Supersymmetric Matrix Model on Z-Orbifold

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Abstract

We find that the IIA Matrix models defined on the non-compact \(C^3/Z_6\), \(C^2/Z_2\) and \(C^2/Z_4\) orbifolds preserve supersymmetry where the fermions are on-mass-shell Majorana-Weyl fermions. In these examples supersymmetry is preserved both in the orbifolded space and in the non-orbifolded space at the same time. The Matrix model on \(C^3/Z_6\) orbifold has the same \(\mathcal{N} = 2\) supersymmetry as the case of \(C^3/Z_3\) orbifold which was pointed out previously. On the other hand the Matrix models on \(C^2/Z_2\) and \(C^2/Z_4\) orbifold have a half of the \(\mathcal{N} = 2\) supersymmetry. We further find that the Matrix model on \(C^2/Z_2\) orbifold with a parity-like identification preserves \(\mathcal{N} = 2\) supersymmetry.

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Introduction

The dynamics of the extended objects, D-branes [1], the key ingredients to understand the nonperturbative effect of strings and their property of duality, can be described in terms of the dimensionally reduced Supersymmetric Yang-Mills (SYM) Theory [2]. In particular, the dynamics of the simplest point-like D-particles may play the fundamental role [3] and it is described by the matrix description of the 11d super-membrane theory [4]. On the other hand the dynamics of D-particles can be viewed as the SYM theory dimensionally reduced to one temporal dimension, that is the quantum mechanics of the SUSY Matrix model. Here, the bosonic fields of the SUSY Matrix model describe the noncommutative bosonic coordinates consisting of $N$ D-particles. They are represented by $N \times N$ hermitian matrices. We call this the 11d M-theory.

The 11d M-theory compactified on a simple orbifold space, $S^1/Z_2$, was shown to be connected with the 10d $E_8 \times E_8$ heterotic string theory [5]. While the 6d compactification on the Calabi-Yau manifold of the usual 10d heterotic string theory was given by [6], and the compactification on the various orbifold spaces was studied in [7],[8],[9].

The type IIA Matrix models compactified on the $Z_2$ orbifold spaces were explicitly investigated by [10], [11], [12], [13], [14], and others. On the other hand, the type IIB Matrix model was formulated by [15], and the type IIB $USp(2k)$ Matrix model compactified on the $Z_2$ orbifold spaces was investigated by [16].

In order to understand the realistic world in terms of the string dynamics or the M-theory, we need to find Matrix models defined on the flat 4d space-time and non-flat 6d spaces.

Explicit examples of the Matrix models defined on the 4d flat space-time and non-compact 6d orbifold spaces are demonstrated by [17], [18].

In the supersymmetric Matrix model defined on $C^3/Z_3$ orbifold [18], the number of supersymmetry is reduced to $\mathcal{N} = 2$. In this paper we generalize the previous study on the Matrix model on $C^3/Z_3$ to more general orbifold spaces. We first formulate the Matrix model on the generic $C^i/Z_n$ orbifold, by using the 't Hooft algebra for $SU(n)$ monopoles. After examining carefully the structure of fermions satisfying the Majorana conditions, the Supersymmetric IIA Matrix models on the non-compact $C^i/Z_n$ orbifold spaces are found to be limited in number. Examples of these supersymmetric Matrix models are given by $C^3/Z_6$, $C^2/Z_2$ and $C^2/Z_4$ orbifold. In these examples supersymmetry is preserved both in the
orbifolded space and in the (non-orbifolded) flat space at the same time. The number of supersymmetry, however, depends on the models. The Matrix model on \( C^3/Z_6 \) orbifold has the same \( \mathcal{N} = 2 \) supersymmetry as the case of \( C^3/Z_3 \) orbifold investigated previously. On the other hand the Matrix models on \( C^2/Z_2 \) and \( C^2/Z_4 \) orbifolds have the same number as \( \mathcal{N} = 1 \) supercharges.

We further study the case in which the parity symmetry (the reverse of the space coordinates) is supplemented to the \( Z_2 \) symmetry, where the parity-like \( Z_2 \) identification is defined. We then find that the Matrix model on \( C^2/Z_2 \) orbifold with this parity-like identification differs from the usual \( C^2/Z_2 \) orbifold, and the former model preserves \( \mathcal{N} = 1 \) supersymmetry both in the orbifolded space and non-orbifolded space, and has \( \mathcal{N} = 2 \) supersymmetry parameters in the total space.

**IIA Matrix model**

We treat \( nN \) D-particles. The action of the IIA Matrix model is given by

\[
S = \frac{1}{2g^2} \int dt \, \text{Tr} \left\{ (D_t X^I)^2 + \frac{1}{2} [X^I, X^J]^2 - ig^2 \Theta^T D_t \Theta - g^2 \Theta^T \tilde{\Gamma}^I [X_I, \Theta] \right\}, \tag{1}
\]

where \( I, J = 1, \cdots, 9 \) label the Minkowski space-time indices. \( \Gamma^I \) is the 10-dimensional gamma matrices with \( \tilde{\Gamma}^I = \Gamma^0 \Gamma^I \), \( g \) is the Yang-Mills coupling and \( D_t \) is the covariant derivative defined by

\[
D_t = \partial_t - i [X_t, \;]. \tag{2}
\]

The bosonic fields \( X_I \) and the fermionic fields \( \Theta \) are \( nN \times nN \) hermitian matrices. The fermionic fields \( \Theta \) are given by the Majorana-Weyl fermions and have sixteen degrees of freedom on the mass shell. These fields depend on temporal dimension. The diagonal parts of the bosonic fields \( X_I \) are given by the solutions of equations of motion and represents the classical coordinates. The other parts of the bosonic, fermionic and ghost fields have the fluctuating fields which represent interaction among the \( nN \)-body D-particles.

The supersymmetric transformation of IIA Matrix model is described as follows:

\[
\delta X_t = ig \epsilon^T \Theta, \tag{3}
\]

\[
\delta X_I = ig \epsilon^T \tilde{\Gamma}^I \Theta, \tag{4}
\]

\[
\delta \Theta = \frac{1}{g} \left( \tilde{\Gamma}^I (D_t X_I) \epsilon - \frac{i}{2} \Gamma^{IJ} [X_I, X_J] \epsilon \right) + \xi, \tag{5}
\]

\[3\]
where $\Gamma^{IJ} = \frac{1}{2}[\Gamma^I, \Gamma^J]$. $\epsilon$ is the supersymmetric transformation parameter of $\mathcal{N} = 1$ SYM theory and $\xi$ is the translation parameter of the fermions. This model has $\mathcal{N} = 2$ SUSY.

**IIA Matrix model on the $C^i/Z_n$ Orbifold**

It is convenient to use the complex notations for the orbifolded dimensions. The complex coordinates $Z_j$ is defined as follows:

\[ Z_i \equiv X^{2i} + iX^{2i+1}, \quad (6) \]

where $i = 2, 3, 4$ for three complex spaces $C^3$, $i = 3, 4$ for the two complex spaces $C^2$ and $i = 4$ for the one complex space $C^1$. We keep the flat space-time coordinates at least as 4 dimensions, and take the orbifold spaces for the other dimensions. Let us impose a $Z_n$ symmetry about the complex coordinates $Z_j$ and obtain the $Z_n$-orbifold. The complex coordinates $Z_j$ have $Z_n$ symmetry under

\[ Z_j \simeq \omega Z_j, \quad (7) \]

where $\omega = e^{2\pi i/n}$.

