AdS/CFT correspondence.

### 5 Particle creation

Time-dependent backgrounds often imply multiple vacua. In such cases one vacuum should be an excited state of another vacuum. In particular when the out-vacuum, corresponding to an out-state, is different from the in-vacuum, corresponding to an in-state, the particle creation effects occur. For example, a real scalar field may be expanded by a set of solutions \(\{u_k^{\text{in}}, u_k^{\text{in*}}\}\) or \(\{u_k^{\text{out}}, u_k^{\text{out*}}\}\) of the Klein-Gordon equation as

\[
\phi = \sum_k \left\{ a_k^{\text{in}} u_k^{\text{in}} + a_k^{\text{in*}} u_k^{\text{in*}} \right\} = \sum_k \left\{ a_k^{\text{out}} u_k^{\text{out}} + a_k^{\text{out*}} u_k^{\text{out*}} \right\}.
\]  

(102)
Here annihilation operators are denoted by $a^{\text{in, out}}_k$ and creation operators by $a^{\text{in, out*}}_k$. If two sets of solutions are related by
\begin{equation}
    u_{\text{in}}^k = \alpha_k u_{\text{out}}^k + \beta_k u_{\text{out*}}^k ,
\end{equation}
we have
\begin{equation}
    a_{\text{out}}^k = \alpha_k a_{\text{in}}^k + \beta_{k}^* a_{\text{in*}}^k .
\end{equation}
Since the number operator corresponding to $k$ in the out-mode is $N_{\text{out}}^k = a_{\text{out*}}^k a_{\text{out}}^k$, the number of particles of this mode in the in-vacuum is given by $|\beta_k|^2$.

One may consider the two-point functions of two $\phi$, as in [27]:
\begin{equation}
    \langle 0 | \phi(1) \phi(2) | 0 \rangle_{\text{in}} = \sum_k u_{\text{in}}^k(1) u_{\text{in*}}^k(2) .
\end{equation}
Here we denote the coordinates of the respective fields by 1 or 2. On the other hand,
\begin{equation}
    \langle 0 | \phi(1) \phi(2) | 0 \rangle_{\text{in}} = \sum_k \frac{u_{\text{out}}^k(1) u_{\text{out*}}^k(2)}{\alpha_k} \langle 0 | 0 \rangle_{\text{in}}
    = \left\{ \langle 0 | \phi(1) \phi(2) | 0 \rangle_{\text{in}} + \sum_k \frac{\beta_k}{\alpha_k} u_{\text{out}}^k(1) u_{\text{in*}}^k(2) \right\} \langle 0 | 0 \rangle_{\text{in}} .
\end{equation}
For the propagator, the same relation follows:
\begin{align}
    iG_{\text{out-in}}(1, 2) &= iG_{\text{in-in}}(1, 2) + \sum_k \frac{\beta_k}{\alpha_k} u_{\text{out}}^k(1) u_{\text{in*}}^k(2) \langle 0 | 0 \rangle_{\text{in}} \\
    iG_{\text{out-in}}(1, 2) &\equiv \langle 0 | T \phi(1) \phi(2) | 0 \rangle_{\text{in}} ,
    iG_{\text{in-in}}(1, 2) &\equiv \langle 0 | T \phi(1) \phi(2) | 0 \rangle_{\text{in}} .
\end{align}
Here $T$ is the time ordering operator.

Recent interest in the AdS/CFT correspondence in time-dependent backgrounds [4, 28], motivated the study [29, 30] of particle creation for the 4d deSitter brane, which can be regarded as an inflationary Universe, embedded in the 5d AdS bulk. Within the AdS/CFT correspondence, the partition function in the bulk gravity corresponds to the generating function of the dual CFT. The CFT generating function for the operator $O$ between the in- and the out-vacuum is given by the bulk partition function $Z_{\text{in-out}}(\phi_0)$,
which is calculated by imposing proper initial and final boundary conditions for the field in the bulk

\[ \langle 0 | T e^{i \int_{\text{boundary}} \phi_0 \partial_t} | 0 \rangle_{\text{in}} = Z_{\text{in} \rightarrow \text{out}} (\phi_0) . \]  

Let us assume that the metric of the bulk space-time has the following form:

\[ ds^2 = \frac{1}{z^2} \left( dz^2 + \tilde{g}_{mn} dx^m dx^n \right) \]  

and assume that there is a boundary at \( z = 0 \). Then the bulk-boundary propagator \( G_B (x^m, z; x^m') \), which connects the boundary value \( \phi_0 \) of the scalar field \( \phi \), is given by the bulk propagator \( G_F \) [31]

\[ G_B (x^m, z; x^m') \phi_0 (x^m') = 2 \hat{\nu} \lim_{z' \to 0} z'^{-2 h_+} G_F (x^m, z; x^m, z') \phi_0 (x^m') . \]  

Here \( h_\pm = 1 \pm \frac{\nu}{2} \), \( \hat{\nu} = \sqrt{4 + m^2 l^2} \) (\( m \) is the mass of the scalar field). In [27], by combining (107), (108), and (111), the following expression is obtained

\[ \langle 0 | \phi(1) \phi(2) | 0 \rangle_{\text{in}} = \frac{\langle 0 | \phi(1) \phi(2) | 0 \rangle_{\text{in}}}{\langle 0 | 0 | 0 \rangle_{\text{in}}} \]

\[ + 2 i \hat{\nu} \lim_{z' \to 0, \epsilon \to 0} e^{-2 h_+} \frac{\partial}{\partial z'} \left\{ z'^{-2 h_+} u_k^* (x^m, z) u_k^* (x^m', z') \right\} \bigg|_{z' = \epsilon} . \]

The second term is responsible for particle creation effect when applying AdS/CFT correspondence to the time-dependent backgrounds.

In this section, we investigate the particle production for the scalar field \( \phi \) satisfying the Klein-Gordon equation (80) in the space-time background (57). Eqs.(80) and (82) indicate that \( \phi \) satisfies

\[ - \frac{3}{t_c^3} \partial_s (s \partial_s \phi) + \frac{1}{3 t_c^3} \partial_r^2 \phi = \frac{3}{4 t_c^3} \left( - \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \phi = \omega^2 \phi . \]  

24
In the coordinates $\xi$ and $\eta$, the 2d space-time is flat. When one regards $\xi$ as a time coordinate, there is no particle creation. However, the natural time coordinate should be $s$, which is related to the global time coordinate $t$. As the coordinate system given by $s$ and $r$ is not inertial one, the particle creation takes place.

For $\phi \propto e^{-ipr}$, where $p$ is a constant corresponding to the momentum conjugate to $r$, Eq.(113) reduces as

$$0 = \frac{3}{t_c^3} \partial_s (s \partial_s \phi) + \frac{p^2}{3t_c^5 s} \phi + \omega^2 \phi . \quad (114)$$

With the redefinition of $s$ in terms of $\sigma$:

$$s = \frac{\sigma^2}{\omega^2} , \quad (115)$$

one obtains the Bessel differential equation:

$$0 = \frac{\partial^2 \phi}{\partial \sigma^2} + \frac{1}{\sigma} \frac{\partial \phi}{\partial \sigma} + \left( \tilde{\nu}^2 - \frac{1}{\sigma^2} \right) \phi , \quad \tilde{\nu}^2 = \frac{4p^2}{9t_c^2} , \quad (116)$$

whose solution can be obtained via the Hankel functions $H^{(1)}_{i\tilde{\nu}}(\sigma)$ and $H^{(2)}_{i\tilde{\nu}}(\sigma)$. Since $0 \leq s < +\infty$, then $0 \leq \sigma < +\infty$. In the following, however, we analytically continue $\sigma$ into the region where $\sigma$ is negative, i.e. $-\infty < \sigma < \infty$, and regard the limit $\sigma \to +\infty$ to correspond to the out-state and $\sigma \to -\infty$ to the in-state.

