Matrix Models, Argyres-Douglas singularities and double scaling limits

Gaetano Bertoldi *

Center for Theoretical Physics,
 Massachusetts Institute of Technology
 Cambridge MA 02139
 bertoldi@mit.edu

ABSTRACT: We construct an $\mathcal{N} = 1$ theory with gauge group $U(nN)$ and degree $n + 1$ tree level superpotential whose matrix model spectral curve develops an Argyres-Douglas singularity. The calculation of the tension of domain walls in the $U(nN)$ theory shows that the standard large $N$ expansion breaks down at the Argyres-Douglas points, with tension that scales as a fractional power of $N$. Nevertheless, it is possible to define appropriate double scaling limits which are conjectured to yield the tension of 2-branes in the resulting $\mathcal{N} = 1$ four dimensional non-critical string theories as proposed by Ferrari.

KEYWORDS: matrix models, Argyres-Douglas points, double scaling limits, non-critical strings.

*Research supported in part by the CTP and the LNS of MIT and the U.S. Department of Energy under cooperative research agreement # DE-FC02-94ER40818. G. B. is also supported in part by the INFN “Bruno Rossi” Fellowship.
1. Introduction

In [1, 2, 3], Dijkgraaf and Vafa conjectured that the exact superpotential and gauge couplings of a class of $\mathcal{N} = 1$ super Yang-Mills theories can be calculated by doing perturbative computations in an auxiliary matrix model. They considered theories with a polynomial superpotential $W(\Phi)$ for the chiral adjoint field $\Phi$ and proposed that $W$ is actually the potential in the related matrix model. Furthermore, only planar diagrams in the matrix model contribute to the effective superpotential. This striking result was later proved with perturbative field theory arguments in [4] and by the analysis of the generalized Konishi anomaly in [5]. The solution of the matrix model in the planar limit is captured by the so-called spectral curve, which is given by

\[ \begin{align*}
    \text{In [6], Ferrari studied an } \mathcal{N} = 1 \ U(N) \text{ gauge theory with cubic superpotential and using the results of [7] discovered that, in the phase where the gauge group is unbroken, there are critical values of the superpotential couplings where the effective superpotential is non-analytic and the standard large } N \text{ expansion is singular. In fact, the tension of supersymmetric domain walls scales as a fractional power of } N \text{ at the critical points. This breakdown of the } 1/N \text{ expansion can be compensated by taking the limit } N \to \infty \text{ and approaching the critical points in a correlated way. Furthermore, these double scaling limits are conjectured to define an } \mathcal{N} = 1 \text{ four dimensional non-critical string theory. This relies on a proposal made by Ferrari on how to generalize the old matrix model approach to non-critical strings [8] to the four dimensional case [9, 10, 11, 12, 13]. In this approach the matrix model integral is replaced by the four dimensional gauge theory path integral itself and the continuous worldsheets are built out of the large Feynman diagrams in the } N \to \infty \text{ limit. It was also shown in [6] that, from the matrix model point of view, the singularity corresponds to a transition from a two-cut solution to a one-cut solution. A cycle of the genus 1 spectral curve that describes the two-cut solution shrinks to zero size.}

    \text{Ferrari also raised the question of the structure of higher order critical points à la Argyres-Douglas [14]. A matrix model spectral curve undergoes such a degeneration when two or more cycles with non-vanishing intersection shrink to zero size simultaneously. These singularities were first investigated in the context of } \mathcal{N} = 2 \text{ super} \end{align*} \]
Yang-Mills theories, whose low-energy physics is encoded by Seiberg-Witten hyperelliptic curves [15, 16, 17]. Their importance lies in the fact that, since the vanishing cycles have non-trivial intersection, the low-energy theory contains both electric and magnetic charges [14]. Furthermore, they are non-trivial interacting $\mathcal{N}=2$ conformal field theories [14, 18, 19, 20] and they provide the first quantitative check of the scenario advocated by Ferrari. In fact, it was shown in [10, 11], that these $\mathcal{N}=2$ CFTs can be used to define double scaling limits that yield dual non-critical string theories.