To impose the $Z_n$ symmetry on the $nN \times nN$ matrices, we use the 't Hooft matrices; $U$ and $V$:

\[
U = \begin{pmatrix}
1 & \omega \\
\omega & \ddots \\
\vdots & \ddots & \ddots \\
\omega^{n-1} & \cdots & \cdots & \cdots & \omega^{n-1}
\end{pmatrix}_{nN \times nN}, \\
V = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & 1 & 0
\end{pmatrix}_{nN \times nN}, \quad (8)
\]

where the block matrices is $N \times N$ matrices. $U$ and $V$ satisfy $UU^\dagger = U^\dagger U = U^n = 1$, $VV^\dagger = V^\dagger V = VV^T = V^T V = V^n = 1$ and the following relation:

\[ UV = \omega VU. \quad (9) \]

We start with the $nN$-body system of the D-particles in the complex space(s) $Z_j$. And we divide the complex space(s) $Z_j$ into $n$ equal parts. Under the $Z_n$ symmetry, $N$ D-particles are distributed into a small part of the same size. There are $n$ mirror images in the complex space(s) $Z_j$. Let us use a $SO(nN)$ group $V$ and impose the $Z_n$ invariance on the bosonic and fermionic fields

\[ X^\mu_\parallel = VX^\mu_\parallel V^\dagger, \quad (10) \]
\[ Z_j = \omega_j V Z_j V^\dagger, \quad \Theta = \hat{\omega} V \Theta V^\dagger, \] 

where

\[ \omega_j = \exp \left( 2\pi i \frac{n_j}{n} \right), \]

\[ \hat{\omega} = \exp \left( 2\pi i \sum_{j=2}^{4} \frac{n_j}{n} b_j^\dagger b_j \right). \]

\( n_j \) is an integer. \( \hat{\omega} \) represents the \( Z_n \) symmetry for 10 dimensional Majorana-Weyl spinors on the mass shell. The 16 Majorana-Weyl spinors can be represented by using the raising and the lowering operator; \( b_\mu \) and \( b^\dagger_\mu \). \( b_\mu \) are defined as follows:

\[ b_0 = \frac{1}{2} (\Gamma^1 - \Gamma^0), \]

\[ b_j = \frac{1}{2} (\Gamma^{2j} - i\Gamma^{2j+1}), \quad j = 1, \ldots, 4. \]

The Gamma matrices \( \Gamma^\mu \) satisfy the Clifford algebra \( \{ \Gamma^\mu, \Gamma^\nu \} = 2\eta^{\mu\nu} \), where \( (\Gamma^0)^2 = -1 \) and \((\Gamma^j)^2 = 1 \).

The 16 Majorana-Weyl spinors on the mass shell are written as

\[ \Theta^a = \begin{pmatrix} \psi^a \vert 0 \rangle \\ \psi^{a*} \vert 0 \rangle \\ \psi^{[ij]a} b^\dagger_i b^\dagger_j \vert 0 \rangle \\ \psi^{[ij]a*} b_i^\dagger b_j^\dagger \vert 0 \rangle \end{pmatrix}, \]

where \( i = 1, \ldots, 4 \). The superscript \( a \) denotes the number of the supersymmetry; \( a = 1, 2 \). The ground state of the spinors is defined as follows:

\[ \vert 0 \rangle \equiv \vert -,-,-,-,- \rangle, \]

where the first minus sign is fixed on the mass shell and the others can be changed into the plus sign using the raising operator \( b_i^\dagger \).

We impose the Majorana condition:

\[ B \Theta = \Theta^*, \]

where \( B = \Gamma^3 \Gamma^5 \Gamma^7 \Gamma^9 \). From this condition, we obtain

\[ \sum_{i=2}^{4} \frac{n_j}{n} = \text{integer or half-integer}. \]
IIA Matrix model on the special $C^i/Z_n$ Orbifold

In this section, we treat the special $C^i/Z_n$ orbifolded spaces where $n_j$ does not depend on the subscript $j$. In other words, the phases $\omega_j$ are equal in the orbifolded complex spaces; $n_2 = n_3 = n_4$ for the $C^3/Z_n$ orbifold, $n_3 = n_4$ for the $C^2/Z_n$ orbifold and $n_4$ for the $C^1/Z_n$ orbifold. From eq. (19), The combination $(\frac{n_2}{n}, \frac{n_3}{n}, \frac{n_4}{n})$ is given as follows:

$$
\left(\frac{1}{2}, \frac{1}{2}, 1\right), \left(\frac{1}{3}, \frac{1}{3}, 1\right), \left(\frac{1}{6}, \frac{1}{6}, 1\right), \text{ on } C^3/Z_{2,3,6},$
$$
$$
\left(1, \frac{1}{2}, 1\right), \left(1, \frac{1}{4}, 1\right), \text{ on } C^2/Z_{2,4},$
$$
$$
\left(1, 1, \frac{1}{2}\right), \text{ on } C^1/Z_{2}.
$$

(20)

Using eqs. (10), (11) and (12), we can get the bosonic fields and the fermionic fields on the special $C^i/Z_n$ orbifolds. We decompose the $nN \times nN$ matrices into the sum of the tensor products ($N \times N$ matrices) $\otimes (n \times n$ matrices) and then we obtain

$$X^\mu_\parallel = H^\mu_1 \otimes 1 + A^\mu_1 \otimes V^\dagger + A^\mu_2 \otimes (V^\dagger)^2 + \cdots + A^\mu_m \otimes (V^\dagger)^m + A^\mu_m \otimes V^m + \ldots + A^\mu_2 \otimes V^2 + A^\mu_1 \otimes V,
$$

for $Z_{2m+1}$, (21a)

$$X^\mu_\parallel = H^\mu_1 \otimes 1 + A^\mu_1 \otimes V^\dagger + \cdots + A^\mu_m \otimes (V^\dagger)^{2m-3} + H^\mu_m \otimes (V^\dagger)^{2m-2}
$$
$$+ H^\mu_{m+1} \otimes (V^\dagger)^{2m-1} + H^\mu_{m} \otimes V^{2m-2} + A^\mu_1 \otimes V^2 + A^\mu_1 \otimes V, \text{ for } Z_{4m-2}, (21b)

$$X^\mu_\parallel = H^\mu_1 \otimes 1 + A^\mu_1 \otimes V^\dagger + \cdots + H^\mu_m \otimes (V^\dagger)^{2m-2} + A^\mu_m \otimes (V^\dagger)^{2m-1}
$$
$$+ H^\mu_{m+1} \otimes (V^\dagger)^{2m} + A^\mu_m \otimes V^{2m-1} + H^\mu_m \otimes V^{2m-2} + \ldots + H^\mu_2 \otimes V^2 + A^\mu_1 \otimes V, \text{ for } Z_{4m}, (21c)
$$

$$Z_i = B_{1i} \otimes U + B_{2i} \otimes UV^\dagger + B_{3i} \otimes U(V^\dagger)^2 + \cdots + B_{(n-1)i} \otimes UV^2 + B_{ni} \otimes UV, (22)$$

$$\Theta_1 = \hat{H}_1 \otimes 1 + \hat{A}_1 \otimes V^\dagger + \hat{A}_2 \otimes (V^\dagger)^2 + \cdots + \hat{A}_m \otimes (V^\dagger)^m
$$
$$+ \hat{A}_m \otimes V^m + \cdots + \hat{A}_2 \otimes V^2 + \hat{A}_1 \otimes V, \text{ for } Z_{2m+1}, (23a)$$

