Topological anomalies from the path integral measure in superspace

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Abstract

A fully quantum version of the Witten-Olive analysis of the central charge in the $\mathcal{N} = 1$ Wess-Zumino model in $d = 2$ with a kink solution is presented by using path integrals in superspace. We regulate the Jacobians with heat kernels in superspace, and obtain all superconformal anomalies as one Jacobian factor. The conserved quantum currents differ from the Noether currents by terms proportional to field equations, and these terms contribute to the anomalies. We identify the particular variation of the superfield which produces the central charge current and its anomaly; it is the variation of the auxiliary field. The quantum supersymmetry algebra which includes the contributions of superconformal anomalies is derived by using the Bjorken-Johnson-Low method instead of semi-classical Dirac brackets. We confirm earlier results that the BPS bound remains saturated at the quantum level due to equal anomalies in the energy and central charge.

1 Introduction and brief summary

Supersymmetry and topology are intimately linked. For example, instantons play an important role in the effective action for rigidly supersymmetric (susy) models[1]. The Donaldson invariants, which characterize topological properties of compact manifolds, can be computed by using a particular topological field theory which is obtained by twisting a Euclidean supersymmetric $\mathcal{N} = (2, 2)$ model[2]. We shall consider here the surface terms in the supersymmetry algebra[3] which form the central charges.

The supersymmetry algebra of the kink, an $\mathcal{N} = (1, 1)$ rigidly supersymmetric model in $1 + 1$ dimensions with a soliton solution, reads as follows at the classical level [3]

$$\{Q_{cl}, Q_{cl}\} = 2H_{cl} - 2Z_{cl}$$

(1.1)

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Here $Q_{cl}$ is the classical supersymmetry charge which leaves the classical kink solution invariant and, properly extended to the quantum level, should leave the kink vacuum invariant. $H_{cl}$ is the classical Hamiltonian which gives the classical mass of the kink solution $\varphi_K(x)$, and $Z_{cl}$ is the integral of a total derivative

$$Z_{cl} = \int_\infty^\infty U(\varphi_K) \partial_x \varphi_K dx$$  

which is non-vanishing because the kink solution has a topological twist ($\varphi_K(\infty)$ differs from $\varphi_K(-\infty)$). The result in (1.1) can be derived by using semi-classical Dirac brackets.

In the 1970’s and 1980’s solitons were studied in detail [4], and the issue whether for supersymmetric solitons $Z$ is modified by quantum corrections was studied in several articles, with conflicting results [5]. The kink model breaks conformal symmetry explicitly and is nonintegrable, and hence methods used for exactly soluble models were of no avail. Six years ago the issue whether the BPS bound $H_{cl} = Z_{cl}$ remains satisfied at the quantum level was again raised [6], and subsequently in a series of articles by several authors the quantum corrections to $\langle H \rangle$ and $\langle Z \rangle$ were calculated, where by $\langle H \rangle$ we mean the expectation value of the quantum Hamiltonian in the kink vacuum, and similarly for $\langle Z \rangle$. It was found that even though $Z$ is classically the integral of a total divergence, there are nonvanishing quantum corrections to $\langle Z \rangle$ which are equal to those to $\langle H \rangle$, so that the BPS bound remains saturated at the quantum level [7, 8, 9, 10]. The nonvanishing corrections to $\langle Z \rangle$ come from a new anomaly, whose existence was conjectured in [7] and subsequently found and evaluated in [8]. This result was in fact in conflict with the result of [9] where BPS saturation and nonvanishing quantum corrections to $\langle H \rangle$ and $\langle Z \rangle$ were obtained apparently without the need for the anomalous term in $Z$ found in [8]. However, as has been clarified recently in [10], this was due to manipulations of unregularized expressions; consistent dimensional regularization indeed reproduces the anomaly in $Z$. The existence of an anomaly in $Z$ is therefore by now beyond doubt. Qualitatively, the reason is that $Z$ is a composite operator which should be regularized at the quantum level by, for example, point-splitting, and although both the nonanomalous and the anomalous corrections to the central charge density $\zeta_0(x)$ are still total divergences in this regularization scheme, the space integral of the anomalous contributions no longer vanishes, being proportional to $\int_\infty^\infty U''(\varphi_K) \partial_x \varphi_K dx$. However, the details are quite subtle; different regularization schemes give unexpected contributions to $\zeta''$, which consist of nonanomalous and anomalous contributions. For example, using ordinary ('t Hooft-Veltman) dimensional regularization, parity violation due to massless chiral domain wall fermions in the extra dimension is responsible for the anomaly in the central charge in 2 dimensions, whereas using dimensional reduction, the anomalies reside not in loop graphs but in the evanescent counter terms which renormalize the currents [10]. Using the higher space-derivative regularization scheme, there is an extra term in the central charge current which produces the anomaly [8], while using heat kernel methods, subtle boundary contributions produce anomalies [11].

