A note on the Casimir energy of a massive scalar field in positive curvature space

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June 6, 2003

Abstract
We re-evaluate the zero point Casimir energy for the case of a massive scalar field in \( R^1 \times S^3 \) space, allowing also for deviations from the standard conformal value \( \xi = 1/6 \), by means of zero temperature zeta function techniques. We show that for the problem at hand this approach is equivalent to the high temperature regularization of the vacuum energy, as conjectured in a previous publication. Two different, albeit equally valid, ways of doing the analytic continuation are described.

PACS numbers 04.62,+v; 11.10.Wx; 98.80.Hw

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In recent papers, the vacuum and finite temperature energies for massless and massive scalar fields in positive curvature spaces, as $S^3$, were considered, with special emphasis being put on the analysis of entropy bounds corresponding to those cases [see Refs. [1] and [2], respectively]. Specifically, in Ref. [2], dealing with the massive case and arbitrary coupling, the vacuum (Casimir) energy and the finite temperature energy were obtained through the application of a generalized zeta function technique [3], which provides a neat formal separation of the logarithm of the partition function into the non-thermal and thermal sectors. This approach holds generally and, in particular, it does for $S^1 \times S^d$ spaces. For the special case $d = 3$, the result for the logarithm of the partition function can be combined with the Abel-Plana rescaled sum formula [5], leading to

$$\log Z(\beta) = -\frac{\beta}{2r} \sum_{n=1}^{\infty} n^2 \left(n^2 + \mu_{\text{eff}}^2\right)^{1/2} + \frac{r \mu_{\text{eff}}^2}{\beta} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(m_{\text{eff}} \beta n)$$
$$+ \frac{\mu_{\text{eff}}^2 \beta}{(2\pi)^2 r} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2\pi n \mu_{\text{eff}}) - \frac{\mu_{\text{eff}}^3 \beta}{2\pi r} \sum_{n=1}^{\infty} \frac{1}{n} K_3(2\pi n \mu_{\text{eff}}). \quad (1)$$

From this result the present authors conjectured that the renormalized vacuum energy could be inferred as

$$E_0 = -\frac{\mu_{\text{eff}}^2}{(2\pi)^2 r} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2\pi n \mu_{\text{eff}}) + \frac{\mu_{\text{eff}}^3}{2\pi r} \sum_{n=1}^{\infty} \frac{1}{n} K_3(2\pi n \mu_{\text{eff}}), \quad (2)$$

where $r$ is the radius of $S^3$, $\beta$ is the reciprocal of the temperature, and the parameter $\mu_{\text{eff}}$, to be defined below, plays the role of an ‘effective mass’. The conjecture stems from the fact that in Eq. (1), the terms linear in $\beta$ give rise to temperature independent terms when we calculate basic thermodynamics quantities such as the free energy and the energy. Also, in the very-high temperature limit, it is physically plausible to expect the Stefan-Boltzmann term—which is contained in the second term in Eq. (1)— to be the only surviving one. It must be stressed that the renormalized vacuum energy was calculated with a representation derived from the finite temperature sector of the problem at hand, an approach advocated in [6] and made easier, in the present case, by the absence of the zero mode.

In this note, we wish to show that the above-mentioned conjecture holds indeed true and that the standard zero-temperature zeta function regularization procedure (as prescribed in [3], for instance) works very nicely, yielding the correct vacuum energy for the non-conformal case. In what follows we will also show that the zeta function technique is free from the disturbing difficulties met by Green function techniques [1] —and also by other techniques— when applied to the same problem. In all evaluations natural units will be employed.

The vacuum energy for a massive scalar field in $S^3$ and arbitrary conformal parameter $\xi$, which can also be read out from Eq. (1), is given by