$$\Theta_1 = \hat{H}_1 \otimes 1 + \hat{A}_1 \otimes V^\dagger + \cdots + \hat{A}_{m-1} \otimes (V^\dagger)^{2m-3} + \hat{H}_m \otimes (V^\dagger)^{2m-2}
$$
$$+ \hat{H}_{m+1} \otimes (V^\dagger)^{2m-1} + \hat{H}_m \otimes V^{2m-2} + \hat{A}_m \otimes V^{2m-3} + \ldots$$
\[
\Theta_1 = \hat{H}_1 \otimes V + \hat{A}_1^\dagger \otimes V + \ldots + \hat{H}_m \otimes (V^\dagger)^{2m-2} + \hat{A}_m \otimes (V^\dagger)^{2m-1} + \ldots
\]
\[
\Theta_1 = \hat{H}_m \otimes V^2 + \hat{A}_m \otimes V, \quad \text{for } Z_{4m-2},
\]

and
\[
\Theta_4 = 2 \Theta_1
\]

\[
\Theta_\omega = \hat{B}_1 \otimes U + \hat{B}_2 \otimes UV^\dagger + \hat{B}_3 \otimes U(V^\dagger)^2 + \ldots + \hat{B}_{n-1} \otimes UV^2
\]
\[
+ \hat{B}_n \otimes UV, \quad \text{for } Z_{4m},
\]

where \( H_i \) and \( \hat{H}_i \) are \( N \times N \) hermitian matrices and \( A_i, B_i, \hat{A}_i \) and \( \hat{B}_i \) are arbitrary matrices. The degrees of freedom of \( N \times N \) hermitian matrix is equal to \( N^2 \), and the degrees of freedom of \( N \times N \) arbitrary matrix is equal to \( 2N^2 \). Therefore we can obtain the degrees of freedom of the bosonic fields and the fermionic fields for eq. (20).

Table 1: The degrees of freedom on \( C^i/Z_n \) \( (i = 2, 3, n = 2, 3, 4, 6) \)

<table>
<thead>
<tr>
<th>( H/\Gamma )</th>
<th>( X_\mu^i )</th>
<th>( Z_i )</th>
<th>( \Theta_1 )</th>
<th>( \Theta_\omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C^3/Z_{2m+1} )</td>
<td>( 2(2m+1)N^2 )</td>
<td>( 12(2m+1)N^2 )</td>
<td>( 4(2m+1)N^2 )</td>
<td>( 24(2m+1)N^2 )</td>
</tr>
<tr>
<td>( C^2/Z_{2m+1} )</td>
<td>( 4(2m+1)N^2 )</td>
<td>( 8(2m+1)N^2 )</td>
<td>( 8(2m+1)N^2 )</td>
<td>( 16(2m+1)N^2 )</td>
</tr>
<tr>
<td>( C^3/Z_{4m-2} )</td>
<td>( 2(3m-1)N^2 )</td>
<td>( 12(4m-2)N^2 )</td>
<td>( 4(3m-1)N^2 )</td>
<td>( 24(4m-2)N^2 )</td>
</tr>
<tr>
<td>( C^2/Z_{4m-2} )</td>
<td>( 4(3m-1)N^2 )</td>
<td>( 8(4m-2)N^2 )</td>
<td>( 8(3m-1)N^2 )</td>
<td>( 16(4m-2)N^2 )</td>
</tr>
<tr>
<td>( C^3/Z_{4m} )</td>
<td>( 2(3m+1)N^2 )</td>
<td>( 12 \cdot 4mN^2 )</td>
<td>( 4(3m+1)N^2 )</td>
<td>( 24 \cdot 4mN^2 )</td>
</tr>
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<td>( C^2/Z_{4m} )</td>
<td>( 4(3m+1)N^2 )</td>
<td>( 8 \cdot 4mN^2 )</td>
<td>( 8(3m+1)N^2 )</td>
<td>( 16 \cdot 4mN^2 )</td>
</tr>
</tbody>
</table>

As we can see in the table 1, we find that the fermionic fields are twice of the degrees of freedom of the bosonic fields, where \( m \) is an integer and \( m \geq 1 \). Especially, we find that the fermionic fields \( \Theta_{2\pi i} \) in the flat spaces are twice of the degrees of freedom of the bosonic fields \( X_\mu^i \) in the flat spaces and the fermionic fields \( \Theta_\omega (\Theta_\omega^\dagger) \) in the orbifolded spaces are twice of the degrees of freedom of the bosonic fields \( Z_j (Z_j^\dagger) \).

We find that the result in the table 1 may implicitly mean the existence of the supersymmetry. Therefore we will actually check whether the supersymmetry exists or not in the next subsection. However we notice that for the degrees of freedom of fermions in the case of \( c = -1 \) we find that \( \Theta_{2\pi i}^* \) is equal to \( \Theta_{2\pi i/2} \) from eq. (49) and in table 4 of the appendix. We then find that matrices \( \Theta_{2\pi i/2} \) have the same degrees of freedom as matrices \( \Theta_{2\pi i} \). Namely we find that degrees of freedom
of bosonic fields are not same as that of fermionic fields on the $C^3/Z_2$ orbifolds using the representation of $(1, 1, 1)$ and on the $C^1/Z_2$ orbifolds using the representation of $(1, 1, 1)$ shown in (20). Therefore we need to check the supersymmetry on the $C^2/Z_2, C^2/Z_4$ and $C^3/Z_6$ orbifolds except $C^3/Z_2$ and $C^1/Z_2$ orbifolds of eq. (20).

**SUSY of IIA Matrix model on the special $C^i/Z_n$ Orbifold**

We confirm the existence of the supersymmetry on the $C^2/Z_2, C^2/Z_4$ and $C^3/Z_6$ orbifolds in the similar way as we checked for the $C^3/Z_3$ orbifold [18]. The supercharge of IIA Matrix model in 10 dimensions is given by

$$\bar{\epsilon}' Q = \frac{1}{g} \text{Tr} \left\{ (i\Theta^\dagger)_\alpha (\Gamma^{0/J} D_t X_I - i\Gamma^{I/J} [X_I, X_J]) \epsilon\alpha + (i\Theta^\dagger)_\alpha \xi\alpha \right\}, \quad (25)$$

and

$$\epsilon' = \left( \begin{array}{c} \epsilon[i]^{[a]} b_0^\dagger b_m^\dagger |0\rangle \\ \epsilon[jk]^{[a]} b_0^\dagger b_i^\dagger |0\rangle \\ \epsilon[ij]^{[a]} b_0^\dagger b_i^\dagger b_j^\dagger |0\rangle \end{array} \right), \quad (26)$$

where $a = 1, 2$, $i, j, k = 1, \cdots, 4$, $\epsilon' \equiv (\epsilon^1, \epsilon^2)^T \equiv (\epsilon, \xi)^T$ and $\epsilon'$ are sixteen Majorana-Weyl fermions on the mass shell, i.e., $\epsilon[i]^{[a]}|0\rangle = \frac{1}{2} \epsilon[ijk]^{[a]} |0\rangle$. Since there are no the bosonic fields in the second term on the right hand side of eq. (25), we can not verify the supersymmetric transformation. We then omit the second term of eq. (25):

$$\bar{\epsilon} Q = \frac{1}{g} \text{Tr} \left[ (i\Theta^\dagger)_\alpha (\Gamma^{0/J} D_t X_I - i\Gamma^{I/J} [X_I, X_J]) \right] \epsilon\alpha \right], \quad (27)$$

Taking the trace with respect to $n \times n$ matrices, we obtain the supercharge on the special $C^2/Z_2, C^2/Z_4$ and $C^3/Z_6$ orbifolds. The supercharges are given in eqs. (53), (54) and (55) in the appendix. From the result of the supercharges on $C^2/Z_2, C^2/Z_4$, and $C^3/Z_6$ orbifolds in eqs. (53), (54) and (55) of the appendix, respectively, we find that supersymmetry exists. We especially point out that supersymmetric parameters of the standard Matrix model with respect to supercharges on $C^2/Z_2$ and $C^2/Z_4$ orbifolds have half as many supersymmetric parameters as that of IIA Matrix model on flat spaces in 10 dimensions.

