It is shown that the gravitational redshift as predicted by Einstein’s theory, is modified in the presence of second rank antisymmetric tensor (Kalb-Ramond) field in a string inspired background spacetime. In presence of extra dimensions, the Randall-Sundrum brane world scenario is found to play a crucial role in suppressing this additional shift. The bound on the value of the warp factor is determined from the redshift data and is found to be in good agreement with that determined from the requirements of Standard model.

The massless antisymmetric tensor Kalb-Ramond (KR) field $B_{\mu \nu}$, that appears in the string spectrum [1], implements in a natural way the spacetime background for string theory possessing torsion in addition to curvature. Quantum consistency of the theory further demands the augmentation of the KR field strength $\partial_{[\mu}B_{\nu \lambda]}$ with the Chern-Simons (CS) three-form. The CS extension, in turn, plays the crucial role in restoring the $U(1)$ gauge symmetry [2] that is apparently lost when torsion is coupled minimally with the gauge field in the original Einstein-Cartan theory. Solutions of the modified general relativistic field equations in various situations may now provide new results leading to a possible quantitative assessment of string inspired torsion models. In an earlier work [3], the study of geodesics in static spherically symmetric KR spacetimes with or without gravitating matter provided evidences for effects of the KR field in the context of bending of light trajectories as well as perihelion precession of planetary orbits which differ from the general relativistic estimates. Although the prospect of detecting the KR field is quite pronounced in an otherwise empty spacetime (which incidentally resembles a wormhole or a naked singularity [3,4]), in presence of gravitating matter the KR field produces little effects on the above-mentioned phenomena. One possible reason for such smallness of torsion (or, the KR field) can be given in the context of theories with large extra dimensions [5,6] where torsion is supposed to co-exist with gravity in the bulk, while all the standard model fields are confined to a 3-brane. For models of Randall-Sundrum type [6], it has been shown recently [7] that inspite of having the same status as gravity in the bulk, effects of massless torsion becomes heavily suppressed on the standard model brane, thus producing the illusion of a torsionless Universe.

With an aim of estimating this arguably weak KR field we examine here it’s effect on another important general relativistic phenomenon, namely the gravitational redshift of signals from distant radio sources. The spacetime structure is taken to be static spherically symmetric. The absence of any experimental evidence of the contribution to the redshift due to the KR field as well as imperceptible ambient temperature of the KR field with respect to the Cosmic Microwave Background temperature yield specific bounds on the KR field energy. From a higher dimensional point of view such bounds may enable one to estimate the compactification parameters in a certain scheme. We study such circumstances when torsion (or KR) field resides in the higher dimensional bulk spacetime alongwith gravity while all standard model fields are supposed to be confined to a 3-brane [7].

According to the formalism in [2], the four dimensional effective action for Einstein-Cartan-KR coupling with torsion (identified as the modified KR field strength) is given by

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R(g)}{\kappa^2} - \frac{1}{12} H^2 \right\}$$

where $\kappa^2 = 8\pi G$ is the gravitational coupling constant and the three-form $H$ is the strength of the antisymmetric KR field $B_{\mu \nu}$ plus the $U(1)$ electromagnetic CS term: $H = dB + \kappa A \wedge F$ ($A$ being the electromagnetic field and $F = dA$ it’s strength). The field equations are given by

$^*$Electronic address: tpssg@iacs.res.in
$^\dagger$Electronic address: saurabh@juphys.ernet.in
where $D$ stands for the covariant derivative defined in terms of the usual Christoffel connection and $T_{\mu}^{\nu\,(KR)}$, the KR energy-momentum tensor is given by
\[
T_{\mu}^{\nu\,(KR)} = -\frac{1}{6} \left( 2 \delta_{\mu}^{\nu} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - H_{\mu\alpha\beta} H^{\nu\alpha\beta} \right)
\]  
(3)

We consider a general static spherical symmetric metric structure
\[
d\text{s}^2 = B(r)dt^2 - \frac{dr^2}{A(r)} - r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]  
(4)

Relating the KR field strength to the derivative of a pseudoscalar axion field $\xi$ via the duality: $\partial_\mu B_{\nu\lambda} = \epsilon_{\mu\nu\lambda} \partial_\sigma \xi$, one can check that $\xi$ satisfies the massless Klein-Gordon equation $\partial_\mu D^\mu \xi = 0$. For the above metric this yields $(d\xi/dr)^2 = b/(4\pi r^4 AB)$, where the integration constant $b$ determines the measure of the torsion energy density $\rho_{KR}$ which is given by
\[
\rho_{KR}(r) = \frac{b}{4\pi r^4 B(r)} + O(\kappa).
\]  
(5)

Dropping the $O(\kappa)$ contributions that arise by virtue of the Chern-Simons extension, we write the gravitational field equations as
\[
B(rA)^{\prime} - B = - \kappa^2 \frac{b}{8\pi r^2}
\]  
(6)
\[
A(rB)^{\prime} - B = \kappa^2 \frac{b}{8\pi r^2}
\]  
(7)
\[
AB'' - \frac{AB'^2}{2B} + \frac{A'B'}{2} + \frac{(AB)'r}{r} = - \kappa^2 \frac{b}{4\pi r^2}.
\]  
(8)

where $\kappa$ denotes differentiation with respect to $r$.