In this paper, higher critical points à la Argyres-Douglas in $\mathcal{N}=1$ theories are constructed and studied. In particular, a $U(nN)$ gauge theory breaking to $U(N)^n$ in the presence of the one-parameter superpotential

\[ y^2 = (x^n - u)^2 - 4\Lambda^{2n}, \quad n \geq 3. \]

2. Argyres-Douglas singularities

Argyres-Douglas singularities were originally investigated in the context of $\mathcal{N}=2$ super Yang-Mills theories, whose low-energy physics is encoded by Seiberg-Witten hyperelliptic curves [15, 16, 17]. The fact that the vanishing cycles have non-trivial intersection implies that the low-energy $\mathcal{N}=2$ theory has massless solitons with mutually non-local charges [14]. Namely, these solitons are both electrically and magnetically charged under the same $U(1)$ factor. The theories at these points are actually superconformal [18]. As an illustration, let us consider the $\mathcal{N}=2$ Seiberg-Witten curve for $SU(3)$ [15, 16, 17]

\[ y^2 = (x^3 - ux - v)^2 - 4\Lambda^6. \]

The above genus 2 hyperelliptic curve is singular whenever the polynomial on the r.h.s. has at least a double root, which is equivalent to the vanishing of its discriminant

\[ \Delta = 2^{12}\Lambda^{18} \left( 4u^3 - 27(v + 2\Lambda^3)^2 \right) \left( 4u^3 - 27(v - 2\Lambda^3)^2 \right). \]

For instance for $v = 0$, $u = 3e^{2\pi ik/3}\Lambda^2$, $k = 0, 1, 2$, the curve reduces to

\[ y^2 = (x - e^{\pi ik/3}\Lambda)^2(x + e^{\pi ik/3}\Lambda)^2(x^2 - 4e^{2\pi ik/3}\Lambda^2), \]

which has two double roots. This is the limit where two mutually local dyons become massless. Argyres-Douglas points in moduli space, however, correspond to higher order singularities. An example is given by $u = 0$ and $v = \pm 2\Lambda^3$, where the curve becomes [14]

In the following, a simple generalization of the above singularity will be considered. In particular, the on shell spectral curve of the $\mathcal{N}=1$ system studied in the paper is going to be

\[ y^2 = (x^n - u)^2 - 4\Lambda^{2n}, \quad n \geq 3. \]
It is easy to recognize that for \( u = \pm 2\Lambda^n \), \( n \) out of the \( 2n \) branch points coalesce leading to an Argyres-douglas singularity. Before introducing the specific model which is object of study, it is necessary to review the strong coupling approach to the study of \( \mathcal{N} = 1 \) gauge theories with polynomial superpotentials \( \mathcal{W}(\Phi) \) [21, 22].

3. The strong coupling approach

The dynamics of \( \mathcal{N} = 1 \) \( U(N) \) gauge theories with polynomial superpotentials can be studied by treating \( \mathcal{W}(\Phi) \) as a perturbation of the underlying strongly coupled gauge theory with \( \mathcal{W} = 0 \). The latter system has \( \mathcal{N} = 2 \) supersymmetry and a Coulomb moduli space of vacua described by a Seiberg-Witten curve [15, 16, 17]

\[
y^2 = P^2_N(x) - 4\Lambda^{2N},
\]

where the coefficients of \( N \)-th order polynomial \( P_N(x) \) depend on the \( N \) moduli \( \langle tr\Phi^r \rangle, r = 1, \ldots, N \). In this strong coupling approach, which was developed in [21] using the methods of [22], \( \mathcal{W} \) is regarded as an effective superpotential on the moduli space. The generic low energy group on the Coulomb moduli space is \( U(1)^N \). Vacua in which the low energy group of the \( \mathcal{N} = 1 \) theory is \( U(1)^n \), for \( n < N \), can be found by extremizing the superpotential on submanifolds of the Coulomb branch where \( N - n \) monopoles of the \( \mathcal{N} = 2 \) theory are massless. The superpotential lifts all of the moduli space except for a finite set of vacua. At points where \( N - n \) mutually local monopoles become massless, the Seiberg-Witten curve has the following factorization