SUSY parameters on $C^2/Z_2$ and $C^2/Z_4$ orbifolds are described as follows:

$$\left( \begin{array}{c} \epsilon[i]^{[a]} b_0^\dagger b_m^\dagger |0\rangle \\ \frac{1}{2} \epsilon[ij]^{[a]} b_0^\dagger b_i^\dagger b_j^\dagger |0\rangle \end{array} \right), \quad (28)$$

where $m, n = 1, 2$ and $i, j = 3, 4$. According to eqs. (53) and (54), we find that the supersymmetric parameters $\epsilon[i]^{[a]}$ are only used and the remainders $\epsilon[i]$ are
not. Then $\mathcal{N} = 2$ supersymmetry of IIA Matrix model in flat spaces turns into $\mathcal{N} = 1$ supersymmetry of IIA Matrix model on $C^2/Z_2$ and $C^2/Z_4$ orbifolds. We find that the bosonic fields $X_\theta$ are transformed to fermionic fields $\Theta_{2\pi i}$, by SUSY transformation with the parameter $\epsilon^{[m]n}$, while the bosonic fields $Z_i$ are transformed to the fermionic fields $\Theta_\omega$ by the SUSY transformation with the parameter $\epsilon^{[m34]}$ and $Z_i^\dagger$ are transformed to $\Theta_{\omega2}$ with $\epsilon^{[m]}$.

We similarly find that the supercharge on $C^3/Z_6$ orbifold has the same number of supersymmetric parameters as IIA Matrix model on the flat space in 10 dimensions. SUSY parameters are described as follows:

$$
\begin{pmatrix}
\epsilon^{[1]}b_0^\dagger b_1^\dagger [0] \\
\epsilon^{[i]}b_0^\dagger b_i^\dagger [0] \\
\epsilon^{[234]}b_0^\dagger b_2^\dagger b_3^\dagger b_4^\dagger [0] \\
\frac{1}{2}\epsilon^{[1ij]}b_0^\dagger b_i^\dagger b_j^\dagger [0]
\end{pmatrix},
$$

(29)

where $i = 2, \cdots, 4$. The supersymmetric parameters on $C^3/Z_6$ orbifold are equal to the supersymmetric parameters on $C^3/Z_3$ orbifold [18]. Namely the bosonic fields $X_\theta$ are transformed to fermionic fields $\Theta_1$, by SUSY transformation with the parameter $\epsilon^{[i]}$, while the bosonic fields $Z_i$ are transformed to the fermionic fields $\Theta_\omega$ by the SUSY transformation with the parameter $\epsilon^{[i]}$ and $Z_i^\dagger$ are transformed to $\Theta_{\omega2}$ with $\epsilon^{[i]}$.

**IIA Matrix model on a parity-like $Z_2$ Orbifold**

Finally, we mention about another $C^i/Z_2$ orbifold, i.e. a *paritylike* $C^i/Z_2$ orbifold. Due to N. Kim and S. J. Rey, the fields of IIA Matrix model on $Z_2$ orbifold is described as follows:

$$
X^\mu_\theta = MX^\mu T M^\dagger,
$$

$$
X^i_\perp = -MX^i T M^\dagger,
$$

$$
\Theta = PM\Theta M^\dagger,
$$

(30)

where $\mathcal{P}$ is a parity transformation matrix and the bosonic and fermionic fields have $2N \times 2N$ hermitian matrices [12]. The bosonic and fermionic fields have $2N \times 2N$ hermitian matrices. $\mathcal{P}$ has the same peculiarity to the parity-like $Z_2$ orbifold because the parity transformation operator is equal to the phase of $Z_2$ symmetry:

$$
\mathcal{P} \equiv (i\Gamma^6 \Gamma^7)(i\Gamma^8 \Gamma^9)
$$
\[
\exp \left( -2\pi i \sum_{i=3}^{4} \left( -\frac{n_i}{2} \right) b_i^\dagger b_i \right) = \hat{\omega},
\]

where the matrix \( M \) satisfy \( M^2 = 1 \) and \( MM^T = M^T M = \pm 1 \), and so \( M \) belongs to \( SO(2N) \) or \( USp(2N) \) group. A parity-like \( Z_2 \) orbifold can be defined not only in even dimensions but also defined in any dimensions. In the case of \( SO(2N) \) group, the matrix \( M \) is defined as

\[
M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{2N \times 2N}.
\]

Then the matrices \( X_\mu^\mu, X_i^i, \Theta_{2\pi i}, \Theta_\omega \) are written as follows:

\[
X_\mu^\mu = \begin{pmatrix} H_1^{\mu} \\ S^{\mu} \\ H_1^{T\mu} \end{pmatrix}, \quad X_i^i = \begin{pmatrix} H_2^i \\ -J^i \\ -H_2^{Ti} \end{pmatrix},
\]

\[
\Theta_{2\pi i} = \begin{pmatrix} \hat{H}_1 \\ \hat{S} \\ \hat{H}_1^T \end{pmatrix}, \quad \Theta_\omega = \begin{pmatrix} \hat{H}_2 \\ -\hat{J} \\ -\hat{H}_2^T \end{pmatrix},
\]

where \( H_1, H_2, \hat{H}_1 \) and \( \hat{H}_2 \) are \( N \times N \) hermitian matrices, \( S \) and \( \hat{S} \) are \( N \times N \) symmetric matrices and \( J \) and \( \hat{J} \) are \( N \times N \) anti-symmetric matrices. While, in the case of \( USp(2N) \) group the matrix \( M \) is defined as

\[
M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2N \times 2N}.
\]

Then the matrices \( X_\mu^\mu, X_i^i, \Theta_{2\pi i}, \Theta_\omega \) are written as

\[
X_\mu^\mu = \begin{pmatrix} H_1^{\mu} \\ J^{\mu} \\ H_1^{T\mu} \end{pmatrix}, \quad X_i^i = \begin{pmatrix} H_2^i \\ S^i \\ -H_2^{Ti} \end{pmatrix},
\]

\[
\Theta_{2\pi i} = \begin{pmatrix} \hat{H}_1 \\ \hat{J} \\ \hat{H}_1^T \end{pmatrix}, \quad \Theta_\omega = \begin{pmatrix} \hat{H}_2 \\ \hat{S} \\ -\hat{H}_2^T \end{pmatrix}.
\]

From the above fields we can obtain the degrees of freedom of the bosonic fields and the fermionic fields on the parity-like \( Z_2 \) orbifold. Then we find that the degrees of freedom of the bosonic fields is not equal to that of fermionic fields on \( C^1/Z_2 \) and \( C^3/Z_2 \) orbifolds. On the other hand the degrees of freedom of bosonic and fermionic fields are same for \( C^2/Z_2 \) orbifold. Hereafter we only treat \( C^2/Z_2 \) orbifold.
Table 2: The degrees of freedom of $SO(2N)$ group on parity-like $Z_2$ orbifold

<table>
<thead>
<tr>
<th></th>
<th>$X^\mu_\parallel$</th>
<th>$X^i_\perp$</th>
<th>$\Theta_{2\pi i}$</th>
<th>$\Theta_{\pi i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^4/Z_2$</td>
<td>$2N(2N+1)/2 \times 4$</td>
<td>$2N(2N-1)/2 \times 4$</td>
<td>$4N(2N+1)/2 \times 8$</td>
<td>$4N(2N-1)/2 \times 8$</td>
</tr>
</tbody>
</table>