Assuming general series’ of the forms $B(r) = 1 + \sum_{i=1}^{\infty} b_i/r^i$; $A(r) = 1 + \sum_{j=1}^{\infty} a_j/r^j$, and demanding asymptotic flatness $B, A \to 1$ as $r \to \infty$, we obtain the general solution for the metric coefficients $B$ and $A$ as
\[
B(r) = 1 - \frac{2m}{r} + \left( \frac{m}{3r^3} + \frac{2m^2}{3r^4} + \frac{24m^3 - 3mb}{20r^5} + \cdots \right)
\]  
(9)
\[
A(r) = 1 - \frac{2m}{r} + \left( \frac{1}{r^2} + \frac{m}{r^3} + \frac{4m^2}{3r^4} + \frac{24m^3 - mb}{12r^5} + \cdots \right).
\]  
(10)

where $m$ is the usual gravitating mass and we have redefined $b$ by absorbing a factor of $G$ in it.

Now, for a typical atomic transition, the fractional shift $z$ in the frequency ($\nu^*$) of a signal emitted on the surface of a distant object $O^*$ and the frequency ($\nu$) of the signal emitted via the same transition occurring at the observer’s location $O$ is given by the well-known formula [8]
\[
z = \frac{\nu^* - \nu}{\nu} = \left\{ \frac{g_n(r^*)}{g_n(r)} \right\}^{1/2} - 1
\]  
(11)

where $r^*$ is the radius of the distant object $O^*$ (which is considered to be spherical) and $r$ is the distance between $O^*$ and $O$.

In pure Schwarzschild geometry (without torsion) the frequency shift is given by
\[
z_s = \left( \frac{m}{r} - \frac{m}{r^*} \right) - \left( \frac{m^2}{2r^*} \right) + \left( \frac{3m^2}{2r^2} - \frac{m^2}{2r^*} \right) + \cdots
\]  
(12)

Since $r^*$ is always less than $r$, the leading term on the right is negative implying redshift.

In presence of torsion, an otherwise empty spacetime ($m = 0$) has the structure of a naked singularity or a wormhole (depending on the sign of $b$) [3,4], viz, $B = 1, A = 1 + b/r^2$, and the above formula yields zero frequency shift $z = 0$. 

2
However, in general, for non-zero \( m \), considering the smallness of the torsion parameter \( b \), we can relate the total frequency shift \( z \) with the pure Schwarzschild shift \( z_s \) by

\[
\left( \frac{1+z}{1+z_s} \right)^2 = \frac{1 + b f(r^{*}) + O(b^2)}{1 + b f(r) + O(b^2)} ; \quad f(r) = \frac{1}{3r^2} \left( \frac{m}{r} + \frac{4m^2}{r^2} + \cdots \right).
\]  

(13)

In most astrophysical observations, the source and destination objects \( O^* \) and \( O \) are quite far apart, i.e., \( r^* \ll r \). Also, for light to emit from \( O^* \) its mass \( m < r^* \). Therefore we can approximately express the total frequency shift \( z \) as

\[
z = z_s \left( 1 + \frac{\Delta z}{z_s} \right) ; \quad z_s \approx \frac{m}{r^*} + \cdots ; \quad \frac{\Delta z}{z_s} \approx -\frac{b}{6r^{*2}}.
\]

(14)

The maximum KR field energy density that can be obtained using the above expression for the fractional departure \( \Delta z/z_s \) is

\[
|\rho_{KR}^{\text{max}}| = \frac{b}{4\pi Gr^4} = 2 \left( \frac{M^*}{V^*} \right) \left( \frac{r^{*}}{r} \right)^4 |\Delta z/z_s|^2
\]

(15)

where \( M^* \) and \( V^* \) are the mass and volume of the source object \( O^* \), \( r^* \) is its radius and \( r \) is the distance of the source \( O^* \) from the observer \( O \).