For positive and large $\sigma$, the Hankel functions $H^{(1)}_{i\tilde{\nu}}(\sigma)$ and $H^{(2)}_{i\tilde{\nu}}(\sigma)$ behave as

$$H^{(1)}_{i\tilde{\nu}}(\sigma) \sim \sqrt{\frac{2}{\pi \sigma}} e^{i \left( \sigma - \frac{2i\tilde{\nu} + 1}{4} \right)} , \quad H^{(2)}_{i\tilde{\nu}}(\sigma) \sim \sqrt{\frac{2}{\pi \sigma}} e^{-i \left( \sigma - \frac{2i\tilde{\nu} + 1}{4} \right)} . \quad (117)$$

In this limit $H^{(1)}_{i\tilde{\nu}}(\sigma)$ correspond to the negative frequency modes of the out-vacuum and $H^{(2)}_{i\tilde{\nu}}(\sigma)$ to positive frequency ones. Since the Bessel differential equation (116) is invariant under analytical continuation: $\sigma \to -\sigma$, then $H^{(1)}_{i\tilde{\nu}}(-\sigma)$ and $H^{(2)}_{i\tilde{\nu}}(-\sigma)$ are also solutions. Eq.(117) then implies that $H^{(1)}_{i\tilde{\nu}}(-\sigma)$ correspond to positive frequency modes of the in-vacuum and
\[ H_{i\tilde{\nu}}^{(2)}(\sigma) \] to negative frequency ones. Therefore

\[
\phi = \sum_{\nu} \left( b_{\nu}^{\text{out}*} H_{i\tilde{\nu}}^{(1)}(\sigma) + a_{\nu}^{\text{out}} H_{i\tilde{\nu}}^{(2)}(\sigma) \right)
= \sum_{\tilde{\nu}} \left( b_{\tilde{\nu}}^{\text{in}*} H_{i\tilde{\nu}}^{(2)}(-\sigma) + a_{\tilde{\nu}}^{\text{in}} H_{i\tilde{\nu}}^{(1)}(-\sigma) \right)
\].

(118)

The out-vacuum \(|0\rangle_{\text{out}}\) and in-vacuum \(|0\rangle_{\text{in}}\) can be defined by

\[
\begin{align*}
\begin{array}{ll}
a_{\nu}^{\text{out}}|0\rangle_{\text{out}} &= b_{\nu}^{\text{out}}|0\rangle_{\text{out}} = 0, \\
a_{\nu}^{\text{in}}|0\rangle_{\text{out}} &= b_{\nu}^{\text{in}}|0\rangle_{\text{in}} = 0.
\end{array}
\end{align*}
\]

(119)

The analytic continuation \(\sigma \rightarrow e^{i\theta}\sigma\) and taking \(\theta = \{0, 1\}\) implies that for \(\sigma \rightarrow -\sigma\) one gets

\[
\begin{align*}
H_{i\tilde{\nu}}^{(1)}(-\sigma) &= -e^{\tilde{\nu}\pi} H_{i\tilde{\nu}}^{(2)}(\sigma) \\
H_{i\tilde{\nu}}^{(2)}(-\sigma) &= 2 \cosh (\tilde{\nu}\pi) H_{i\tilde{\nu}}^{(2)}(\sigma) + e^{\tilde{\nu}\pi} H_{i\tilde{\nu}}^{(1)}(\sigma)
\end{align*}
\]

(120)

Then the coefficients of the Bogolubov transformation between the in- and out-vacuum are of the form:

\[
\begin{align*}
b_{\nu}^{\text{out}*} &= e^{-\tilde{\nu}\pi} b_{\nu}^{\text{in}*}, \\
a_{\nu}^{\text{out}} &= 2 \cosh (\tilde{\nu}\pi) b_{\nu}^{\text{in}*} - e^{\tilde{\nu}\pi} a_{\nu}^{\text{in}},
\end{align*}
\]

(121)

or

\[
\begin{align*}
b_{\nu}^{\text{in}*} &= e^{\tilde{\nu}\pi} b_{\nu}^{\text{out}*}, \\
a_{\nu}^{\text{in}} &= -e^{-\tilde{\nu}\pi} a_{\nu}^{\text{out}} + 2 \cosh (\tilde{\nu}\pi) b_{\nu}^{\text{out}*}
\end{align*}
\]

(122)

The number \(N_{\tilde{\nu}}\) of created out-particles corresponding to \(\tilde{\nu}\) in in-vacuum is given by

\[
N_{\tilde{\nu}} = \text{in} \langle 0 | \left\{ a_{\nu}^{\text{out}*} a_{\nu}^{\text{out}} + b_{\nu}^{\text{out}*} b_{\nu}^{\text{out}} \right\} |0\rangle_{\text{in}} = 4 \cosh^2 (\tilde{\nu}\pi)
\]

(123)

which becomes large for large \(\tilde{\nu}\). From Eq.(116), large \(\tilde{\nu}\) implies large momentum \(p\).

Note that this calculation is done by analytically continue \(\sigma\) into unphysical region with negative \(\sigma\), while the original region for \(\sigma\) was: \(0 \leq \sigma < +\infty\). The restriction to the physical region \(0 \leq \sigma < +\infty\), would imply that \(N_{\tilde{\nu}}\) is now smaller. While we were unable to obtain analytic results for \(N_{\tilde{\nu}}\) in the physical region \(0 \leq \sigma < +\infty\), let us make the following comment. Since the Hankel functions \(H_{i\tilde{\nu}}^{(1,2)}(\sigma)\) behave as \(\sigma^{\pm i\nu}\) when \(\sigma\) is small, one could define
a new time coordinate $\tilde{\sigma}$ by $\tilde{\sigma} = \ln \sigma + \sigma$. The behavior of the Hankel functions when $\sigma$ or $\tilde{\sigma}$ is large is not changed: $H_{i\tilde{\nu}}^{(1,2)}(\sigma) \sim H_{i\tilde{\nu}}^{(1,2)}(\tilde{\sigma})$ but when $\sigma$ is small, we find
\[
\sinh (\tilde{\nu} \pi) H_{i\tilde{\nu}}^{(1)}(\tilde{\sigma}) \sim -2i\tilde{\nu} e^{-i\tilde{\nu} \tilde{\sigma}} + 2^{-i\tilde{\nu}} e^{i\tilde{\nu} \tilde{\sigma}} \\
\sinh (\tilde{\nu} \pi) H_{i\tilde{\nu}}^{(2)}(\tilde{\sigma}) \sim 2i\tilde{\nu} e^{-i\tilde{\nu} \tilde{\sigma}} + 2^{-i\tilde{\nu}} e^{i\tilde{\nu} \tilde{\sigma}} .
\] (124)

The part behaving as $e^{-i\tilde{\nu} \tilde{\sigma}}$ ($e^{i\tilde{\nu} \tilde{\sigma}}$) can be identified with the positive (negative) frequency part. Then the Bogolubov coefficient $\beta$ in (103) behaves exponentially $\beta \sim e^{\tilde{\nu} \pi}$ again. In any case, since for the region $-\infty < \sigma < \infty$, there is a symmetry $\sigma \rightarrow -\sigma$, we expect, that in the interval $0 < \sigma < \infty$, $N_{\tilde{\nu}}$ is reduced by a factor of two relative to (123).