The discovery that an anomaly is present in the central charge has led to precise calculations which restore the BPS bound at the quantum level, but raises profound questions concerning topological symmetries at the quantum level. In [10], a preliminary
analysis was made of ordinary and conformal multiplets of currents, and ordinary and conformal multiplets of anomalies; in particular, a conformal central charge current was identified whose divergence contained, in addition to terms due to explicit symmetry breaking, the anomaly in the central charge. Thus the central charge contains an anomaly and is itself the anomaly of another current.

In this article, we intend to study the anomaly structure of the currents of the supersymmetric kink model in superspace. We use the path integral formulation of anomalous Ward identities [12, 13], and the present work extends a superspace analysis of conformal anomalies in QCD in 3 + 1 dimensions [14][15]. A superspace approach to the anomalies in the central charge and the kink energy was first given in [8]. In that article several regularization schemes were used to evaluate the one-loop corrections, in particular a higher derivative regularization scheme in superspace. We shall start with a path integral approach, and compute the Jacobians for generalized supersymmetry transformations using a superspace heat kernel. This will lead to a multiplet of anomalies which contains the trace anomaly and the central charge anomaly in addition to the supersymmetry anomaly and other terms. Crucial in this approach is that careful regularization of terms proportional to the field equation of the auxiliary field yields nonvanishing contributions to the Ward identities. In the literature, some articles deal directly with the central charge current while in some other articles the central charge anomaly is obtained from a supersymmetry transformation of the conformal anomaly in the supercurrent. We shall derive the central charge anomaly both directly by evaluating the Jacobian, and by a supersymmetry transformation of the conformal anomaly in the supercurrent, and show that the results agree with each other.

We begin with a local \((x \text{ and } \theta\) dependent) supersymmetry transformation of the scalar superfield \(\phi(x, \theta)\) and apply the Noether theorem in superspace. Using the quadratic part of the superspace action as regulator, we obtain a Ward identity in superspace (corresponding to a hierarchy of Ward identities in \(x\)-space) which contains the one-loop anomalies in certain quantum currents \(\tilde{J}_\mu(x), \tilde{T}_{\mu\nu}(x), \tilde{\zeta}_\mu(x)\), in addition to explicit symmetry breaking terms. The use of a superfield formulation of heat kernels in strictly \(d = 2\) Minkowski space-time to regularize the Jacobians in the path integral formulation manifestly preserves ordinary rigid supersymmetry at all stages. A subtle point is the proper identification of the quantum currents. The naive Noether currents are not conserved, but the Ward identities contain the currents \(\tilde{J}_\mu(x), \tilde{T}_{\mu\nu}(x), \tilde{\zeta}(x)_\mu\) which are conserved, so these can be used to construct time-independent charges. If we were to substitute the field equations in these conserved quantum currents, we would obtain the Noether currents, but this is not allowed at the quantum level. In fact, the contributions proportional to field equations produce anomalies, as we already mentioned.

We also derive the supersymmetry algebra at the quantum level. This requires a full quantum operator approach which incorporates anomalies, rather than a semiclassical approach based on Dirac brackets. Fortunately there exists such an approach: the Bjorken-Johnson-Low (BJL) method which automatically incorporates the effects of superconformal anomalies. The BJL method has been widely used for current algebras in the 1960’s [16]. It allows one to rewrite results obtained from path integrals into operator relations. To go from the path integral results to operator results one must use field
equations, but these field equations sometimes yield anomalies. The BJL method takes such anomalies into account, and this is crucial in our case. For readers who are not familiar with this technique we give a discussion of this method in the appendix. We thus present a fully quantum version of the Witten-Olive analysis. The deformation of the supersymmetry algebra by anomalies is such that the BPS bound remains saturated due to uniform shifts in energy and central charge. In our algebraic approach, we initially formulate all results in terms of a total superfield, and do not decompose this superfield into a background and a quantum part. Only at the end do we need to use some properties of the kink background, namely the fact that the vacuum is annihilated by one of the supersymmetry charges (the time-independent charge).

We would like to summarize our results briefly. It is shown that the Noether current for ordinary (nonconformal) supersymmetry

\[ j_\mu(x) = -[\hat{\theta}\phi + U(\phi)]\gamma^\mu \psi(x) \]  

with \( U(\phi) = g(\phi^2 - v_0^2) \) contains in the present formulation an apparent anomaly

\[ \partial_\mu j^\mu(x) = \frac{hg}{2\pi} \hat{\theta}\psi(x). \]  

(Only mass renormalization is necessary, \( v_0^2 = v^2 + \delta v^2 \), and the mass counter term \( \delta v^2 \) is fixed by requiring that tadpoles vanish in the trivial vacuum [6]. For more general renormalization conditions, see [17]).