Conversely, given a polynomial \( P_N(x) \) with the above factorization, one may look for a superpotential consistent with this vacuum. This inverse technique was used in [23] to rederive the \( \mathcal{N} = 2 \) solution using the geometric engineering approach of [21]. It was also used in [24] to study the various phases of such \( \mathcal{N} = 1 \) gauge theories and the structure of their parameter space. The same authors also provided a generalization of the above approach to include the cases \( \text{deg} \mathcal{W}(\Phi) > n + 1 \) and \( \text{deg} \mathcal{W}(\Phi) > N \). Therefore, in order to construct examples of matrix model spectral curves which develop Argyres-Douglas singularities, one can start from a \( p \)-parameter family of \( \mathcal{N} = 2 \) hyperelliptic curves that displays such a degeneration in some appropriate limit. Then, using the inverse technique, it is possible to determine a superpotential consistent with these curves. The procedure yields the corresponding on shell family of \( \mathcal{N} = 1 \) spectral curves and once this is given one can study the gauge theory along the lines of [21, 1, 2, 3]. In the following, a one-parameter family of genus \( n - 1 \) hyperelliptic curves that can develop Argyres-Douglas singularities is introduced. Then, a consistent order \( (n + 1) \) superpotential \( \mathcal{W}(\Phi) \) is determined. Finally, the effective superpotential and glueball superfields \( S_k \) will be evaluated.
3.1. The model
Consider a $U(n)$ Seiberg-Witten curve of the following form

4. The glueball superfields
The glueball superfields $S_k$ are given by the period integrals of the meromorphic one-form $y dx$ along the closed loops $A_k$ surrounding the $k$-th branch cut [21, 1, 2, 3]

By (??), we can also see that $S_k$ is a homogeneous function of degree $n + 1$, which yields

4.1. Solution of the Picard-Fuchs equation
By the following change of variables

$$z = \frac{u^2}{4\Lambda^{2n}}$$

Eq.(??) becomes a hypergeometric equation

In the limit $u \to \infty$, which is dual to $\Lambda \to 0$, the solutions of Eq.(??) are asymptotic to

$$u^{\alpha_{\pm}} f_{\pm}(u),$$

where

$$\alpha_{\pm} = \frac{1}{n} \pm 1, \quad \lim_{u \to \infty} f_{\pm}(u) \neq 0.$$  

Setting $S_k = u^{\alpha_{\pm}} f_k(u)$ and changing variables to $z = \frac{4\Lambda^{2n}}{u^2}$, Eq.(??) is equivalent to

4.2. Non-analytic behaviour close to the Argyres-Douglas points
By analytic continuation of (??)(??), we find [26]

$$S_k = C_{3, k} u^{-1 + \frac{1}{n}} \left( A_1 F \left( \frac{1}{2}, 1 \frac{1}{2n}; 1 \frac{1}{2n}, \frac{1}{2} \frac{1}{n}, 1 - \frac{4\Lambda^{2n}}{u^2} \right) \right)$$
5. The multiplication map

By the so-called multiplication map by \(N\) introduced in [21], the above \(U(n)\) theory with superpotential (??) can be mapped to a \(U(nN)\) theory with the same tree level superpotential. In particular, the vacuum considered up to now, which is a Coulomb vacuum with unbroken \(U(1)^n\) gauge group, is associated to \(N\) different vacua with unbroken \(U(N)^n\). In fact, given a set of polynomials \(P_n(x), F_{2m}(x)\) and \(H_{n-m}(x)\), all with the highest coefficient equal to 1, that satisfy the following relations