Table 3: The degrees of freedom of $USp(2N)$ group on parity-like $Z_2$ orbifold

<table>
<thead>
<tr>
<th></th>
<th>$X^\mu_\parallel$</th>
<th>$X^i_\perp$</th>
<th>$\Theta_{2\pi i}$</th>
<th>$\Theta_{\pi i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^4/Z_2$</td>
<td>$2N(2N-1)/2 \times 4$</td>
<td>$2N(2N+1)/2 \times 4$</td>
<td>$2N(2N-1)/2 \times 8$</td>
<td>$2N(2N+1)/2 \times 8$</td>
</tr>
</tbody>
</table>

We similarly check the supersymmetry of a parity-like $C^2/Z_2$ orbifold and obtain eq. (56) in the appendix. SUSY parameters on parity-like $C^2/Z_2$ orbifold are described as

$$
\begin{pmatrix}
\epsilon^{[m]}b^\dagger_m b^\dagger_0 |0\rangle \\
\epsilon^{[i]}b^\dagger_i b^\dagger_0 |0\rangle \\
\frac{1}{2}\epsilon^{[mij]}b^\dagger_m b^\dagger_i b^\dagger_j |0\rangle \\
\frac{1}{2}\epsilon^{[mnj]}b^\dagger_m b^\dagger_n b^\dagger_j |0\rangle
\end{pmatrix},
$$

(38)

where $m, n = 1, 2$ and $i, j = 3, 4$. This result on a parity-like $C^2/Z_2$ orbifold is different from that on a normal $C^2/Z_2$ orbifold. Namely this parity-like $C^2/Z_2$ orbifold has $\mathcal{N} = 2$ supersymmetry. The bosonic fields $X^\mu_\parallel$ are transformed to the fermionic fields $\Theta_{2\pi i}$ and $\Theta_\omega$, by SUSY transformation with the parameter $\epsilon^{[m]}$ and $\epsilon^{[i]}$, respectively, while the bosonic fields $X^i_\perp$ are transformed to the fermionic fields $\Theta_{2\pi i}$ and $\Theta_\omega$ by the SUSY transformation with the parameter $\epsilon^{[i]}$ and $\epsilon^{[m]}$, respectively.

**Conclusions**

In this paper, we have obtained the supersymmetric Matrix models on some non-compact orbifolds, namely $C^2/Z_2$, $C^2/Z_4$ and $C^3/Z_6$ with a cyclic identification and $C^2/Z_2$ with a parity-like identification. We considered that the bosonic and fermionic fields consisting of $nN$ D-particles had the $Z_n$ symmetry, and these fields were represented by $nN \times nN$ matrices. The fields in 4 dimensions are remained on the non-orbifolded space-time. Imposing the Majorana condition on the orbifolded Matrix model, we have got eq. (19) and been able to restrict to the
special orbifolds, e.g., the $C^2/Z_2$, $C^2/Z_4$, $C^3/Z_3$ and $C^3/Z_6$ orbifolds. The degrees of freedom of the bosonic and fermionic fields are equal.

Firstly, the number of SUSY parameters of the $C^2/Z_2$ and $C^2/Z_4$ orbifolds with a cyclic identification becomes the half of those of Matrix models. So the number of the supercharges becomes equal to that of $\mathcal{N} = 1$ SUSY in 10 dimensions.

Secondly the number of SUSY parameters of the $C^3/Z_6$ orbifold with a cyclic identification becomes equal to the case of Matrix models. Then the number of the supercharges remains the same as that of $\mathcal{N} = 2$ SUSY in 10 dimensions.

In the case of first and second cases, with respect to the $C^2/Z_2$, $C^2/Z_4$ and $C^3/Z_6$ orbifolds with a cyclic identification, the bosonic fields $X^\mu_\parallel$ without imposing the orbifold on the space-time are transformed to the fermionic fields $\Theta_{2\pi i}$ without imposing the orbifold on the space-time by the SUSY parameters. While the bosonic fields $Z_i$ with orbifolding spaces are transformed to the fermionic fields $\Theta_\omega$ with imposing the orbifolding spaces by the SUSY parameters.

Finally the number of degrees of freedom of the $C^2/Z_2$ orbifold with a parity-like identification is equal to the case of Matrix models. So the number of the supercharges also remains the same as $\mathcal{N} = 2$ SUSY in 10 dimensions. Then SUSY of the $C^2/Z_2$ orbifold with a parity-like identification has SUSY of the $C^2/Z_2$ and $C^2/Z_4$ orbifolds with a cyclic identification and furthermore the bosonic fields $X^\mu_\parallel (Z_i)$ are transformed to the fermionic fields $\Theta_\omega (\Theta_{2\pi i})$.

Furthermore compactifying for the $C^2/Z_2$ orbifold spaces, we may find that $\mathcal{N} = 4$ SUSY in 6 dimensions appears.

**Acknowledgment**

I would deeply like to thank Prof. A. Sugamoto for collaborating the previous work and a part of this work with many useful discussions and encouraging greatly in the course of this work. Thanks are also due to Prof. N. Kawamoto for useful suggestions in the completion of this paper and reading the manuscript.

**Appendix**

**Gamma matrices**

Gamma matrices which belong to the Clifford algebra are defined as

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu},$$
The gamma matrices in 10 dimensions are given as

\[
\begin{pmatrix}
\Gamma^0 \\
\Gamma^1 \\
\Gamma^2 \\
\Gamma^3 \\
\Gamma^4 \\
\Gamma^5 \\
\Gamma^6 \\
\Gamma^7 \\
\Gamma^8 \\
\Gamma^9
\end{pmatrix}
= \begin{pmatrix}
i\sigma_2^{(0)} \\
\sigma_1^{(0)} \\
1^{(0)} \otimes \sigma_1^{(1)} \\
1^{(0)} \otimes \sigma_2^{(1)} \\
1^{(0)} \otimes (1^{(1)} \otimes \sigma_1^{(2)}) \\
1^{(0)} \otimes (1^{(1)} \otimes (1^{(2)} \otimes \sigma_1^{(3)})) \\
1^{(0)} \otimes (1^{(1)} \otimes (1^{(2)} \otimes (1^{(3)} \otimes \sigma_2^{(4)}))
\end{pmatrix},
\]

where \(1^{(\mu)}\) are 2 \(\times\) 2 unit matrices and \(\sigma_i\) \((i = 1, 2, 3)\) are Pauli matrices;

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

A complex conjugate matrix \(B\) is defined by \(\Gamma^\mu \equiv B\Gamma^\mu B^{-1}\) and is written as follows:

\[
B = c\Gamma^3\Gamma^5\Gamma^7\Gamma^9,
\]

where \(c\) is a phase which satisfies \((BB^\dagger = B^\dagger B = BB^* = B^*B = )|c|^2 = 1\) with Majorana condition:

\[
\psi^c \equiv \psi,
\]

\[
\psi = B^{-1}\psi^*.
\]

**Spinors**

Spinors in 10 dimensions are composed of raising and lowering operators of gamma matrices. The lowering operators are defined as

\[
b_0 = \frac{1}{2}(\Gamma^1 - \Gamma^0), \quad b_j = \frac{1}{2}(\Gamma^{2j} - i\Gamma^{2j+1}),
\]
where \( j = 1, \ldots, 4 \). The eigen state of spinors has all down spin states and is defined as follows:

\[
|0\rangle \equiv |-,-,-,-,-\rangle.
\]  

(45)

Imposing the raising operator \( b^\dagger_\mu \) (\( \mu = 0, \ldots, 4 \)) on eq. (45), the \( \mu \)-th down spin state turns into the excited up spin state. From these states we can give type II 16 Majorana-Weyl fermions in 10 dimensions, and the 0-th down spin is fixed on the mass shell.

\[
|\psi\rangle = \begin{pmatrix}
\psi|0\rangle \\
\frac{1}{2}\epsilon_{ij}|\psi^{ij}b^\dagger_ib^\dagger_j|0\rangle \\
\psi^{[1234]}b^\dagger_1b^\dagger_2b^\dagger_3b^\dagger_4|0\rangle
\end{pmatrix},
\]  

(46)

where \( |\psi\rangle \) are Majorana-Weyl spinors, i.e., \( \psi^{[1234]*}|0\rangle = \psi|0\rangle \) and \( \psi^{ij}*|0\rangle = -\frac{1}{2}\epsilon_{ijk}\psi^{[kl]}|0\rangle \).