As a result, the supercharge \( Q^\alpha = \int dx j^{0,\alpha} \) is time dependent, though \( \gamma_\mu j^\mu \) is shown to contain no anomalous component \(^1\) but only an nonanomalous component

\[ (\gamma_\mu j^\mu(x))_{anomaly} = 0. \]  

We thus define the conserved supercurrent

\[ \tilde{J}_\mu(x) = [j_\mu(x) - \frac{hg}{2\pi} \gamma_\mu \psi(x)]^\alpha; \quad \partial_\mu \tilde{J}^\mu(x) = 0 \]  

and the associated time-independent supercharge

\[ \tilde{Q}^\alpha = \int dx \tilde{J}^{0,\alpha}(x) \]  

which generates ordinary supersymmetry. This conserved current contains the following anomalous component

\[ (\gamma_\mu \tilde{J}^\mu(x))_{anomaly} = -\frac{hg}{\pi} \psi(x). \]

Because the model we consider breaks superconformal symmetry explicitly, there is also a nonanomalous contribution to \( \gamma_\mu \tilde{J}^\mu \) both at the tree and at the loop level.

\(^1\)Divergences with respect to \( \theta \) in superspace lead to anomalies in \( x \)-space which are of the form \( \gamma_\mu j^\mu \) instead of \( \partial_\mu j^\mu \). Similarly, the anomaly in the energy density is due to the trace anomaly, which itself is due to another \( \theta \) divergence in superspace. From a superspace point of view, \( \theta \) divergences and \( x \) divergences are equally fundamental.
The supersymmetry algebra for the conserved charge $\tilde{Q}$ is then derived by using the BJL method. We rewrite Ward identities which are derived from path integrals and which contain the covariant $T^*$ time ordering, in terms of Ward identities at the operator level which contain the $T$ time ordering symbol. We obtain

\[ i\{\tilde{Q}^\alpha, \tilde{Q}^\beta\} = -2(\gamma^\mu)^\alpha\beta \tilde{P}_\mu - 2\tilde{Z}(\gamma_5)^\alpha\beta \] (1.9)

where

\[ \tilde{P}_\mu = \int dx \tilde{T}_{0\mu}(x), \quad \tilde{H} = \tilde{P}_0, \]
\[ \tilde{Z} = \int dx \tilde{\zeta}_0(x) = -\int dx \tilde{\zeta}^0(x). \] (1.10)

The BJL method, unlike the semi-classical Dirac bracket, incorporates all the quantum effects, in particular superconformal anomalies. The operators $\tilde{T}_{\mu}^\mu$ and $\tilde{\zeta}^\mu$ are conserved quantities

\[ \partial_\mu \tilde{T}_\nu^\mu = 0, \quad \partial_\mu \tilde{\zeta}^\mu = 0 \] (1.11)

but contain superconformal anomalies

\[ \tilde{T}_{\mu}^\mu(x) = F(x)U(x) - g\varphi\bar{\psi}(x) + \frac{hg}{2\pi}F(x) \]
\[ = T_{\mu}^\mu(x) + \frac{hg}{2\pi}F(x), \]
\[ \gamma_\mu\tilde{\zeta}^\mu(x)\gamma_5 = \partial_\mu\varphi(x)U\gamma^\mu + \frac{hg}{2\pi}\partial_\mu\varphi(x)\gamma^\mu \]
\[ = (\gamma_\mu\tilde{\zeta}^\mu(x))\gamma_5 + \frac{hg}{2\pi}\partial_\mu\varphi(x)\gamma^\mu. \] (1.12)

We derive these equations from the path integral formulation. In these equations, $T_{\mu}^\mu(x)$ and $(\gamma_\mu\tilde{\zeta}^\mu)$ contain only the terms which explicitly break superconformal symmetry, as we shall show. These arise from the superpotential, are “soft” (they have lower dimension because they are proportional to the dimensionful $g$), and there are no anomalous contributions to these quantities.

The relations in (1.5) and (1.6) can be combined to give a similar result as in (1.12)

\[ \gamma_\mu\tilde{J}_\mu(x) = \gamma_\mu j_\mu(x) - \frac{hg}{\pi}\bar{\psi}(x). \] (1.13)

The other conserved operators themselves can also be decomposed in a similar way as in (1.6)

\[ \tilde{T}_{\mu\nu}(x) = T_{\mu\nu}(x) + \eta_{\mu\nu}\frac{hg}{4\pi}F(x), \]
\[ \tilde{\zeta}_\mu(x) = \zeta_\mu(x) + \frac{hg}{2\pi}\epsilon_{\mu\nu}\partial_\nu\varphi(x). \] (1.14)