$$E_0 = \frac{1}{2r} \sum_{\ell=0}^{\infty} D_\ell M_\ell, \quad (3)$$
where
\[ M'_\ell := (\ell + 1)^2 + \mu_{\text{eff}}^2 \]  
and the dimensionless parameter \( \mu_{\text{eff}}^2 \) is defined as
\[ \mu_{\text{eff}}^2 = \mu^2 + \chi - 1, \]
where \( \mu := mr \), \( m \) being the mass of an elementary excitation of the scalar field, \( r \) the radius of \( S^3 \), and \( \chi := \xi R r^2 \); \( R \) is the Ricci curvature scalar. It is easy to see that the parameter \( \mu_{\text{eff}}^2 \) plays a role similar to that of an effective mass. Note, moreover, that \( \mu_{\text{eff}}^2 \) and \( \chi \) are real numbers and that \( \mu^2 \geq 0 \). The degeneracy factor is \( D_\ell = (\ell + 1)^2 \). Hence, for a massive scalar field the Casimir energy is formally given by
\[ E_0(\mu_{\text{eff}}^2) = \frac{1}{2r} \sum_{\ell=0}^{\infty} (\ell + 1)^2 \sqrt{(\ell + 1)^2 + \mu_{\text{eff}}^2}, \]
an expression which needs to be conveniently regularized. This will be always done in what follows by using zeta function techniques. Thus, the standard conformal case corresponds to the values \( \mu^2 = 0 \) and \( \chi = 1 \) (\( \xi = 1/6 \), \( R = 6/r^2 \)) and, using the zeta function of the Hamiltonian operator, we obtain in this simple case
\[ E_0(\mu^2 = 0, \chi = 1) = \frac{1}{2r} \zeta (-3) = \frac{1}{240r}. \]
Our problem now is, however, to find an analytical continuation in terms of the zeta function for the more difficult series in Eq. (6) and give thereby both a mathematical and a physical meaning to it.

Starting from Eq. (6) we can straightforwardly write:
\[ \sum_{\ell=0}^{\infty} (\ell + 1)^2 \left[(\ell + 1)^2 + \mu_{\text{eff}}^2\right]^{-s}\bigg|_{s=-\frac{1}{2}} = F(-3/2, \mu_{\text{eff}}^2) - \mu_{\text{eff}}^2 F(-1/2, \mu_{\text{eff}}^2), \]
being, by definition,
\[ F(s, \mu_{\text{eff}}^2) := \sum_{\ell=0}^{\infty} \left[(\ell + 1)^2 + \mu_{\text{eff}}^2\right]^{-s} \]
or alternatively, after simple manipulations,
\[ \sum_{\ell=0}^{\infty} (\ell + 1)^2 \left[(\ell + 1)^2 + \mu_{\text{eff}}^2\right]^{-s}\bigg|_{s=-\frac{1}{2}} = \frac{1}{1 - s} \frac{\partial}{\partial t} G(s - 1, t, \mu_{\text{eff}}^2)\bigg|_{t=1, s=-\frac{1}{2}}, \]
with
\[ G(s, t, \mu_{\text{eff}}^2) := \sum_{\ell=0}^{\infty} \left[(\ell + 1)^2 t + \mu_{\text{eff}}^2\right]^{-s} \]
It is not difficult to prove that both expressions, (8) and (10), lead to the same result, as described below.

The sum over \( \ell \) can be identified as an Epstein series [3, 4]

\[
E^c_N(z; \vec{a}; \vec{c}) := \sum_{n_1 \ldots n_N=0}^{\infty} \left[ a_1 (n_1 + c_1)^2 + \cdots + a_N (n_N + c_N)^2 + c \right]^{-z}
\]

(12)

where \( a_i > 0, c_i > 0 \) and \( c > 0 \). The analytical continuation in the special case \( N = 1 \) is given by

\[
E^c_N(z; \vec{a}; \vec{c}) = \frac{c^{-z}}{\Gamma(z)} \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma(z + p)}{p!} \left( \frac{a_1}{c} \right)^p \zeta(-2p, c_1) + \frac{e^{\frac{-z}{2}}}{2} \sqrt{\pi c_1} \frac{\Gamma(z - \frac{1}{2})}{\Gamma(z)}
\]

\[
+ \frac{2\pi^z}{\Gamma(z)} \cos(2\pi c_1) a_1^{-\frac{1}{2} - \frac{1}{4}} e^{-\frac{1}{4} + \frac{1}{4}} \sum_{n_1=0}^{\infty} n_1^{\frac{s-\frac{1}{2}}{2}} K_{\frac{s-\frac{1}{2}}{2}} \left( 2\pi n_1 \sqrt{\frac{c}{a_1}} \right).
\]

(13)