### Spinors on the Orbifolds

We impose the \( Z_n \) symmetry for some complex spaces in 10 dimensions and then we represent spinors corresponding to the complex spaces.

\[
\hat{\alpha}|\psi\rangle \equiv \exp \left( 2\pi i \sum_{\mu=0}^{4} \frac{n_\mu}{n} b^\dagger_\mu b_\mu \right)|\psi\rangle,
\]  

(47)

where \( \mu = 0, \ldots, 4, n_\mu \) and \( n \) are the integer and \( \hat{\alpha}' \) is the operator for \( Z_n \) orbifold. Imposing Majorana condition, we obtain a condition:

\[
\exp \left( 2\pi i \sum_{j=1}^{4} \frac{n_j}{n} \right) = c.
\]  

(48)

Because of \( c^2 = 1 \) (\( B^2\psi = B\psi^* = \psi \)), we find that \( \sum \frac{n_j}{n} \) in eq. (48) has the integer or half-integer.

We substitute eq. (46) into eq. (47) and show the states of orbifolded spinors in table 4, 5, 6, which are obtained by imposing the symmetry of the orbifold with respect to spinors in 10 dimensions, and 7 in the case of \( n = 2, 3, 4 \), and then we write the rotated spinors for the phase \( \omega_j (= \exp(2\pi in_j/n)) \) as \( \Theta_{2\pi in_j/n} \) on the mass shell. From the table 4, 5, 6 and 7, we find that spinors have the next relations:

\[
\Theta_{(2\pi i-a)j}^* = \Theta_{ai}, \quad (c = 1),
\]

\[
\Theta_{(\pi-a)j}^* = \Theta_{ai}, \quad (c = -1).
\]

(49)

Then we find that \( Z_{2m+1} \) orbifolded spaces in the case of \( c = -1 \) do not exist, which can be understood from the left part of eq. (48).
The conjugate fields for the bosonic fields \((X^I)_{ij}\) and the fermionic fields \((\Theta)_{ij}\) in 10 dimensions are written as

\[
\begin{align*}
(P_i^I)_{ji} & \equiv \frac{\partial L}{\partial (D_t X^I)_{ij}} = \frac{1}{g^2} (D_t X_I)_{ji}, \\
(P_{\Theta})_{ji} & \equiv \frac{\partial L}{\partial (D_t \Theta)_{ij}} = i (\Theta^I)_{ji},
\end{align*}
\]

(50)
Therefore, the supersymmetric charge is given as follows:

\[ \epsilon' Q = \text{Tr} \left( \Pi_I \delta X^I + 2 \Pi_{\Theta} \delta \Theta \right), \]

where \( \Theta \) are Majorana-Weyl fermions. We find that supercharge of IIA Matrix model in 10 dimensions is written as eq. (25).
Supercharge on orbifolds

Supercharge on $C^2/Z_2$ orbifold is written as follows:

\[
\bar{\epsilon} Q = -\frac{2}{g} \text{Tr} \left[ i \langle \phi | \hat{H}_1 \hat{B}_{1m} + \hat{H}_2 \hat{B}_{2m} | \epsilon^{[m]} \rangle \delta_{km} \\
+ i \langle \psi^{[12]} | \hat{H}_1 \hat{B}_{1m} + \hat{H}_2 \hat{B}_{2m} | \epsilon^{[m]} \rangle \epsilon_{km} \\
+ i \langle \psi^{[mi]} | \hat{B}_1^i \hat{B}_1^j + \hat{B}_2^i \hat{B}_2^j | \epsilon^{[n]} \rangle \epsilon_{ij} \delta_{mn} \\
+ \langle \psi | \hat{H}_1 ( [H_{10}, B_{1m}] + [H_{20}, B_{2m}] ) | \epsilon^{[n]} \rangle \delta_{mn} \\
+ \langle \psi^{[12]} | \hat{H}_1 ( [H_{10}, B_{1m}] + [H_{20}, B_{2m}] ) | \epsilon^{[n]} \rangle \epsilon_{mn} \\
+ \langle \psi^{[mi]} | \hat{B}_1^i \left( [H_{10}, B_{1j}] + [H_{20}, B_{2j}] \right) | \epsilon^{[n]} \rangle \delta_{mn} \epsilon_{ij} (-1) \\
+ \langle \psi^{[mi]} | \hat{B}_2^i \left( \epsilon \omega H_{20} B_{1j} - B_{1j} H_{20} \right) + \epsilon \omega H_{20} B_{2j} - B_{2j} H_{20} \right) | \epsilon^{[n]} \rangle \delta_{mn} \epsilon_{ij} (-1) \\
+ \text{h.c.} \right],
\]

(53)

where $m, n = 1, 2, i, j = 3, 4,$ and we define $B_{1m}$ as $H_{1(2m)} + i H_{1(2m+1)}$, $B_{2m}$ as $H_{2(2m)} + i H_{2(2m+1)}$ and $\hat{B}_{1m}$ as $\partial_t B_{1m}$ and so on. We find that the bosonic fields are transformed to the fermionic fields with supersymmetric parameters $\epsilon^{[n]}$ on $C^4/Z_2$ orbifold and the number of the parameters $\epsilon^{[n]}$ is the half of the number of the parameters $(\epsilon^{[n]}, \epsilon^{[i]})$ for IIA Matrix model.

Supercharge on $C^2/Z_4$ orbifold is written as

\[
\bar{\epsilon} Q = -\frac{4}{g} \text{Tr} \left[ i \langle \psi | \hat{H}_1 \hat{B}_{1m} + \hat{A}_1 \hat{B}_{2m} + \hat{A}_1^\dagger A_{10} | \epsilon^{[n]} \rangle \delta_{mn} \\
+ i \langle \psi^{[12]} | \hat{H}_1 \hat{B}_{1m} + \hat{A}_1 \hat{B}_{2m} + \hat{A}_1^\dagger \hat{A}_{10} | \epsilon^{[n]} \rangle \epsilon_{mn} \\
+ i \langle \psi^{[mi]} | \hat{B}_1^i \hat{B}_1^j + \hat{B}_2^i \hat{B}_2^j + \hat{B}_3^i \hat{B}_3^j + \hat{B}_4^i \hat{B}_4^j | \epsilon^{[n]} \rangle \delta_{mn} \epsilon_{ik} \\
+ \langle \psi | \hat{H}_1 \left( [H_{10}, B_{1m}] + [A_{10}, B_{2m}] \right) + [H_{20}, B_{3m}] + [A_{10}^\dagger, B_{2m}] \right) | \epsilon^{[n]} \rangle \delta_{mn} \\
+ \langle \psi^{[12]} | \hat{H}_1 \left( [H_{10}, B_{1m}] + [A_{10}, B_{2m}] \right) + [H_{20}, B_{3m}] + [A_{10}^\dagger, B_{2m}] \right) | \epsilon^{[n]} \rangle \epsilon_{mn} \\
+ \langle \psi^{[12]} | \hat{A}_1^\dagger \left( [H_{10}, B_{2m}] + [A_{10}, B_{1m}] + [H_{20}, B_{4m}] + [A_{10}^\dagger, B_{3m}] \right) | \epsilon^{[n]} \rangle \delta_{mn} \\
+ \langle \psi^{[12]} | \hat{A}_1^\dagger \left( [H_{10}, B_{2m}] + [A_{10}, B_{1m}] + [H_{20}, B_{4m}] + [A_{10}^\dagger, B_{3m}] \right) | \epsilon^{[n]} \rangle \epsilon_{mn} \\
+ \langle \psi | \hat{H}_2 \left( [H_{10}, B_{3m}] + [A_{10}, B_{2m}] + [H_{20}, B_{1m}] + [A_{10}^\dagger, B_{3m}] \right) | \epsilon^{[n]} \rangle \delta_{mn} \\
+ \langle \psi^{[12]} | \hat{H}_2 \left( [H_{10}, B_{3m}] + [A_{10}, B_{2m}] + [H_{20}, B_{1m}] + [A_{10}^\dagger, B_{3m}] \right) | \epsilon^{[n]} \rangle \epsilon_{mn} \\
+ \text{h.c.} \right],
\]