\[ \text{The symbols } (\gamma^0)^\alpha\beta \text{ and } (\gamma_5)^\alpha\beta \text{ denote the matrices } (\gamma^0)^\alpha\gamma(C^{-1})\gamma^\beta \text{ and } (\gamma_5)^\alpha\gamma(C^{-1})\gamma^\beta \text{ and are in our conventions equal to } -i \text{ and } i\tau_3, \text{ respectively, see Section 2. The indices } \alpha \text{ and } \beta \text{ are equal to } + \text{ or } -, \text{ and for } \alpha = \beta = + \text{ one finds that the quantum anticommutator has the same form as the classical relation (1.1).} \]
The operators $T_{\mu\nu}(x)$ and $\zeta_{\mu}(x)$ are specified by

$$T_{\mu\nu}(x)_{\text{anomaly}} = 0, \quad \zeta_{\mu}(x)_{\text{anomaly}} = 0 \quad (1.15)$$

corresponding to the absence of an anomaly in $\gamma_{\mu}\beta$, see (1.5), but although both $\tilde{\zeta}_{\mu}$ and $\zeta_{\mu}$ are conserved, $T_{\mu\nu}$ is not conserved

$$\partial_{\nu}T_{\mu\nu}(x) \neq 0, \quad (1.16)$$
similar to the non-conservation of $j_{\mu}$.

In terms of $T_{\mu\nu}$ and $\zeta_{\mu}$, the supersymmetry algebra reads

$$i\{\tilde{Q}^\alpha, \tilde{Q}^\beta\} = -2(\gamma^\mu)^{\alpha\beta}P_{\mu} + \int dx \frac{h}{2\pi} F(x)(\gamma^0)^{\alpha\beta}$$
$$-2Z(\gamma_5)^{\alpha\beta} + \int dx \frac{h}{\pi} \partial_1 \varphi(x)(\gamma_5)^{\alpha\beta} \quad (1.17)$$

where we used $\epsilon^{01} = 1$ and

$$P_{\mu} = \int dx T_{0\mu}(x), \quad H = P_0,$$
$$Z = \int dx \zeta_0(x) = -\int dx \zeta^0(x). \quad (1.18)$$

We see that the supersymmetry algebra in terms of time-independent charges has the same form at the quantum level as at the classical level, see (1.10). This agrees with [8], whose analysis is based on this observation. In (1.16) we have used charges $P$ and $Z$; $Z$ is free from anomalies and the anomaly of $\tilde{Z}$ explicitly appears on the right-hand side, but $P$ still contains a superconformal anomaly though $T_{\mu\nu}$ is free from the trace anomaly. (In other words, $T_{00}$ and $T_{11}$ have equal anomalies, and the anomaly in $T_{00}$ doubles the contribution in (1.14) proportional to $F$, see Section 6. Using (1.6) to write $\tilde{Q}$ in terms of $Q$, all the anomaly terms in (1.17) cancel separately if one splits off the anomaly from $P_{\mu}$. In this way, also in terms of $Q$, $P_{\mu}$ and $Z$ the quantum anticommutator has the same form as classically. However, $Q$ and $P$ are time-dependent, so they are physically less relevant.) These two alternative ways of writing the algebra give rise to the same physical conclusion, namely, uniform shifts in energy and central charge in the vacuum of the time independent kink solution. Both maintain the BPS bound, since they describe the same algebra on a different basis.

2 The model and the superspace regulator

We briefly summarize some of the features of the model which describes the supersymmetric model; the $N = (1, 1)$ supersymmetric Wess-Zumino model in $d = 2$ Minkowski space. The model is defined in terms of the superfield

$$\phi(x, \theta) = \varphi(x) + \theta \psi(x) + \frac{1}{2} \theta \theta F(x) \quad (2.1)$$
where $\theta^\alpha$ is a Grassmann number, and $\theta^\alpha$ and $\psi^\alpha(x)$ are two-component Majorana spinors; $\varphi(x)$ is a real scalar field, and $F(x)$ is a real auxiliary field. We define $\bar{\theta} = \theta^T C$ with $C$ the charge conjugation matrix, and the inner product for spinors is defined by

$$\bar{\theta} \theta \equiv \bar{\theta}^\alpha C_{\alpha\beta} \theta^\beta \equiv \bar{\theta} \theta$$

(2.2)

with the Dirac matrix convention

$$\gamma^0 = -\gamma_0 = -i\tau^2, \quad \gamma^1 = \gamma_1 = \tau^3, \quad C = \tau^2, \quad \gamma_5 = \gamma^0 \gamma^1$$

(2.3)

The choice $\gamma^1 = \tau^3$ has certain advantages for the evaluation of the spectrum of the fermions; we shall not evaluate this spectrum, but still use $\gamma^1 = \tau^3$ in order to agree with the literature. We use the metric $\eta_{\mu\nu} = (-1, 1)$ for $\mu = (0, 1)$, hence $(\gamma^0)^2 = -1$ but $\gamma_5^2 = 1$. Useful identities are

$$\epsilon^{\nu\mu} \gamma^\mu \gamma^5 = -\gamma^\nu$$

and

$$\gamma^\mu \gamma^5 = -\epsilon^{\mu\nu} \gamma^\nu$$

with $\epsilon^{01} = 1$. Since in this representation the charge conjugation matrix equals $\tau^2$, the Majorana condition $\psi^\dagger \tau^2 = \psi^T C$ reduces to the statement that all Majorana spinors are real. We frequently use the relations $\bar{\epsilon} \psi = \bar{\psi} \epsilon$, $\bar{\epsilon} \gamma^\mu \psi = -\bar{\psi} \gamma^\mu \epsilon$ and $\bar{\epsilon} \gamma_5 \psi = -\bar{\psi} \gamma_5 \epsilon$ (the sign in the last relation is opposite to the 4 dimensional case).