At first sight the result that $N_{\tilde{\nu}}$ grows exponentially with $\tilde{\nu}$ is puzzling, since for example in the deSitter space, creation of particles with high momenta is exponentially suppressed [32]. The resolution to this puzzle lies in the fact that the calculation was performed originally (see Eq.(79)) only the region $|s| \ll t_c$ and it turns out the values of $\tilde{\nu}$ are bounded from below to be $\leq \mathcal{O}(\infty)$. This can be understood in the following way. Note that $t_c$ plays a role of a cutoff scale. When $t \sim t_c$, the 4d slice of the metric (56) with constant $y$ has the following form:
\[
\bar{g}_{mn}(x)dx^m dx^n \sim -\frac{ds^2}{4\pi T_H s} + 4\pi T_H sdr^2 + t_c^2 \sum_{i,j=1}^{d-2} \gamma_{ij} dx^i dx^j .
\] (125)

The form of the metric (125) implies that in this limit the the scale factor associated with $s$ is large, and for the the radial direction $r$ the scale factor is small (inversely proportional to the scale factor of $s$). Thus, the large scale of $s$ (or $t$) corresponds to the small (inverse) scale of $r$. Since $p$ in (114) is a momentum, dual to the radial direction $r$, it would scale as $t$, which is consistent with the dimensionless expression of $\tilde{\nu}$ in (116). The expression of $\tilde{\nu}$ in (116) also implies that $\tilde{\nu}$ cannot be large. Thus, the calculation of $N_{\tilde{\nu}}$ in (123) is valid only for $\tilde{\nu} \leq \mathcal{O}(1)$. On the other hand for large $\tilde{\nu}$, one should consider the region $t \gg t_c$, where the metric of the slice with constant $y$ is pure deSitter. In this case, the particle production for large $\tilde{\nu}$ should be suppressed exponentially [32], which is consistent with the result in [27]. Thus, the total number of the produced particles should become finite. While the above considerations shed light on the qualitative picture
of particle production for this background, it would be interesting to obtain
the quantitative result for $N_\nu$ in the full range of $t$.

Let us now consider the last term in (112). From the metric (57), one
may identify $z = l^{-1} \sinh^{-1} \frac{y}{l} \sim 2e^{-\frac{y}{l}}$. We also assume the conformal weight $h_+$ in (112) as $h$ in (94) with (+)-sign and $\hat{\nu}$ as $\sqrt{4 + l^2m^2}$. It follows that $u_k^*$ is given by combining (86) and (118), etc., as:

$$u_k^* \sim \sqrt{C} e^{ipr} H^{(2)}_{iv}(-\sigma) e^{itc} \sum_{i=1,2} k_ix^i \left( \sinh \frac{y}{l} \right)^{-\frac{1}{2}} \hat{Q}_\nu^\mu \left( \cosh \frac{y}{l} \right).$$ (126)

Here $C$ is a normalization constant and $x^i$'s ($i = 1, 2$) are coordinates defined in (57). It follows from (93), that $\phi_Q(y)$ corresponds to the conformal field with the weight $h$ with (+) signs. In this case the last term in (112) has the following form:

$$2i\sqrt{4 + l^2m^2} C e^{ip(r+r')} H^{(2)}_{iv}(-\sigma) H^{(2)}_{iv}(\sigma') e^{itc} \sum_{i=1,2} k_i(x^i + x'^i)$$
$$\times \left( \frac{5}{2} + \sqrt{4 + l^2m^2} \right) 2^{-2-2\sqrt{4+l^2m^2}}$$
$$\times \frac{\pi e^{2\mu i} \Gamma(\nu + \mu + 1)}{\Gamma \left( \nu + \frac{3}{2} \right)^2}. \tag{127}$$

Here $\nu$ and $\mu$ are defined in (85). $C$ should also depend on $\nu$ and $\mu$ as in (91). If $C$ is given by $|C|$ in (91), Eq.(127) is rewritten as

$$2i\sqrt{4 + l^2m^2} \frac{\Gamma(\nu - \mu + 1)}{\Gamma \left( \nu - \mu + \frac{1}{2} + 1 \right) \Gamma \left( \frac{\nu - \mu}{2} + 1 \right)}$$
$$\times e^{ip(r+r')} H^{(2)}_{iv}(-\sigma) H^{(2)}_{iv}(\sigma') e^{itc} \sum_{i=1,2} k_i(x^i + x'^i)$$
$$\times \left( \frac{5}{2} + \sqrt{4 + l^2m^2} \right) 2^{-2-2\sqrt{4+l^2m^2}}$$
$$\times \frac{\pi e^{2\mu i} \Gamma(\nu + \mu + 1)}{\Gamma \left( \nu + \frac{3}{2} \right)^2}. \tag{128}$$

In the above expressions (127) and (128), $y$-dependence is cancelled. In order to obtain the final expression, we have to integrate the obtained expressions with respect to $p$ (or $\tilde{\nu}$), $\omega$, and $k$. However, we were unable to obtain the

28
explicit integration results. One may be able to determine dominant regions of the integral and obtain an approximate expression. In any case, (127) or (128) imply that there will be specific modifications on the CFT side, due to particle creation.

The geometry of the boundary where the CFT is given by (56), with \( \tilde{k} = 0 \) and \( \mu = -\tilde{\mu} \). When \( t = t_c \), the coordinate \( r \) is degenerate and the spatial region becomes 2-dimensional flat space given by the coordinates \( x^i \). If we compactify \( x^i \), the spatial topology can be a two-torus. When \( t \) is large, the space-time geometry becomes 3-dimensional deSitter space. If we compactify the coordinates \( r \) and \( x^i \), the geometry of the spatial part is \( S^1 \times T^2 \sim T^3 \). Here \( S^1 \) corresponds to \( r \) and \( T^2 \) to \( x^i \). The radius of \( S^1 \) is proportional to the inverse of time \( t \) and that of \( T^2 \) to time \( t \) itself.

Creation of particles gives an extra contribution to the energy-momentum tensor and it will shift the energy of the vacuum. Then it is natural to expect that the surface stress-tensor (70) will also be modified by the contribution from the created particles, which might resolve the discrepancy between the expressions of energy in (69) and (78). It is interesting, however, that the trace of the stress-tensor (conformal anomaly) will be the same for any (in- or out-) state which was actually demonstrated in section 3.

6 Wilson loop

In this section the Wilson loop [15] is found in the background (57) with \( k = 0 \). The formal procedure treating the Wilson loop is a standard one. First one considers a loop on the boundary \( (y \to \infty \) in (57)) of the bulk space-time and finds a configuration of the string, whose boundary is the loop, so that the Nambu-Goto action (190) has its minimum. For the case that the space-time is static, we may take a rectangular loop, whose length is \( T \) and \( L \) in the (Euclidean) time direction and in the spatial direction, respectively. If \( T \gg L \), the potential energy between a quark and an anti-quark is given by dividing the value of the action, evaluated for the string configuration, by \( T \). Since the space-time under discussion is now time-dependent, the metric inside the Nambu-Goto action (190) becomes time-dependent as well. Moreover, we cannot consider only rectangular loops.