By Eqs.(??),(??) and (??), the spectral curve relative to one of the \(N\) vacua of the \(U(nN)\) theory breaking to \(U(N)^n\) is given by

![Figure 1: The \(B_k\) cycles and path \(\gamma\) for \(n = 3, (Im u = 0)\).](image)

6. The effective superpotential

In this section, we are going to evaluate the effective superpotential for each of the \(N\) vacua of the \(U(nN)\) theory considered above. The supersymmetric domain walls
interpolating between two such vacua are interesting objects to study and their tension can be calculated exactly once the expression of $W_{\text{eff}}$ is known \[28\]. In section 7, the large $N$ limit of the tension will be considered. It follows that, due to the non-analyticity of $W_{\text{eff}}$ at the Argyres-Douglas points, the standard $1/N$ expansion breaks down at the singularities. However, there exist double scaling limits of the tension that yield well-defined results. The general matrix model formula for the effective superpotential of a $U(N)$ theory breaking to $\prod_{k=1}^{n} U(N_k)$ is \[24, 21, 1, 2, 3\]

Let us note that

\[N_k = \frac{1}{2\pi i} \int_{A_k} T,\]

and

\[\tau_0 = \frac{1}{2\pi i} \int_{B_1} \tilde{T}, \quad b_k = -\frac{1}{2\pi i} \int_{B_k} \tilde{T} - \tau_0,\]

where the $A_k$’s are the closed cycles surrounding the branch cuts of the spectral curve and the $B_k$’s are the non-compact cycles connecting the points at infinity on the two sheets of the spectral curve passing through the $k$-th branch cut. The intersection pairings of these cycles are $A_j \cap A_k = B_j \cap B_k = 0$ and $A_j \cap B_k = \delta_{jk}$. In Fig.(1), we depicted the $A$ and $B$ paths on one of the two sheets for the case $n = 3$. By $\Lambda_0$ we denote the point at infinity. The path $\gamma$ enters in the rigorous definition of the matrix model conjecture recently studied by Lazaroiu \[27\] involving holomorphic matrix models. By definition, $\gamma$ threads the branch cuts and fixes the basis of $A$ and $B$ cycles on the spectral curve. The glueball superfields and the derivatives of the prepotential $F$ are given by \[21, 1, 2, 3\]

Besides the closed cycles $A_k$ it is useful to consider the closed cycles

Let us show that the symmetry (\ref{eq:1}) implies a simple relation between $\frac{\partial F}{\partial S_k}$ and $\frac{\partial F}{\partial S_{k-1}}$. First of all

\[\frac{\partial F}{\partial S_k} = \int_{B_k} ydx = \lim_{\Lambda_0 \to \infty} 2 \int_{x_{k,+}}^{\Lambda_0} ydx - 2W(\Lambda_0),\]

where

\[x_{k,\pm} = e^{2\pi ik/n} (u \pm 2\Lambda^n)^{1/n},\]

are the branch points of the spectral curve (\ref{eq:2}) and $\Lambda_0$ is the point at infinity on the upper sheet. Setting $x = e^{2\pi i/n} \tilde{x}$, the above integral becomes

\[
\int_{x_{k,+}}^{\Lambda_0} ydx = \int_{x_{k,+}}^{\Lambda_0} \sqrt{(x^n - u)^2 - 4\Lambda^{2n}} dx = e^{2\pi i/n} \int_{x_{k-1,+}}^{\Lambda_0} \sqrt{\tilde{x}^n - u)^2 - 4\Lambda^{2n}} d\tilde{x},
\]
where $\tilde{\Lambda}_0 = e^{-2\pi i/n}\Lambda_0$. Then

$$\frac{\partial F}{\partial S_k} = \lim_{\Lambda_0 \to \infty} 2 \int_{x_{k,+}}^{\Lambda_0} ydx - 2 W(\Lambda_0) = \lim_{\Lambda_0 \to \infty} 2 e^{2\pi i/n} \int_{x_{k-1,+}}^{\tilde{\Lambda}_0} ydx - 2 e^{2\pi i/n} W(\tilde{\Lambda}_0)$$