The supersymmetry transformation is induced by its action on the coordinates in superspace, $\phi'(x', \theta') = \phi(x, \theta)$. One has

$$\begin{align*}
\theta' &= \theta - \epsilon, \\
x'^\mu &= x^\mu - \bar{\theta} \gamma^\mu \epsilon
\end{align*}$$

(2.4)

leading in first order of $\epsilon$ to

$$\begin{align*}
\phi'(x, \theta) &= \phi(x^\mu + \bar{\theta} \gamma^\mu \epsilon, \theta + \epsilon) = \\
&= \phi(x, \theta) + \bar{\theta} \gamma^\mu \epsilon \partial_\mu \phi(x) + \bar{\epsilon} \psi(x) + \bar{\theta} \epsilon F(x) + \frac{1}{2} (\bar{\theta} \theta) \epsilon \gamma^\mu \partial_\mu \psi
\end{align*}$$

(2.5)

where we used a Fierz rearrangement

$$((\bar{\theta} \partial_\mu \psi)(\bar{\theta} \gamma^\mu \epsilon) = \frac{1}{2} (\bar{\theta} \theta) \bar{\epsilon} \gamma^\mu \partial_\mu \psi.$$  

(2.6)

In terms of components one obtains from $\delta \phi = \phi'(x, \theta) - \phi(x, \theta)$

$$\begin{align*}
\delta \varphi &= \bar{\epsilon} \psi(x), \\
\delta \psi &= \partial_\mu \varphi(x) \gamma^\mu \epsilon + F(x) \epsilon = \bar{\theta} \varphi(x) \epsilon + F(x) \epsilon, \\
\delta F &= \bar{\epsilon} \gamma^\mu \partial_\mu \psi = \bar{\epsilon} \bar{\theta} \psi(x).
\end{align*}$$

(2.7)

The supercharge which generates (2.4)

$$\bar{\epsilon} Q \equiv \epsilon^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha} - \bar{\epsilon} \gamma^\mu \theta \partial_\mu$$

(2.8)

and the covariant derivative

$$\bar{\eta} D \equiv \bar{\eta}^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha} + \bar{\eta} \gamma^\mu \theta \partial_\mu$$
anti-commute with each other. We have

\[ D^\alpha \phi(x, \theta) = \psi^\alpha + \theta^\alpha F + (\gamma^\mu \theta)^\alpha \partial_\mu \varphi + (\gamma^\mu \theta)^\alpha \bar{\vartheta} \partial_\mu \psi \] (2.9)

and

\[ D\phi(x, \theta) D\phi(x, \theta) = \bar{\psi} \psi + 2\bar{\psi} \theta F + 2(\bar{\psi} \gamma^\mu \theta) \partial_\mu \varphi + \bar{\theta} \theta [FF - \partial^\mu \varphi \partial_\mu \varphi - \bar{\psi} \gamma^\mu \partial_\mu \psi] \] (2.10)

where we used \( \bar{\vartheta} \gamma^\mu \theta = 0 \).

We next note that

\[ \phi^3(x, \theta) = \varphi^3 + 3(\bar{\theta} \psi) \varphi^2 + \frac{1}{2}(\bar{\theta} \theta)[3F \varphi^2 - 3(\bar{\psi} \psi) \varphi] \] (2.11)

by using \( (\bar{\theta} \psi)(\bar{\theta} \psi) = -\frac{1}{2}(\bar{\theta} \theta)(\bar{\psi} \psi) \). We thus choose the action

\[ \int dx d\theta^2 \mathcal{L}(x, \theta) = \int dx d\theta^2 \left[ \frac{1}{4} \bar{D} \phi(x, \theta) D\phi(x, \theta) + \frac{1}{3} g \phi^3(x, \theta) - g v_0^2 \phi(x, \theta) \right] \]

\[ = \int dx \left\{ \frac{1}{2} [FF - \partial^\mu \varphi \partial_\mu \varphi - \bar{\psi} \gamma^\mu \partial_\mu \psi] + g F \varphi^2 - g (\bar{\psi} \psi) \varphi - g v_0^2 F \right\} \] (2.12)

where we used the convention

\[ \int d^2 \theta \frac{1}{2}(\bar{\theta} \theta) = 1 \] (2.13)