In our case, after the identifications: \( a_1 = 1, c_1 = 1, z = s - 1, \) and \( c \equiv \mu_{eff}^2 t^{-1} \), we have

\[
E_0 = \frac{\mu_{eff}^4}{16r} \Gamma(-2) + \frac{3\mu_{eff}^2}{4\pi^2 r} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2\pi n \mu_{eff}) + \frac{\mu_{eff}^3}{2\pi r} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2\pi n \mu_{eff}).
\]

(15)

Alternatively, by using Eq. (10), that is, taking the derivative with respect to \( t \) and setting \( t = 1 \) and \( s = -1/2 \) we arrive, after some manipulations, at the following (differently looking) expression for the vacuum energy

\[
E_0 = -\frac{\mu_{eff}^4}{16r} \Gamma(-2) + \frac{\mu_{eff}^2}{4\pi^2 r} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2\pi n \mu_{eff}) - \frac{\mu_{eff}^3}{2\pi r} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial u} K_2(u = 2\pi n \mu_{eff}).
\]

(16)

The divergent term in Eqs. (15) and (16) may be taken care of with the minimal subtraction scheme [9], what means to take into account the principal part of \( \Gamma(-2) = \psi(3)/2 \). However, this procedure would imply, in our case, to keep a quartic term in \( \mu_{eff} \) that would spoil the
behavior of the vacuum energy in the classical limit, where the vacuum oscillations are known to vanish. As a consequence, old-style physical renormalization immediately prescribes that this term must be simply discarded, what we will do from now on, without too much ado.

We can now further simplify Eq. (16), by making use of the recursion relation [10]

$$-2 \frac{dK_\nu(z)}{dz} = K_{\nu-1}(z) + K_{\nu+1}(z),$$

which allows us to write

$$E_0 = \frac{\mu_{\text{eff}}^2}{4\pi^2 r} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2\pi n \mu_{\text{eff}}) + \frac{\mu_{\text{eff}}^3}{4\pi r} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2\pi n \mu_{\text{eff}}) + \frac{\mu_{\text{eff}}^3}{4\pi r} \sum_{n=1}^{\infty} \frac{1}{n} K_3(2\pi n \mu_{\text{eff}}).$$

(17)

If we now make use of this other recursion relation:

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z),$$

we easily obtain both Eq. (2), conjectured in a former paper, and also Eq. (15), which are completely equivalent, by the same recursive formulas. We have thus shown that all these approaches are perfectly equivalent.

When we take the limit \(\mu_{\text{eff}}^2 \rightarrow 0\), Eqs. (2) and (17) yield the well known result \(E_0 \approx 1/240\). Moreover, if we make use of the appropriate small argument expansion of the Bessel functions of the second kind, we obtain

$$E_0 \approx \frac{1}{240r} - \frac{\mu_{\text{eff}}^2}{48r} - \frac{1}{2} \left[ \frac{1}{8r} + \frac{1}{16r} \left( -\frac{3}{2} + 2\gamma_E \right) \right] \mu_{\text{eff}}^4. \quad (18)$$

On the other hand, if we take the opposite limit, \(\mu_{\text{eff}}^2 \rightarrow \infty\), the vacuum energy given by Eq. (2)—or (15) or (17)—behaves in the way that one would normally expect of constrained zero-point oscillations of a massive quantum field: the vacuum energy goes to zero in an exponential way.

In Ref. [1] the case of a massless scalar was considered. In order to obtain the vacuum energy, the formalism due to Kantowski and Milton [8], which relies on Green function techniques, was successfully employed. However, when the same techniques were applied to the conformal symmetry breaking case, \textit{ab initio} unsurmountable difficulties were met. The finite temperature energy evaluation is also plagued by difficulties. The good news are that none of these problems are present when we have recourse to the generalized zeta function procedure, as we have here shown.
Acknowledgments

A.C.T. wishes to acknowledge the kind hospitality of the Institut d’Estudis Espacials de Catalunya (IEEC/CSIC) and of the Universitat de Barcelona, Departament d’Estructura i Constituents de la Matèria, where the present work was began. The investigation of E.E. has been supported by DGI/SGPI (Spain), project BFM2000-0810, and by CIRIT (Generalitat de Catalunya), contract 1999SGR-00257.

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