\footnote{Let us stress again, that the interpretation of this quantity, calculated on the gravity side, as a Wilson loop is valid only if the AdS/CFT correspondence is applicable.}
In principle, one should consider a loop of an arbitrary shape, which is the boundary of the string. After that one divides the string by the small time interval. In the time interval, the time distance between quark and anti-quark can be defined. Furthermore, in this time interval one may often neglect the time dependence using an adiabatic approximation. By such a procedure, the potential between a quark and an anti-quark can be defined.

We start with the near-horizon region \((0 \leq s \ll t_c)\) for \((79)\). Then the metric \((57)\) has the following form:

\[
\begin{align*}
\text{ds}^2 &\sim dy^2 + l^2 \sinh^2 \frac{y}{t} \left( -\frac{ds^2}{3t_c s} + 3t_c s dr^2 + t_c^2 \sum_{i,j=1,2} \left( dx^i \right)^2 \right). \\
\end{align*}
\]  

(129)

We also employ Wick rotation and the following redefinition of the coordinates

\[
s = -\frac{3t_c}{4} u^2, \quad r = \frac{2i}{3t_c} v, \quad x^i = \frac{1}{t_c} \hat{x}^i.
\]

(130)

Since the region near the boundary \((y \to +\infty)\) is taken, it is assumed \(y\) is large. The corresponding metric is:

\[
\begin{align*}
\text{ds}^2 &= dy^2 + \frac{l^2}{4} e^{2y} \left( du^2 + u^2 dv^2 + \sum_{i,j=1,2} \left( d\hat{x}^i \right)^2 \right). \\
\end{align*}
\]  

(131)

We consider a short time interval given by \(u_0 \leq u \leq u_0 + \Delta u\) \((0 < \Delta u \ll u_0)\) and thus \(u\) is close to being constant. In this case the Nambu-Goto string action \((190)\) can be evaluated, by choosing \(\tau = u\) and \(\sigma = v\) in \((190)\). As \(u\) is almost constant, the configuration does not depend on \(u\) in the interval \(u_0 \leq u \leq u_0 + \Delta u\), as \(y = y(v)\) and \(\hat{x}^i = 0\) is taken. Then the Nambu-Goto action has the following form:

\[
S_N = \frac{\Delta u l}{2} \int dv \sqrt{\left( \partial_v y \right)^2 + \frac{l^2 u_0^2}{4} e^{2y}}.
\]

(132)

The Euler-Lagrange equation \((132)\),

\[
0 = \frac{e^y}{2} \sqrt{\left( \partial_v y \right)^2 + \frac{l^2 u_0^2}{4} e^{2y}} + \frac{l^2 u_0^2 e^{2y}}{8 \sqrt{\left( \partial_v y \right)^2 + \frac{l^2 u_0^2}{4} e^{2y}}} \\
- \partial_v \left( \frac{le^y \partial_v y}{2 \sqrt{\left( \partial_v y \right)^2 + \frac{l^2 u_0^2}{4} e^{2y}}} \right),
\]

(133)
shows that the following quantity is constant:

$$E = \frac{l^3 u_0^2 e^{\frac{3v}{l}}}{8 \sqrt{(\partial_v y)^2 + \frac{1}{4} u_0^2 e^{\frac{2v}{l}}}}.$$  \hfill (134)

If there is a turning point at \(y = y_0\), where \(\partial_v y = 0\), then

$$e^{\frac{2v_0}{l}} = \frac{4E}{l^2 u_0}. \hfill (135)$$

Therefore, in order for \(y \geq y_0\) to be large, \(\frac{4E}{l^2 u_0} \gg 1\). Now \(y\) is parameterized as

$$e^{-\frac{2v}{l}} = e^{-\frac{2v_0}{l}} \sin \phi. \hfill (136)$$

Then \(\phi = 0\) and \(\phi = \pi\) correspond to the boundary \(y \to \infty\), and \(\phi = \frac{\pi}{2}\) to the turning point \(y = y_0\). Combining (134), (135), and (136), one finds

$$\frac{dv}{d\phi} = \frac{1}{u_0} e^{\frac{2v}{l}} \sqrt{\sin \phi}. \hfill (137)$$

From the metric (131), the distance \(L_{q\bar{q}}\) between a quark and an anti-quark is given by

$$L_{q\bar{q}} = u_0 \int dv = e^{\frac{2v_0}{l}} \int_0^\pi d\phi \sqrt{\sin \phi} = \frac{1}{u_0} \frac{\sqrt{2} \Gamma \left(\frac{3}{4}\right)}{\Gamma \left(\frac{3}{2}\right)}.$$  \hfill (138)

By using (134), (135), (136), and (137), the following expression for the action (132) is obtained:

$$S_N = \frac{\Delta ul^2 e^{\frac{3v}{l}}}{4} \int_0^{2\pi} d\phi \sin^{-\frac{3}{2}} \phi.$$  \hfill (139)

The last expression, however, diverges at \(\phi = 0\) and \(\phi = \pi\). The divergence occurs because it contains the (infinite) self-energy of the quark and the anti-quark. The self-energy can be evaluated by considering the configuration \(0 < y < \infty, x^i = 0\), and \(v\) is constant. Choosing \(\sigma = y\) in the Nambu-Goto action (190), we obtain

$$S_{\text{self}} = \frac{\Delta ul}{2} \int_0^{\infty} e^{\frac{2v}{l}} dy.$$  \hfill (140)
It is natural to divide the region of integration with respect to \( y \) into \( 0 < y < y_0 \) and \( y_0 \leq y < \infty \). In the region \( y_0 \leq y < \infty \), we parameterize \( y \) by (136). The action for the self-energy is

\[
S_{\text{self}} = -\frac{\Delta u l^2}{2} \left( 1 - e^{\frac{y_0}{l}} \right) + \frac{\Delta u l^2}{8} e^{\frac{y_0}{l}} \int_0^{\frac{\pi}{2}} d\phi \frac{\cos \phi}{\sin \frac{\pi}{2} \phi}. \tag{141}
\]

Then the action, subtracted by the self-energy contribution, is given by

\[
S_N - 2S_{\text{self}} = \frac{\Delta u l^2 e^{\frac{y_0}{l}}}{4} \left( 4 + \frac{2\Gamma\left(\frac{3}{2}\right)^2}{\Gamma\left(\frac{3}{2}\right)} \right) + \Delta u l^2 \left( 1 - e^{\frac{y_0}{l}} \right)
= \frac{\Delta u l^2}{2\sqrt{2L}} + \Delta u l^2. \tag{142}
\]

In the last expression, we used (138) and dependence on \( e^{\frac{y_0}{l}} \) is absent. The potential \( V \) between the quark and anti-quark is then given by

\[
V = \frac{l^2}{2\sqrt{2L}} + l^2, \tag{143}
\]

and it does not depend on the time coordinate \( u \) or \( u_0 \).