Note that in our case the low-energy gauge group is $U(1)^n$, namely $N_k = 1, k = 1, \ldots, n$. Then, it follows that

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The $A_k$ and $C_l$ cycles for $n = 3$, $(Im\, u = 0)$.}
\end{figure}

6.1. The periods of $T$

In order to evaluate the effective superpotential (??), we need to evaluate the integer constants $b_k$ or equivalently $c_k$. 

7
Then, by (??), (??), (??) and (??), the effective superpotential (??) of the $U(nN)$ theory in the $\ell$-th vacuum, $\ell = 1, \ldots, N$, is

$$W_{\text{eff}, \ell}(S) = 2\pi i \sum_{k=1}^{n} b_k S_k(u, \eta^{2\ell} \Lambda_0^{2n}) = -2\pi i N \sum_{k=1}^{n} (k-1)e^{2\pi i (k-1)/n} S_1(u, \eta^{2\ell} \Lambda_0^{2n})$$

7. The large $N$ limit

Following [6, 9], we expect a non-trivial behaviour of the large $N$ limit at the Argyres-Douglas critical points. An observable that can be determined exactly from the superpotential (??) is the tension of supersymmetric domain walls between the $N$ vacua of the $U(nN)$ theory. The tension of such a BPS domain wall interpolating between the $p$-th and $q$-th vacua, $p, q = 1, \ldots, N$, is given by [28]

The large $N$ gauge theory is expected to have a dual description as a string theory with string coupling $g_{st}$ proportional to $1/N$ [29]. By (??), we see that the tension of domain walls interpolating between adjacent vacua, where $(p - q)$ is of order $N^0$, is proportional to $N$ or equivalently to $1/g_{st}$. This is the characteristic behaviour of a $D$-brane tension and it was in fact argued in [30] that the confining strings can end on the domain walls.

However, the expansion (??) is singular in the vicinity of the Argyres-Douglas points, $x = x_c = 1$, because

$$F'(x) \approx A_1 + A_2(1 - x)^{\frac{n+2}{2n}},$$

where the constants $A_1$ and $A_2$ are given in (??). Thus, $F'(x)$ is not defined for $x = x_c$. Actually, at the Argyres-Douglas points, the complexified tension becomes

7.1. The double scaling limit

Eq. (??) suggests that the divergences at $x = 1$ can be compensated by taking the limits $N \rightarrow \infty$ and $x \rightarrow x_c = 1$ in a correlated way as follows

In particular, the rescaled tensions

In a series of papers [9, 10, 11, 12], Ferrari made a proposal to generalize the matrix model approach to non-critical strings [8] to the four dimensional case. The basic idea is to replace matrix integrals with four dimensional gauge theory path integrals with
$N \times N$ adjoint Higgs fields. It was argued that a non-trivial low energy physics develops at special values of the Higgs vacuum expectation values or of the couplings of the Higgs potential. At these points the large $N$ expansion suffers from IR divergences. However, these divergences can be compensated by taking the limit $N \to \infty$ and approaching the critical points in a correlated manner. These double scaled theories are conjectured to be string theories. In particular, the continuous worldsheets are constructed from the large 't Hooft diagrams of the parent gauge theory.

These ideas were tested quantitatively on the moduli space of $\mathcal{N} = 2$ supersymmetric gauge theories in [10, 11]. It was shown that double scaling limits exist at Argyres-Douglas singularities. The non-trivial conformal field theories at these critical points are thus dual to a four dimensional string theory. In [6], the analysis was extended to the $\mathcal{N} = 1$ case. In particular, a $U(N)$ gauge theory with cubic superpotential was studied and it was shown that there are critical values of the superpotential couplings where glueballs are massless, there are tensionless domain walls and confinement without a mass gap. At these critical points, the large $N$ expansion is singular and the tension of domain walls scales as a fractional power of $N$. Nevertheless, double scaling limits analogous to (??) exist and are then conjectured to define a four dimensional non-critical string theory.