The delta function is defined by \( \int d^2 \theta_1 \delta(\theta_1 - \theta_2) = 1 \) and given by

\[ \delta(\theta_1 - \theta_2) = \frac{1}{2}(\bar{\theta}_1 - \theta_2)(\bar{\theta}_1 - \theta_2). \] (2.14)

The potential \( V \) in \( \mathcal{L} = T - V \) is given by \( V = -FU + g \bar{\psi} \psi \varphi \) where

\[ U(\varphi) \equiv g(\varphi^2 - v_0^2). \] (2.15)

We use a coupling constant \( g \) which is related to the coupling constant \( \lambda \) used in other articles by

\[ g = \sqrt{\frac{\lambda}{2}}, \quad v_0 = \frac{\mu_0}{\sqrt{\lambda}} = \frac{m_0}{2g}. \] (2.16)

This coupling constant \( g \) has the dimension of a mass, and \( \mu_0^2 = \mu^2 + \Delta \mu^2 \) where \( m = \sqrt{2} \mu \) is the renormalized meson mass.

To apply the background field method, we decompose the field variable as follows

\[ \phi(x, \theta) = \Phi(x, \theta) + \eta(x, \theta) \] (2.17)

where \( \Phi(x, \theta) \) is the background field and \( \eta(x, \theta) \) is the quantum fluctuation. We then consider the parts of the (superfield) Lagrangian which are quadratic in \( \eta \)

\[ \mathcal{L}_2(x, \theta) = \frac{1}{4} \bar{D} \eta(x, \theta) D\eta(x, \theta) + g \Phi(x, \theta) \eta^2(x, \theta) \] (2.18)
or equivalently

\[ \mathcal{L}_2(x, \theta) = \eta(x, \theta) \Gamma(x, \theta) \eta(x, \theta), \]

\[ \Gamma(x, \theta) = -\frac{1}{4} \bar{D}D + g\Phi(x, \theta). \]  

(2.19)

The regulator will be quadratic in \( \Gamma \). To construct it, we first derive some basic properties of the operator \( \bar{D}D \). From the definition \(^3\)

\[ \langle 0 | T^* \eta(x^\mu, \theta_1) \eta(y^\mu, \theta_2) | 0 \rangle = -iG(x - y; \theta_1, \theta_2) \]  

(2.20)

and for a supersymmetric vacuum, we have

\[ \langle 0 | T^* \eta(x^\mu - \bar{\theta}_1 \gamma^\mu \epsilon, \theta_1 - \epsilon) \eta(y^\mu - \bar{\theta}_2 \gamma^\mu \epsilon, \theta_2 - \epsilon) | 0 \rangle = -iG(x - y; \theta_1, \theta_2). \]  

(2.21)

In particular for \( \epsilon = \theta_2 \), by using \( \bar{\theta}_2 \gamma^\mu \theta_2 = 0 \) and \( \bar{\epsilon} \gamma^\mu \epsilon = 0 \), we obtain

\[ \langle 0 | T^* \eta(x^\mu - \bar{\theta}_1 \gamma^\mu \theta_2, \theta_1 - \theta_2) \eta(y^\mu, 0) | 0 \rangle = -iG(x - y; \theta_1, \theta_2). \]  

(2.22)

For the massless theory there are no off-diagonal terms in the propagator, hence the free field propagator in superspace can be written in terms of the propagator of the scalar field \( \varphi \)

\[ G(x - y; \theta_1, \theta_2) = \int \frac{d^2k}{(2\pi)^2} e^{ik(x-y)} \frac{\exp[-i\bar{\theta}_1 \gamma^\mu \theta_2]}{k^2 - i\epsilon} \]  

(2.23)

with \( k^2 = k_1^2 - k_0^2 \). This shows that \(^4\)

\[ G(x - y; \theta_1, \theta_2) = \frac{\exp[-i\bar{\theta}_1 \gamma^\mu \theta_2]}{p^2 - i\epsilon} \delta(x - y), \]

\[ \frac{1}{2} \bar{D}DG(x - y; \theta_1, \theta_2) = \delta(x - y) \delta(\theta_1 - \theta_2) \]  

(2.24)

where

\[ p_\mu = \frac{1}{i} \partial_\mu. \]  

(2.25)

This equation yields the following important relations

\[ \frac{1}{2} \bar{D}D \frac{\exp[-i\bar{\theta}_1 \gamma^\mu \theta_2]}{p^2 - i\epsilon} = \delta(\theta_1 - \theta_2), \]

\[ -\frac{1}{2} \bar{D}D \delta(\theta_1 - \theta_2) = \exp[-i\bar{\theta}_1 \gamma^\mu \theta_2], \]

\[ \exp[-i\bar{\theta}_1 \gamma^\mu \theta_2] = 1 - i\bar{\theta}_1 \gamma^\mu \theta_2 - p^2 \delta(\theta_1) \delta(\theta_2) \]  

(2.26)

\(^3\)The symbol \( T^* \) denotes (covariant) time ordering in the path integral approach. It has the property that it commutes with ordinary derivatives \( \partial_\mu \).