Of course, the assumption that \( u \) is almost constant might not be a correct one for a very small \( u \). However, it should be valid for not too small \( u \). This consideration shows that the Wilson loop (and possible confinement) could be considered in time-dependent AdS/CFT. The potential (143) is Coulomb-like potential except for the last term, which can be re-absorbed into the self-energy of the quark and the anti-quark. The Coulomb-like behavior in the potential does not change from that in the usual AdS/CFT case, and due to this term the quarks would not be confined. Thus, the particle creation seems to affect the potential; it seems to imply the same behaviour as that for usual AdS space.

### 7 Discussion

In summary, we discussed the properties of 5d cosmological AdS spaces obtained from the SAdS black hole cousins by exchanging the temporal and spatial coordinates and employing an analytic continuation when necessary.
There are similarities between such time-dependent spaces and black holes (cosmological horizon, Hawking temperature). An attempt to address the time-dependent AdS/CFT correspondence, by accounting for the particle creation effects and the presence of multiple vacua, is presented. In particular, we investigate in detail the role of surface counter-terms in time-dependent AdS spaces. We evaluate quantities which are identified with the surface energy-momentum tensor and the (adiabatic) Wilson loop on the gravity side (when AdS/CFT correspondence is applicable). An explicit example of AdS cosmological backgrounds which lead to the correct holographic conformal anomaly is presented. We also addressed properties of a massive scalar propagator and found that the scaling dimensions of these fields indicate an agreement with the dual field theory predictions.

The study of these specific time-dependent backgrounds (as solutions of string theory) indicates that certain aspects of a duality between such spaces and dual field theories in one dimension less can be addressed. Of course, it remains as a challenge to address an analog of such a duality within the realistic expanding Universe.

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A Surface stress tensor and Wilson loop for T-dual backgrounds

In this Appendix, we investigate properties of time-dependent spaces obtained via T-duality of the original AdS space-times. The obtained space-time is not, however, asymptotically AdS. That is the reason why we rel-
egated this investigation to the Appendix. Nevertheless, as starting space was AdS, there may be remnants of the holographic correspondence for the T-dual space-time. We shall calculate the analogs of the surface stress tensor and Wilson loop and argue about a potential dual field theory interpretation.

When a $\sigma$ model in two dimensions has an Abelian or non-Abelian isometry, one can introduce gauge fields by gauging the isometry and impose a constraint which determines the gauge curvature to be zero. After imposing a gauge fixing condition and integrating out gauge fields, one obtains a dual model. In the dual model, there appears a shift of the dilaton fields when we use a regularization which keeps the general covariance.

The non-Abelian $T$-duality [9] can be used to show the equivalence of string models corresponding to different topologies [10]. Such a duality for the $SU(2)$ chiral model has been investigated in [11]. By using the $SU(2)$ $T$-duality, from the static 4d space-time, a non-trivial space-time can be constructed like in [12].

The $T$-duality is a symmetry of the string theory. The low-energy effective action (of the Neveu-Schwarz Neveu-Schwarz sector) of string theory contains the dilaton field and the (rank two) anti-symmetric tensor field. By the $T$-duality, not only the metric but also the dilaton field and the anti-symmetric tensor field are non-trivially modified.

Here we apply the $SU(2)$ $T$-duality to the Schwarzschild-AdS black hole (1). One may write the metric $d\Omega^2_3$ in (1) by using the group element $g$ of $SU(2)$, which is topologically $S^3$.

$$d\Omega^2_3 = \frac{1}{2} \text{tr} g^d g d g^d d g.$$  

Here we parameterize $g$ by 3 parameters $\phi_i$ ($i = 1, 2, 3$):

$$g = e^{i \phi_i \sigma_i}.$$  

Here $\sigma_i$s ($i = 1, 2, 3$) are Pauli matrices. In general, string theory can be formulated as the $\sigma$-model on the 2-dimensional Riemann surface. The target space of $\sigma$-model corresponds to the real space-time, where string propagates. Then by rewriting the $\sigma$-model in an equivalent way, one can observe the $T$-duality of the string theory. Now introducing the 2-dimensional $SU(2)$ gauge field as an auxiliary field, we rewrite the $\sigma$-model. By using $g$ (145), the Lagrangian density of the $\sigma$ model corresponding to SAdS metric is given.
by
\[ \mathcal{L}_0 = \frac{1}{2} \left\{ -f(r) \partial_t \bar{\partial}t + \frac{1}{f(r)} \partial r \bar{\partial}r + \frac{r^2}{2} \text{tr} g^i \partial g g^i \partial \bar{g} \right\} . \] (146)

The above Lagrangian has \( SO(4) \sim SU(2)_L \otimes SU(2)_R \) symmetry. The two \( SU(2) \) transformations are given by
\[ g \to hg \ (SU(2)_L), \quad g \to gh \ (SU(2)_R) . \] (147)
\( (h \) is a group element of \( SU(2) \)). In order to obtain a dual Lagrangian, we gauge \( SU(2)_L \) symmetry by introducing gauge fields \( A \) and \( \bar{A} \),
\[ A = A^i T_i, \quad \bar{A} = \bar{A}^i T_i \] (148)
\[ T^i = \frac{1}{2} \sigma^i , \] (149)
replace
\[ \text{tr} g^i \partial g g^i \partial \bar{g} \to \text{tr} g^i (\partial + A) g g^i (\bar{\partial} + \bar{A}) \bar{g} , \] (150)
and add a term which leads to vanishing of the gauge curvature \( F = [\partial + A, \bar{\partial} + \bar{A}] \),
\[ \mathcal{L}^\text{constraint} = \frac{1}{2} \text{tr} \xi F \] (151)
Here \( \xi \) is an element of \( SU(2) \) algebra
\[ \xi = \xi^i T^i . \] (152)

If we first integrate \( \xi \), we obtain a constraint that the gauge curvature should vanish, which implies that gauge fields are pure gauge. Then one can choose the gauge condition that the gauge fields vanish. The total Lagrangian including the terms (151) and (152) is reduced to the original Lagrangian (146). On the other hand, the dual Lagrangian can be obtained by integrating the gauge fields first. In fact, when we integrate out the gauge fields \( A \) and \( \bar{A} \) first by choosing the gauge condition
\[ \phi^i = 0 , \] (153)
the dual Lagrangian is obtained as
\[ \mathcal{L}^\text{dual} = \frac{1}{2} \left\{ -f(r) \partial t \bar{\partial}t + \frac{1}{f(r)} \partial r \bar{\partial}r \\
+ \frac{2}{r^4 + 4(\xi)^2} \left[ 2r^2 \delta_{ij} - 4\epsilon_{ijk} \xi^k + \frac{8}{r^2} \xi^i \xi^j \right] \partial \xi^i \partial \bar{\xi}^j \right\} . \] (154)
Here $(\xi)^2 = \xi^i\xi^i$. Furthermore, by using the regularization which preserves the general covariance, there is a dilaton term in the dual theory [9],

$$L^{\text{dilaton}} = -\frac{1}{4\pi} R^{(2)} \Phi$

$$\Phi = \ln(r^4 + 4(\xi)^2) . \quad \tag{155}$$

Here $\Phi$ is the dilaton field. As the dilaton field blows up when $r$ is large, the space time is asymptotically not AdS, but corresponds to the so-called dilatonic vacuum with the dilaton blowing up. In this region the string theory is strongly coupled and the supergravity description of the string theory in this limit may be invalid. If a holographic interpretation were still valid, the strongly coupled limit of the CFT should correspond to the classical limit of the string theory, and the supergravity description might still be valid. However, those are speculations and in the following, the calculation of the trace anomaly from the gravity side will be employed to test if this specific aspect of AdS/CFT correspondence could be tested.