The double scaling limits (??) fit into the above scenario and are consistent with the $\mathcal{N} = 2$ analysis of [11]. The conjecture is that they define an $\mathcal{N} = 1$ four dimensional non-critical string theory. In particular, the rescaled tensions (??) are the tensions of $D2$-branes in this string theory. In fact, in the weak coupling limit, the above tension goes like $1/\kappa$. It is interesting to note that going through the branch cuts of (??) the $D2$-brane is mapped into a soliton whose tension goes like $1/\kappa^2$ at weak coupling.

8. Conclusion

Using the techniques of [21], we constructed an $\mathcal{N} = 1$ theory with gauge group $U(nN)$ and degree $n + 1$ tree level superpotential whose matrix model spectral curve develops an Argyres-Douglas singularity. This theory is closely related to an underlying $\mathcal{N} = 2$ $U(n)$ model. In fact, the one-dimensional parameter space of the $U(nN)$ theory is actually isomorphic to a slice of the $\mathcal{N} = 2$ Coulomb moduli space of the $U(n)$ theory: $n - 1$ parameters of the $U(n)$ Seiberg-Witten curve are set to zero and the remaining one parametrizes the most relevant deformation away from the singularity. The multiplication by $N$ map [21, 24] allows to take the large $N$ limit while keeping the tree level superpotential fixed. In particular, only a finite, $N$-independent, number of parameters is adjusted. The calculation of the tension of domain walls in the $U(nN)$ theory shows that the $1/N$ expansion breaks down at the Argyres-Douglas points. Nevertheless, it is possible to define appropriate double scaling limits (??) which are conjectured to yield the tension of 2-branes in the resulting $\mathcal{N} = 1$ four dimensional non-critical string theories as proposed by Ferrari in [9, 10, 11, 12, 13].
Acknowledgements

I would like to thank Amihay Hanany, Marco Matone and David Tong for discussions and useful comments. My work is supported in part by the CTP and the LNS of MIT, by the U.S. Department of Energy under cooperative research agreement # DE-FC02-94ER40818, and by the INFN “Bruno Rossi” Fellowship.

9. Appendix A

Performing the analytic continuation of (??) we find

\[ S_k = C_{1,k} F \left( -\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} - \frac{1}{2n}, \frac{1}{2}, \frac{u^2}{4\Lambda^{2n}} \right) + C_{2,k} u F \left( -\frac{1}{2n}, -\frac{1}{2n} + 1, \frac{3}{2}, \frac{u^2}{4\Lambda^{2n}} \right) \]

\[ \times \left( \log \frac{-u^2/4\Lambda^{2n}}{\Gamma(1/2 + 1/2n)} + h_{1,k} \right) + \left( u^2/4\Lambda^{2n} \right)^{1/2 + 1/2n} \frac{\Gamma(1)}{\Gamma(1/2 + 1/2n)} \]

\[ + \frac{C_{2,k} \Gamma(3/2)}{\Gamma(1 - 1/2n)} u \left( \log \frac{-u^2/4\Lambda^{2n}}{\Gamma(3/2 + 1/2n)} + \frac{(1/2 - 1/2n)_{k+1}(1/2n)_{k+1}}{k!(k+1)!} \right) \]

\[ \times \left( \log \frac{-u^2/4\Lambda^{2n}}{\Gamma(3/2 + 1/2n)} + h_{2,k} \right) + \left( u^2/4\Lambda^{2n} \right)^{1/2n} \frac{\Gamma(1)}{\Gamma(3/2 + 1/2n)} \].

In the above expression there is a term proportional to \( u^{1+1/n} \) that we need to set to zero. This determines \( C_{2,k} \) as a function of \( C_{1,k} \)

References