\(^4\)Completing squares in \( Z = \int \pi \det \left( \frac{1}{4} \bar{D}D \eta + J \eta \right) \) by setting \( \eta = (\frac{1}{4} \bar{D}D)^{-1} J \) yields \( Z = \int \pi \det \left( \frac{1}{4} J (\bar{D}D)^{-1} J \right) \). Differentiation with respect to \( J \) yields \( \langle T^* \eta(x, \theta_1) \eta(y, \theta_2) \rangle = -i\hbar (\frac{1}{4} \bar{D}D)^{-1} \delta(x - y) \delta(\theta_1 - \theta_2) \). We set \( \hbar = 1 \) in most places.
where the second relation is derived by performing the integral \( \int d^2\theta_2 \) in
\[
\frac{1}{2} DD \exp\left[ \frac{-i(\theta_1 - \theta_3) \cdot p\bar{\theta}_2}{p^2 - i\epsilon} \right] = \delta(\theta_1 - \theta_2) \exp[i \bar{\theta}_3 \cdot p\theta_2]. \tag{2.27}
\]

We thus have
\[
D \equiv \frac{1}{2} \bar{D} D, \\
D(\theta_1)\delta(\theta_1 - \theta_2) = - \exp[-i \bar{\theta}_1 \cdot p\theta_2], \\
D^2(\theta_1)\delta(\theta_1 - \theta_2) = -p^2\delta(\theta_1 - \theta_2). \tag{2.28}
\]

Using these results, we obtain the following equations
\[
\Gamma(x, \theta) = -\frac{1}{2} D + g\Phi(x, \theta), \\
\Gamma^2(x, \theta) = \frac{1}{4} D^2 - \frac{1}{2} gD\Phi - \frac{1}{2} g\Phi D + (g\Phi)^2, \\
\Gamma^2(x, \theta)\delta(\theta - \theta_1) = [-\frac{1}{4} p^2 - \frac{1}{2} gD\Phi - \frac{1}{2} g\Phi D + (g\Phi)^2]\delta(\theta - \theta_1), \\
\Gamma^2(x, \theta)^2\delta(\theta - \theta_1) = \int d^2\theta_2 \Gamma^2(x, \theta)\delta(\theta - \theta_2)\Gamma^2(x, \theta_2)\delta(\theta_2 - \theta_1) \\
= [-\frac{1}{4} p^2 - \frac{1}{2} gD\Phi - \frac{1}{2} g\Phi D + (g\Phi)^2]^2\delta(\theta - \theta_1). \tag{2.29}
\]

We use the heat kernel \( \exp[(\Gamma/M)^2] \) as regulator in superspace. Using (2.29) we find
\[
\exp\left[ \frac{1}{M^2} \Gamma(x, \theta)^2 \right] \delta(\theta - \theta_1) \\
= \exp\left\{ \frac{1}{M^2} \left[ -\frac{1}{4} p^2 - \frac{1}{2} gD\Phi - \frac{1}{2} g\Phi D + (g\Phi)^2 \right] \right\} \delta(\theta - \theta_1). \tag{2.30}
\]

This last equation will be used to evaluate the Jacobian.

## 3 Evaluation of the Jacobian

The regularized Jacobian \( J \) in the path integral
\[
\int D\phi \exp[i \int d^2xd^2\theta L(x, \theta)] \tag{3.1}
\]
for the infinitesimal transformation
\[
\phi'(x, \theta) = \phi(x, \theta) + \omega(x, \theta)\phi(x, \theta) \tag{3.2}
\]
is given at the one-loop level by
\[
\ln J = \int d^2xd^2\theta \langle x, \theta | \omega(x, \theta) \exp\left\{ \frac{1}{M^2} \Gamma(x, \theta)^2 \right\} | \theta, x \rangle
\]
\[
\int d^2x d^2\theta \omega'(x,\theta)\langle x,\theta | \exp\left\{ \frac{1}{M^2} \left[ -\frac{1}{2}D + g\Phi(x,\theta) \right]^2 \right\} |\theta, x \rangle
\]

\[
= \int d^2x d^2\theta \omega'(x,\theta) \times \lim_{y \to x,\theta_1 \to \theta} \exp\left\{ \frac{1}{M^2} \left[ -\frac{1}{2}D + g\Phi(x,\theta) \right]^2 \right\} \delta(\theta - \theta_1) \delta(x - y) \tag{3.3}
\]

\[
= \int d^2x d^2\theta \frac{d^2k}{(2\pi)^2} \omega'(x,\theta) e^{-ikx} \times \lim_{\theta_1 \to \theta} \exp\left\{ \frac{1}{M^2} \left[ -\frac{1}{2}D + g\Phi(x,\theta) \right]^2 \right\} e^{ikx} \delta(\theta - \theta_1)
\]

We recall

\[
\mathcal{D} = \frac{1}{2} \bar{D}D,
\]

\[
p_\mu = (1/i) \partial_\mu \tag{3.4}
\]

and the parameter \( M \) is sent to \( \infty \) later.