Eq.(154) implies that the target space metric is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{2}{r^4 + 4(\xi)^2} \left[ 2r^2\delta_{ij} + \frac{8}{r^2}\xi^i\xi^j \right] d\xi^id\xi^j , \quad \tag{156}$$

and the antisymmetric tensor field also becomes non-trivial

$$B_{\xi^i\xi^j} = -\frac{8\epsilon_{ijk}\xi^k}{r^4 + 4(\xi)^2} . \quad \tag{157}$$

Furthermore, the polar coordinates for $\xi^i$ may be introduced:

$$\begin{cases} 
\xi^1 = \zeta \sin \theta \cos \varphi \\
\xi^2 = \zeta \sin \theta \sin \varphi \\
\xi^3 = \zeta \cos \theta
\end{cases} . \quad \tag{158}$$

Then, dual metric is of the form:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{4}{r^2}d\zeta^2 + \frac{4r^2\zeta^2}{r^4 + 4\zeta^2} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) . \quad \tag{159}$$

We shall now consider an analytic continuation. By replacing

$$t \rightarrow i\tilde{t}, \quad \varphi \rightarrow i\tilde{\varphi} , \quad \tag{160}$$
the following static metric is obtained
\[ ds^2 = -\frac{4r^2 \zeta^2}{r^4 + 4\zeta^2} \sin^2 \theta d\tilde{t}^2 + f(r) d\tilde{\phi}^2 + \frac{1}{f(r)} dr^2 + \frac{4}{r^2} d\zeta^2 + \frac{4r^2 \zeta^2}{r^4 + 4\zeta^2} d\theta^2 . \] (161)

On the other hand if we consider the following analytic continuation:
\[ t \to i \tilde{\phi}, \quad \theta \to \frac{\pi}{2} + i \hat{t}, \] (162)
one obtains the cosmological (time-dependent) metric which is not asymptotically AdS
\[ ds^2 = -\frac{4r^2 \zeta^2}{r^4 + 4\zeta^2} d\tilde{t}^2 + f(r) d\tilde{\phi}^2 + \frac{1}{f(r)} dr^2 + \frac{4}{r^2} d\zeta^2 + \frac{4r^2 \zeta^2}{r^4 + 4\zeta^2} \cosh^2 \hat{t} d\phi^2 . \] (163)

In order to avoid the orbifold singularity at \( r = r_H \), we need to impose the periodicity for the coordinate \( \tilde{\phi} \) in Eqs.(160) and (162):
\[ \tilde{\phi} \sim \tilde{\phi} + \frac{1}{T_H} . \] (164)

Here \( T_H \) is the Hawking temperature (3). Then, naturally \( r \) may be restricted by (7). The singularity at \( r = r_H \) does not appear in (161) or (163).

In the metric (156) when \( r \) is large, we obtain
\[ ds^2 = \frac{l^2}{r^2} dr^2 - \frac{r^2}{l^2} d\tilde{t}^2 + \frac{4}{r^2} \sum_{i=1}^3 (d\xi_i)^2 . \] (165)

Then the surface with constant \( r \) is flat but the spatial distance scales by the inverse power of \( r \), which is different from the case of usual (Schwarzschild) Ads black hole. On the other hand, in the large \( r \) limit, the metrics (161) and (163) behave as
\[ ds^2 = \frac{l^2}{r^2} dr^2 - \frac{4\zeta^2}{r^2} \sin^2 \theta d\tilde{t}^2 + \frac{r^2}{l^2} d\tilde{\phi}^2 + \frac{4}{r^2} d\zeta^2 + \frac{4\zeta^2}{r^2} d\theta^2 , \] (166)
\[ ds^2 = \frac{l^2}{r^2} dr^2 - \frac{4\zeta^2}{r^2} d\tilde{t}^2 + \frac{r^2}{l^2} d\tilde{\phi}^2 + \frac{4}{r^2} d\zeta^2 + \frac{4\zeta^2}{r^2} \cosh^2 \hat{t} d\phi^2 . \] (167)

The metrics of the surface with constant \( r \) (166) and (167) can be obtained from the metric corresponding to (165) by the analytic continuations (160)
and (162), which do not include the radial coordinate \( r \). It is interesting that the surface metrics are also flat.

By putting a boundary with constant \( r \), we may introduce the surface terms:

\[
S_b = S_b^{(1)} + S_b^{(2)}
\]

\[
S_b^{(1)} = \frac{2}{\kappa^2} \int d^4 x \sqrt{-g} \nabla_{\mu} n^\mu
\]

\[
S_b^{(2)} = -\int d^4 x \sqrt{-g} \left( \eta_1 + \eta_2 R_{(4)} \right).
\] (168)

Here \( S_b^{(1)} \) is the Gibbons-Hawking surface term and \( n^\mu \) is the unit vector, (outward) normal to the boundary, which is now

\[
n^r = \sqrt{f(r)} = \sqrt{\frac{r^2}{l^2} + 1 - \frac{\mu}{r^2}}, \quad \text{(other components) = 0}.
\] (169)

The constants \( \eta_1 \) and \( \eta_2 \) in \( S_b^{(2)} \) are determined so that the total action becomes finite. Then the surface energy-momentum tensor can be derived by [13]

\[
T_{mn} = T_{mn}^1 + T_{mn}^2,
\]

\[
T_{mn}^1 = \frac{1}{\kappa^2} \left( 2 g_{(4)}^{mn} \nabla_{\mu} n^\mu - 2 \nabla_{m} n_{n} \right),
\]

\[
T_{mn}^2 = -\eta_1 g_{(4)}^{mn} - 2 \eta_2 \left( \frac{1}{2} g_{(4)}^{mn} R_{(4)} - R_{(4)}^{mn} \right).
\] (170)

(Of course, the interpretation of the above quantity as an energy momentum tensor is valid only if AdS/CFT is applicable.) Here \( g_{(4)}^{mn} \) is the metric induced on the boundary and the curvatures \( R_{(4)} \) and \( R_{(4)}^{mn} \) are constructed with the help of \( g_{(4)}^{mn} \). Th calculation of stress-energy momentum tensor goes in the standard way. When \( r \) is large, \( T_{1t}^{tt} \) and \( T_{2t}^{tt} \) behave as

\[
T_{1t}^{tt} \sim \frac{6l}{\kappa^2 r^2} \left( 1 - \frac{l^2}{2r^2} + \frac{1}{r^4} \left( \frac{3l^4}{8} + \frac{\mu l^2}{2} - \frac{16}{3} \xi^2 \right) \right)
\]

\[
T_{2t}^{tt} \sim \frac{l^2 \eta_1}{r^2} + \frac{(-\eta_1 l^4 + 18 \xi l^2)}{r^4} + \frac{\eta_1 (l^6 + \mu l^4)}{r^6}.
\] (171)
We choose $\eta_1$ and $\eta_2$ so that the leading and sub-leading terms in the total $T^{tt}$ are canceled:

$$\eta_1 = -\frac{6}{l \kappa^2}, \quad \eta_2 = -\frac{l}{6\kappa^2}. \quad (172)$$