(53)
\[ \langle \psi | \hat{A}_1 \left( [H_{10}, B_{4n}] + [A_{10}, B_{3m}] + [H_{20}, B_{2m}] + [A_{10}^\dagger, B_{1m}] \right) | \epsilon^{[n]} \rangle \delta_{mn} \\
+ \langle \psi^{[12]} | \hat{A}_1 \left( [H_{10}, B_{4m}] + [A_{10}, B_{3m}] + [H_{20}, B_{2m}] + [A_{10}^\dagger, B_{1m}] \right) | \epsilon^{[n]} \rangle \epsilon_{mn} \\
+ \langle \psi^{[m]} | \hat{B}_1 \left( [H_{10}, B_{1j}^\dagger] + (A_{10}B_{2j}^\dagger - \omega B_{3j}^\dagger A_{10}) + (H_{20}B_{3j}^\dagger - \omega^2 B_{3j}^\dagger H_{20}) \\
+ (A_{10}^\dagger B_{1j}^\dagger - \omega^2 B_{1j}^\dagger A_{10}) \right) | \epsilon^{[n]} \rangle \delta_{ij} \delta_{mn} (-1) \\
+ \langle \psi^{[m]} | \hat{B}_2 \left( [H_{10}, B_{2j}^\dagger] + (A_{10}B_{1j}^\dagger - \omega B_{3j}^\dagger A_{10}) + (H_{20}B_{3j}^\dagger - \omega^2 B_{1j}^\dagger H_{20}) \\
+ (A_{10}^\dagger B_{1j}^\dagger - \omega^2 B_{1j}^\dagger A_{10}) \right) | \epsilon^{[n]} \rangle \delta_{ij} \delta_{mn} (-1) \\
+ \langle \psi^{[m]} | \hat{B}_3 \left( [H_{10}, B_{3j}^\dagger] + (A_{10}B_{3j}^\dagger - \omega B_{1j}^\dagger A_{10}) + (H_{20}B_{1j}^\dagger - \omega^2 B_{2j}^\dagger H_{20}) \\
+ (A_{10}^\dagger B_{2j}^\dagger - \omega^2 B_{2j}^\dagger A_{10}) \right) | \epsilon^{[n]} \rangle \delta_{ij} \delta_{mn} (-1) \\
+ \langle \psi^{[m]} | \hat{B}_4 \left( [H_{10}, B_{4j}^\dagger] + (A_{10}B_{4j}^\dagger - \omega B_{2j}^\dagger A_{10}) + (H_{20}B_{2j}^\dagger - \omega^2 B_{2j}^\dagger H_{20}) \\
+ (A_{10}^\dagger B_{3j}^\dagger - \omega^2 B_{3j}^\dagger A_{10}) \right) | \epsilon^{[n]} \rangle \delta_{ij} \delta_{mn} (-1) \\
+ \text{h.c.} \right), \tag{54} \]

where \( m, n = 1, 2, i, j = 3, 4 \), and we define \( B_{1m} \) as \( H_{1(2m)} + iH_{1(2m+1)} \), \( B_{2m} \) as \( A_{1(2m)} + iA_{1(2m+1)} \), \( B_{3m} \) as \( H_{2(2m)} + iH_{2(2m+1)} \) and \( B_{4m} \) as \( A_{1(2m)} + iA_{1(2m+1)} \).