For the light coming from the sun to the earth, the general relativistic predictions give \( z_s \approx 2 \times 10^{-6} \) [8]. The maximum KR field energy density that can be calculated using the standard solar system data turns out to be \(~ 7.5 \times 10^{16} \) for \( \Delta z/z_s | < \approx 10^{-32} \). This implies that unless the fractional change \( |\Delta z/z_s| \) is extremely small the ambient KR temperature which is given by

\[
T_{KR}^{\text{max}} \approx 10^8 |\Delta z/z_s|^{1/4}
\]

(16)

is enormously higher than the Cosmic Microwave Background Radiation temperature \( (T_{ cmb} \approx 3K) \) and could have been detected easily. Lack of experimental evidences in support of such a huge background temperature imposes an extremely low upper bound on the fractional change on the redshift of solar radiation \( |\Delta z/z_s| < \approx 10^{-32} \). This departure is well within the error bar standard redshift experiments [9] (\( \epsilon_z \approx 10^{-2} \)) and its detection in present day experimental setup is beyond question.

In the context of the theories with large extra dimensions the above results are, however, particularly useful especially in making predictions of the energy density of the KR field that resides in the bulk alongwith gravity while the other standard model fields are kept confined on a 3-brane. With reference to the analysis in [7] one finds that in a Randall-Sundrum (RS) scenario the zero mode of the KR field is suppressed enormously by the warp factor. The standard Kaluza-Klien (KK) decomposition of the bulk KR field is given by [7]

\[
B_{\mu\nu}(x,y) = \sum_{n=0}^{\infty} B_{\mu\nu}^n \frac{\chi^n(y)}{\sqrt{r_c}}
\]

(17)

where \( x \) stands for the usual 4-dim spacetime coordinates and \( y \) is the extra dimension, \( r_c \) is the RS compactification radius and \( \chi^n \) are the various KK modes. Using the self-adjointness and normalization conditions the solution for the massless \( (n = 0) \) mode in 5-dim RS scenario is found to be

\[
\chi^0 = \sqrt{kr_c} e^{-kr_c \pi}
\]

(18)

where \( k \) is the higher dimensional Planck scale \( (\sim 10^{18} \text{ GeV}) \). In a 4-dim effective theory the massless KR field thus gets suppressed by the factor of \( \exp(-kr_c \pi) \) with respect to the 4-dimensionally projected massless gravitonic modes on the 3-brane. Accordingly, the The duality relationship between the KR field strength and the axion \( \xi (\partial_{[\mu} B_{\nu\lambda]} = \epsilon_{\mu\nu\lambda} \partial_{\sigma} \xi) \) implies the exponential suppression in the \( \xi \)-field as well. In a static spherically symmetric spacetime the maximum 4-dim effective KR field energy on the 3-brane is now suppressed from it’s previous value \( \rho_{KR}^{(0) \text{ max}} \) by

\[
|\rho_{KR}^{\text{max}}| = e^{-2kr_c \pi} |\rho_{KR}^{(0) \text{ max}}|.
\]

(19)

Estimates of |\rho_{KR}^{(0) \text{ max}}| or, conversely, the warp factor \( kr_c \) can now be made from above analysis as follows:
For the redshift of light coming from the sun, we consider the fractional correction $|\Delta z/z_s|$ due to the exponentially suppressed KR field on the 3-brane to be as small as $\sim 10^{-32}$ so that the maximum ambient KR temperature $T_{KR}^{max} \sim 1K < T_{cmb}$, while the fractional correction $|\Delta z(0)/z_s|$ without any Randall-Sundrum type suppression is of the order of the error bar $\sim 10^{-2}$ [9]. Plugging these values in the above equation one finds $|\rho_{KR}^{(0)}| \sim 10^{15} J m^{-3}$ and $kr_c \approx 11$. This is in fairly good agreement with the value of the compactification radius needed in 5-dim RS scenario ($kr_c = 12$) to solve the hierarchy between Planck and electroweak scales. If, on the other hand, we take $kr_c = 12$ and $|\Delta z/z_s| \sim 10^{-32}$; $|\Delta z(0)/z_s| \sim 10^{-2}$, then the maximum ambient temperature reduces to a fairly low value $\sim 0.1K$ which falls well within the error bars of the Cosmic Microwave Background Temperature $T_{cmb}$.

We have thus shown that in a string inspired background spacetime the measure of gravitational redshift departs significantly from that predicted by Einstein’s theory because of the presence of the second rank antisymmetric KR field in the background. The lack of any experimental support in favour of such an additional shift can be explained by the presence of extra dimensions which in a Randall-Sundrum brane world model suppresses this shift to a value smaller than the error bar of the present redshift measurement experiments. The bound on the value of the warp factor for this suppression agrees remarkably well with that already determined to resolve the well known naturalness problem in the Standard model. We may therefore conclude that this excellent agreement of the values of the warp factor calculated in the context of two totally different physical phenomena may be looked upon as an indirect support in favour of the Randall-Sundrum brane world conjecture.

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