Then

$$T^{tt} \sim T_1^{tt} + T_2^{tt} = \frac{l}{\kappa^2 r^6} \left( -3\mu^2 - 32\xi^2 - \frac{15}{4} j^4 \right). \quad (173)$$

Introducing time-like unit vector $q_{\mu}$,

$$q_t = \sqrt{f(r)}, \quad \text{other components} = 0, \quad (174)$$

the mass can be defined by

$$M = \lim_{r \to \infty} \int d^3 \xi m(r), \quad m(r) \equiv \sqrt{g(4)} f^{\frac{1}{2}} T^{\mu \nu} q_\mu q_\nu. \quad (175)$$

Since

$$g(4) = \frac{64r^2}{(r^4 + 4\xi^2)^2}, \quad (176)$$

for large $r$

$$\sqrt{g(4)} f^{\frac{1}{2}} \sim \mathcal{O} \left( r^{-2} \right). \quad (177)$$

On the other hand, by using (173), one gets

$$T^{\mu \nu} q_\mu q_\nu \sim \mathcal{O} \left( r^{-4} \right). \quad (178)$$

We find $\lim_{r \to \infty} m(r) \to 0$, which indicates that in this case the mass vanishes. Originally the surface counter-term $S_b^{(2)}$ (168) has been introduced to make the total action or mass finite. Eq.(176) seems to indicate that the action might be finite except for the divergence coming from the infinite area of the constant $r$ surface. In this case the surface term $S_b^{(2)}$ may not be necessary what is different from the case of usual AdS space. Namely, if one takes $\eta_1 = \eta_2 = 0$, $T^{tt} = T_1^{tt}$ and $T^{\mu \nu} q_\mu q_\nu$ remain finite even in the limit $r \to \infty$:

$$T^{\mu \nu} q_\mu q_\nu \to \frac{6}{\kappa^2 l}. \quad (179)$$

One may now make an attempt to compare the above (analog of) surface stress tensor with the conformal anomaly on the CFT side. When $r$ is large,
the metric tensor and curvatures on the surface of constant $r$ are
\[ ds^2_{(4)} = \sum_{\mu,\nu=0}^{3} g_{(4)\mu\nu} dx^\mu dx^\nu \sim -\frac{r^2}{l^2} dt^2 + \frac{4}{r^2} (d\xi^i)^2 , \]
\[ R_{(4)\xi^i\xi^j;\xi^k\xi^l} = 0 , \quad R_{(4)\xi^i\xi^j\xi^k\xi^l} \sim \frac{48}{r^6} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) . \] (180)

From (18) it follows that the geometry of the surface is $R^1 \times S^3$ and the radius of $S^3$ is $\frac{\sqrt{3}}{r}$. Then the square of 4d Weyl tensor $F_{(4)}$ and the Gauss-Bonnet invariant $G_{(4)}$ are $O\left(r^{-6}\right)$. Since the trace anomaly generally has the following form:
\[ T^A = b \left( F_{(4)} + \frac{2}{3} \Box R_{(4)} \right) + b' G_{(4)} + b'' \Box R_{(4)} , \] (181)
we find $T^A = O\left(r^{-6}\right)$. On the other hand, from direct calculation it follows
\[ T^{1\,m} = 6 \kappa^2 \left\{ -2 - \frac{2l^2}{r^2} + \frac{1}{r^4} \left( \mu l^2 + \frac{3l^4}{4} + \frac{16l^4}{3} \right) + O\left(r^{-6}\right) \right\} \]
\[ T^{2\,m} = -4\eta_1 + \frac{36\eta_2}{r^2} + O\left(r^{-6}\right) . \] (182)

In this case the anomaly could not be reproduced even if one chose $\eta_1$ and $\eta_2$ appropriately. This indicates that additional (higher derivatives) counter-terms are necessary. It could be also that AdS/CFT correspondence does not occur.

The above discrepancy between (181) and (182) is due to the fact that the scaling for the spatial coordinates $\xi^i$ is different from that of the temporal coordinate $t$ for the dual model (156). When the radial coordinate $r$ becomes large, the scale of the spatial direction becomes small but the temporal one becomes large. This discrepancy may be solved if we could consider the $T$-dual model in the $t$-direction, as in Hull’s type IIB$^*$ model [14]. In this case the time component of the metric changes from $-f(r)dt^2$ in (156) to $-\frac{dt^2}{f(r)}$ and the metric takes the form:
\[ ds^2 = \frac{1}{f(r)} \left( -dt^2 + dr^2 \right) + \frac{2}{r^4 + 4(\xi)^2} \left[ 2r^2 \delta_{ij} + \frac{8}{r^2} \xi^i \xi^j \right] d\xi^i d\xi^j , \] (183)

Then the scaling for the temporal coordinate coincides with that of the spatial coordinates. With the new metric (183) one can repeat the evaluation of
analog of surface stress-tensor. When $r$ is large, $T^{tt}_1$ and $T^{tt}_2$ behave as

$$T^{tt}_1 \sim \frac{6r^2}{\kappa^2 l^3} \left(1 + \frac{3l^2}{2r^2} + \frac{1}{r^4} \left(\frac{3l^4}{8} - \frac{3\mu l^2}{2} - \frac{16\xi^2}{3}\right)\right)$$

$$T^{tt}_2 \sim \frac{r^2 \eta_1}{l^2} + \eta_1 + \frac{18\eta_2}{l^2} + \frac{-\mu \eta_1 + 18\eta_2}{r^2} .$$

(184)

We choose $\eta_1$ and $\eta_2$ so that the leading and sub-leading terms in the total $T^{tt}$ are canceled:

$$\eta_1 = -\frac{6}{l\kappa^2}, \quad \eta_2 = -\frac{l}{6\kappa^2} .$$

(185)

Then

$$T^{tt} \sim T^{tt}_1 + T^{tt}_2 = \frac{l}{\kappa^2 l^3 r^2} \left(-6\mu l^2 - 32\xi^2 + 6l^4\right) .$$

(186)

On the other hand, instead of (182)

$$T^{m}_1 = 6 \frac{\kappa^2 l}{l^2} \left\{-4 - \frac{l^2}{r^2} + \frac{16\xi^2}{r^4} + O\left(r^{-6}\right)\right\}$$

$$T^{m}_2 = -4\eta_1 - \frac{36\eta_2}{r^2} + O\left(r^{-6}\right) .$$

(187)

Then with the same choice of $\eta_1$ and $\eta_2$ as in (185), the leading and sub-leading terms in the total $T^{m}_m$ are canceled

$$T^{m}_m = T^{m}_1 + T^{m}_2 = \frac{96\xi^2}{\kappa^2 l^4} .$$

(188)

The above expression still depends on $\xi$, but if we set $\xi^2 = 0$, the result is consistent with (181). Physical meaning of the $\xi^i$-dependence in (188) is not clear. If Eq.(188) is exact, proposed holographic correspondence may hold only on the subspace, which satisfies $\xi^2 = 0$. Another possibility might be that our choice of the boundary of the space-time may not be the correct one for the investigation of the correspondence. Now the boundary has been given by choosing $r$ to be constant, and is taken to infinity in the final limit.