Supercharge on \( \mathbb{C}^3/\mathbb{Z}_6 \) orbifold is written as

\[ \bar{e}Q = \frac{6}{g} \text{Tr} \left[ i \langle \psi | \hat{H}_1 \hat{B}_{11} + \hat{A}_1^\dagger \hat{B}_{21} + \hat{H}_2 \hat{B}_{31} + \hat{H}_3 \hat{B}_{41} + \hat{H}_2 \hat{B}_{51} + \hat{A}_1 \hat{B}_{61} | \epsilon^{[1]} \rangle \right. \\
+ \langle \psi^{[ij]} | \hat{B}_1 \hat{B}_{1i}^\dagger + \hat{B}_2 \hat{B}_{2i}^\dagger + \hat{B}_3 \hat{B}_{3i}^\dagger + \hat{B}_4 \hat{B}_{4i}^\dagger + \hat{B}_5 \hat{B}_{5i}^\dagger + \hat{B}_6 \hat{B}_{6i}^\dagger | \epsilon^{[i]} \rangle \right] \\
\times (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \\
+ \langle \psi | \hat{H}_1 \left( [H_{10}, B_{11}] + [A_{10}, B_{61}] + [H_{20}, B_{51}] + [H_{30}, B_{41}] \\
+ [H_{20}, B_{31}] + [A_{10}^\dagger, B_{21}] \right) | \epsilon^{[1]} \rangle \\
+ \langle \psi | \hat{A}_1 \left( [H_{10}, B_{21}] + [A_{10}, B_{11}] + [H_{20}, B_{61}] + [H_{30}, B_{51}] \\
+ [H_{20}, B_{41}] + [A_{10}^\dagger, B_{31}] \right) | \epsilon^{[1]} \rangle \\
+ \langle \psi | \hat{H}_2 \left( [H_{10}, B_{31}] + [A_{10}, B_{21}] + [H_{20}, B_{11}] + [H_{30}, B_{61}] \\
+ [H_{20}, B_{51}] + [A_{10}^\dagger, B_{41}] \right) | \epsilon^{[1]} \rangle \\
+ \langle \psi | \hat{H}_3 \left( [H_{10}, B_{41}] + [A_{10}, B_{31}] + [H_{20}, B_{21}] + [H_{30}, B_{51}] \\
+ [H_{20}, B_{61}] + [A_{10}^\dagger, B_{51}] \right) | \epsilon^{[1]} \rangle \\
+ \langle \psi | \hat{H}_2 \left( [H_{10}, B_{51}] + [A_{10}, B_{41}] + [H_{20}, B_{31}] + [H_{30}, B_{21}] \\
+ [H_{20}, B_{11}] + [A_{10}^\dagger, B_{61}] \right) | \epsilon^{[1]} \rangle \\
+ \langle \psi | \hat{A}_1 \left( [H_{10}, B_{61}] + [A_{10}, B_{51}] + [H_{20}, B_{41}] + [H_{30}, B_{31}] \right) \]
\[ + [H_{20}, B_{21}] + [A_{10}, B_{11}] \right) |\epsilon^{[1]} \rangle \]
\[ + \langle \psi^{[ij]} | B_{\hat{1}} \left( [H_{10}, B_{1k}^{\dagger}] + (A_{10} B_{2k}^{\dagger} - \omega B_{2k}^{\dagger} A_{10}) + (H_{20} B_{3k}^{\dagger} - \omega^2 B_{3k}^{\dagger} H_{20}) \right. \]
\[ \left. + (H_{30} B_{4k}^{\dagger} - \omega^3 B_{4k}^{\dagger} H_{30}) + (H_{20} B_{5k}^{\dagger} - \omega^2 B_{5k}^{\dagger} H_{20}) \right. \]
\[ \left. + (A_{10} B_{6k}^{\dagger} - \omega B_{6k}^{\dagger} A_{10}^{\dagger}) \right) |\epsilon^{[\ell]} \rangle \delta_{ij} \delta_{ik} - \delta_{jk} \delta_{il} \]
\[ + \langle \psi^{[ij]} | B_{\hat{2}} \left( [H_{10}, B_{2k}^{\dagger}] + (A_{10} B_{3k}^{\dagger} - \omega B_{3k}^{\dagger} A_{10}) + (H_{20} B_{4k}^{\dagger} - \omega^2 B_{4k}^{\dagger} H_{20}) \right. \]
\[ \left. + (H_{30} B_{5k}^{\dagger} - \omega^3 B_{5k}^{\dagger} H_{30}) + (H_{20} B_{6k}^{\dagger} - \omega^2 B_{6k}^{\dagger} H_{20}) \right. \]
\[ \left. + (A_{10} B_{1k}^{\dagger} - \omega B_{1k}^{\dagger} A_{10}^{\dagger}) \right) |\epsilon^{[\ell]} \rangle \delta_{ij} \delta_{ik} - \delta_{jk} \delta_{il} \]
\[ + \langle \psi^{[ij]} | B_{\hat{3}} \left( [H_{10}, B_{3k}^{\dagger}] + (A_{10} B_{4k}^{\dagger} - \omega B_{4k}^{\dagger} A_{10}) + (H_{20} B_{5k}^{\dagger} - \omega^2 B_{5k}^{\dagger} H_{20}) \right. \]
\[ \left. + (H_{30} B_{6k}^{\dagger} - \omega^3 B_{6k}^{\dagger} H_{30}) + (H_{20} B_{5k}^{\dagger} - \omega^2 B_{5k}^{\dagger} H_{20}) \right. \]
\[ \left. + (A_{10} B_{6k}^{\dagger} - \omega B_{6k}^{\dagger} A_{10}^{\dagger}) \right) |\epsilon^{[\ell]} \rangle \delta_{ij} \delta_{ik} - \delta_{jk} \delta_{il} \]
\[ + \langle \psi^{[ij]} | B_{\hat{4}} \left( [H_{10}, B_{4k}^{\dagger}] + (A_{10} B_{5k}^{\dagger} - \omega B_{5k}^{\dagger} A_{10}) + (H_{20} B_{6k}^{\dagger} - \omega^2 B_{6k}^{\dagger} H_{20}) \right. \]
\[ \left. + (H_{30} B_{1k}^{\dagger} - \omega^3 B_{1k}^{\dagger} H_{30}) + (H_{20} B_{2k}^{\dagger} - \omega^2 B_{2k}^{\dagger} H_{20}) \right. \]
\[ \left. + (A_{10} B_{3k}^{\dagger} - \omega B_{3k}^{\dagger} A_{10}^{\dagger}) \right) |\epsilon^{[\ell]} \rangle \delta_{ij} \delta_{ik} - \delta_{jk} \delta_{il} \]
\[ + \langle \psi^{[ij]} | B_{\hat{5}} \left( [H_{10}, B_{5k}^{\dagger}] + (A_{10} B_{6k}^{\dagger} - \omega B_{6k}^{\dagger} A_{10}) + (H_{20} B_{1k}^{\dagger} - \omega^2 B_{1k}^{\dagger} H_{20}) \right. \]
\[ \left. + (H_{30} B_{2k}^{\dagger} - \omega^3 B_{2k}^{\dagger} H_{30}) + (H_{20} B_{5k}^{\dagger} - \omega^2 B_{5k}^{\dagger} H_{20}) \right. \]
\[ \left. + (A_{10} B_{4k}^{\dagger} - \omega B_{4k}^{\dagger} A_{10}^{\dagger}) \right) |\epsilon^{[\ell]} \rangle \delta_{ij} \delta_{ik} - \delta_{jk} \delta_{il} \]
\[ + \langle \psi^{[ij]} | B_{\hat{6}} \left( [H_{10}, B_{6k}^{\dagger}] + (A_{10} B_{1k}^{\dagger} - \omega B_{1k}^{\dagger} A_{10}) + (H_{20} B_{2k}^{\dagger} - \omega^2 B_{2k}^{\dagger} H_{20}) \right. \]
\[ \left. + (H_{30} B_{3k}^{\dagger} - \omega^3 B_{3k}^{\dagger} H_{30}) + (H_{20} B_{5k}^{\dagger} - \omega^2 B_{5k}^{\dagger} H_{20}) \right. \]
\[ \left. + (A_{10} B_{5k}^{\dagger} - \omega B_{5k}^{\dagger} A_{10}^{\dagger}) \right) |\epsilon^{[\ell]} \rangle \delta_{ij} \delta_{ik} - \delta_{jk} \delta_{il} \]
\[ + h.c., \]

(55)

where \( i, j, k, l = 1, 2, 3, \) and we define \( B_{11} \) as \( H_{1(2m)} + iH_{1(2m+1)} \), \( B_{21} \) as \( A_{1(2m)} + iA_{1(2m+1)} \), \( B_{31} \) as \( H_{2(2m)} + iH_{2(2m+1)} \), \( B_{41} \) as \( H_{3(2m)} + iH_{3(2m+1)} \), \( B_{51} \) as \( H_{2(2m)} + iH_{2(2m+1)} \) and \( B_{61} \) as \( A_{1(2m)} + iA_{1(2m+1)} \).

Supercharge on a parity like \( C^2 / Z_2 \) orbifold included in \( USp(2N) \) symmetry is written as

\[
\tilde{Q} = - \frac{1}{g} \text{Tr} \left[ 2 i \langle \psi | \hat{H}_1 \hat{B}_{1m} - \frac{1}{2} \left( \hat{J} \hat{B}_{2m}^* + \hat{J}^* \hat{B}_{2m} \right) |\epsilon^{[m]} \rangle \right] \delta_{mn}
+ 2 i \langle \psi | \hat{H}_1 \hat{B}_{1m} - \frac{1}{2} \left( \hat{J} \hat{B}_{2m}^* + \hat{J}^* \hat{B}_{2m} \right) |\epsilon^{[m]} \rangle \epsilon_{mn}
+ 2 i \langle \psi | \hat{H}_2 \hat{B}_{1j} + \frac{1}{2} \left( \hat{S} \hat{B}_{2j}^* + \hat{S}^* \hat{B}_{2j} \right) |\epsilon^{[m]} \rangle \delta_{ij} \delta_{mn} (-1)
\]

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where \( m, n = 1, 2 \) and \( i, j = 3, 4 \), and we define \( B_{1m} \) as \( H_{1(2m)} + iH_{1(2m+1)} \), \( B_{2m} \) as \( J_{2m} + iJ_{2m+1} \), \( B_{1i} \) as \( H_{1(2i)} + iH_{1(2i+1)} \) and \( B_{2i} \) as \( S_{2i} + iS_{2i+1} \).

References