In this limit we find the geometry of the surface to be $R^1 \times S^3$ and the radius of the $S^3$ to be $\frac{\sqrt{3}}{r}$. With this choice of the boundary, the boundary is independent of $\xi^i$. Note that we can always choose the boundary so that its shape could be different. Generally one may choose the boundary that depends on the coordinates $\xi^i$. If the shape of the boundary is changed, the
extrinsic curvature $2\nabla_m n_n$ and the intrinsic curvature $R_{(4)mn}$ of the boundary can be changed, which may result in the change of the surface energy momentum tensor. If the difference of the surface energy momentum tensor from the original parameterization of the boundary could survive in the limit of $r \to \infty$, the expression of the holographic trace anomaly should also change. These observations again indicate that there are specific modifications that are needed in the application of the AdS/CFT correspondence to time-dependent backgrounds.

In the following we shall consider the Wilson loop [15] as determined on the gravitational side. Of course, this is a rather formal calculation which may have Wilson line interpretation only if the holographic correspondence is applicable. Nevertheless, we use AdS/CFT terminology for simplicity. In the case when the standard AdS/CFT correspondence is applicable, the Wilson line describes the potential between a quark and an anti-quark.

We start with the Euclidean signature space-time. By Wick-rotating $(t \to it)$ the metric (159) one arrives at

$$ds_E^2 = g_{\mu\nu} dx^\mu dx^\nu = f(r) dt^2 + \frac{1}{f(r)} dr^2 + \frac{4 r^2 \zeta^2}{r^4 + 4 \zeta^2} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).$$

(189)

In order to consider the gravitational Wilson loop, we start with the Nambu-Goto string action:

$$S_N = \frac{1}{2\pi} \int d\sigma d\tau \sqrt{\det \left( g_{\mu\nu} (x^\rho (\sigma, \tau)) \partial_\alpha x^\mu (\sigma, \tau) \partial_\beta x^\nu (\sigma, \tau) \right)}. \quad (190)$$

Here $\sigma$ and $\tau$ are the coordinates on the string world-sheet and $\{\partial_\alpha, \partial_\beta\}$ correspond to the derivative with respect to $\{\sigma, \tau\}$.

The starting configuration of the string world-sheet is

$$\tau = t, \quad \sigma = \varphi, \quad r = r(\sigma), \quad \theta = \frac{\pi}{2}, \quad \zeta = \zeta_0 \text{ (constant)}.$$

(191)

As it is a static configuration, the obtained result remains the same, including the case with the T-dual metric (183). The action (190) reduces to the form:

$$S_N = \frac{1}{2\pi} \int d\sigma d\tau \sqrt{\left( \frac{dr}{d\sigma} \right)^2 + V(r)}, \quad V(r) = \frac{4 r^2 \zeta_0^2 f(r)}{r^4 + 4 \zeta_0^2}. \quad (192)$$
Using the Euler-Lagrange equation obtained from the action (190), the following quantity is conserved on the string world-sheet:

\[ E = \frac{V(r)}{\sqrt{\left(\frac{dr}{d\sigma}\right)^2 + V(r)}}. \]  

(193)

Let us assume that there is a turning point at \( r = r_m \), where \( \frac{dr}{d\sigma} = 0 \), and

\[ E = \sqrt{V(r_m)}. \]  

(194)

For the case that the boundary exists at \( r = r_0 \) and both of \( r_0 \) and \( r_m \) are large, we have

\[ V(r) = \frac{4\zeta_0^2}{l^2} \left(1 + \frac{r^2}{r_m^2} + O \left(\frac{1}{r_0^4}\right)\right). \]  

(195)

Then Eq.(193) with (194) can be rewritten as

\[ \left(\frac{dr}{d\sigma}\right) = 4\zeta_0^2 \left(\frac{1}{r^2} - \frac{1}{r_m^2}\right). \]  

(196)

When we consider the bulk space-time in the region where \( r \leq r_0 \), the r.h.s. of (196) is negative although the quantity in the l.h.s. is non-negative and thus in this regime Eq.(196) is inconsistent. Thus, there is no non-trivial solution. On the other hand the region \( r \geq r_0 \) in the bulk space is not a usual one, however it may correspond to the T-dual theory, where the large scale region \((r \gg 1)\) is replaced with the small scale region \((\sim \frac{1}{r} \ll 1)\). Then the solution of (196) is:

\[ r = r_m \sqrt{1 - \frac{4\zeta_0^2}{r_0^4} \sigma^2}. \]  

(197)

Here the constant of integration is chosen so that \( \sigma = 0 \) corresponds to \( r = r_m \). Then the string end-points, which correspond to a quark and an anti-quark, exist at

\[ \sigma = \sigma_\pm \equiv \pm \frac{r_m}{2\zeta_0} \sqrt{1 - \frac{r_0^2}{r_m^2}}. \]  

(198)
Using (189) and (191), the geodesic distance $L$ between $\sigma_+$ and $\sigma_-$ is given by

$$L = \frac{2r_0\zeta_0 (\sigma_+ - \sigma_-)}{\sqrt{r_0^4 + 4\zeta_0^2}} = \frac{2r_0r_m^2 \sqrt{1 - \frac{r_0^2}{r_m^2}}}{\sqrt{r_0^4 + 4\zeta_0^2}}.$$  \hfill (199)

Since $r_m \sim r_0$ for large $r_0$, $V(r)$ in (196) is almost constant on the string world-sheet:

$$V(r) \sim \frac{4\zeta_0^2}{l^2} \left(1 + \frac{l^2}{r_0^2}\right).$$ \hfill (200)

As a result, using (193) the action (192) can be evaluated as

$$S_N = \frac{1}{2\pi} \int d\sigma d\tau V E \sim \frac{\zeta_0}{\pi l} \left(1 + \frac{l^2}{2r_0}\right) \frac{1}{T_H} (\sigma_+ - \sigma_-).$$ \hfill (201)

Here $T_H$ is the Hawking temperature (3), which comes from the periodicity of the Euclidean time. By using (199), we may further rewrite (201) as

$$S_N \sim \frac{\zeta_0 L}{\pi l} \left(1 + \frac{l^2}{2r_0}\right) \frac{1}{T_H} \frac{\sqrt{r_0^4 + 4\zeta_0^2}}{2r_0\zeta_0}.$$ \hfill (202)

The (would be gravitational side) potential between quark and anti-quark is found by the analogy with usual AdS calculation as

$$E(L) = 2\pi T_H S_N = \frac{2\zeta_0}{l} \left(1 + \frac{l^2}{2r_0}\right) \frac{\sqrt{r_0^4 + 4\zeta_0^2}}{2r_0\zeta_0},$$ \hfill (203)

It is linear with respect to the geodesic distance. It indicates the confinement of quarks.

Instead of configuration (191), we may consider another configuration of subject to:

$$\tau = t, \quad \sigma = \zeta, \quad r = r(\sigma), \quad \theta = \theta_0, \quad \phi = \phi_0 (\theta_0, \phi_0 \text{ : constant}).$$ \hfill (204)

Then by the similar calculations, we obtain the potential between the quark and the anti-quark, which is again linear with respect to the geodesic distance:

$$E(L) = \frac{r_0}{l} \left(1 + \frac{l^2}{2r_0}\right) L.$$ \hfill (205)

We have discussed the analog of supergravity side Wilson loop that accounts for the particle creation effects in section 6.

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