When one considers a transformation of the form

\[
\phi'(x,\theta) = \phi(x,\omega) + [\bar{\Omega}(x,\theta)D\phi(x,\theta) + \bar{\epsilon}(x,\theta)Q\phi(x,\theta) + O(x,\theta)]\phi(x,\theta)
\]

\[
\equiv \phi(x,\theta) + \omega(x,\theta)\phi \tag{3.5}
\]

the parameter \( \omega(x,\theta) \) in the first line in (3.3) must be replaced by \( \omega'(x,\theta) \) in the second line where

\[
\omega'(x,\theta) = \frac{1}{2} \bar{\Omega}(x,\theta)D + \frac{1}{2} \bar{\epsilon}(x,\theta)Q + O(x,\theta). \tag{3.6}
\]

The reason for the factor \( 1/2 \) is that when one extracts the factor \( \omega(x,\theta) \) outside the bracket symbol, the derivative operators \( D^\alpha \) and \( Q^\alpha \) (and also the spatial derivative \( \partial_\mu \)) act on both the bra- and ket-vectors.

We start with the evaluation of

\[
\int \frac{d^2k}{(2\pi)^2} e^{-ikx} \exp\left\{ \frac{1}{M^2} \left[ -\frac{1}{4}p^2 - \frac{1}{2}Dg\Phi(x,\theta) - \frac{1}{2}g\Phi(x,\theta)D + g^2\Phi^2 \right] \right\} e^{ikx} \delta(\theta - \theta_1). \tag{3.7}
\]

Passing the factor \( e^{ikx} \) through the integrand replaces \( p \to p + k \), and the operator \( \mathcal{D} \) is modified as follows

\[
e^{-ikx}\mathcal{D}e^{ikx} = \frac{1}{2} [\bar{D} + i(\gamma^\mu \theta)k_\mu][D + i(\gamma^\mu \theta)k_\mu]
\]

\[
= \frac{1}{2} \bar{D}D + i(k\theta)D - \frac{1}{2}(k\theta)(k\theta)
\]

\[
= \mathcal{D} - i\theta kD + \frac{1}{2}k^2(\theta\theta). \tag{3.8}
\]
Replacing \( k_\mu \rightarrow M k_\mu \) (3.9)

we obtain the integral

\[
M^2 \int \frac{d^2 k}{(2\pi)^2} \left\{ \exp\left\{ -\frac{1}{4} (k^2 + 2 \frac{k p}{M} + \frac{p^2}{M^2}) - \frac{1}{2} \left[ \frac{D}{M^2} - i \frac{\bar{\theta} k D}{M} + \frac{1}{2} k^2 (\bar{\theta} \theta) \right] g \Phi(x, \theta) \right\} \right. \\
- \frac{1}{2} g \Phi(x, \theta) \left[ \frac{D}{M^2} - i \frac{\bar{\theta} k D}{M} + \frac{1}{2} k^2 (\bar{\theta} \theta) \right] + \frac{g^2}{M^2} \Phi^2 \right\} \delta(\theta - \theta_1). \tag{3.10}
\]

We now expand the exponent into a power series except for the factor \( \exp[-\frac{1}{4} k^2] \). By using that according to (2.28)

\[
\lim_{\theta_1 \to \theta} \frac{D(\theta)}{\delta(\theta - \theta_1)} = -1 \tag{3.11}
\]

while terms without \( D \) acting on \( \delta(\theta - \theta_1) \) vanish for \( \theta_1 \to \theta \), and noting that only the terms in the integral of order \( 1/M^2 \) or larger survive when \( M \) tends to infinity; one can confirm that only the terms to second order in the expansion survive. In fact, the second order terms completely cancel because the term

\[
k^2 (\bar{\theta} \theta) D/M^2 \tag{3.12}
\]

from the cross terms of \( D/M^2 \) and \( \frac{1}{2} k^2 (\bar{\theta} \theta) \) cancels the square

\[
[-i\theta k D]^2 / M^2 = -\frac{1}{2} (\bar{\theta} \theta) k^2 D D / M^2 = -k^2 (\bar{\theta} \theta) D/M^2. \tag{3.13}
\]

We thus need to evaluate only the first order terms

\[
\int \frac{d^2 k}{(2\pi)^2} \exp\left\{ -\frac{1}{4} k^2 \right\} \left\{ -g \Phi(x, \theta) D \right\} \delta(\theta - \theta_1) = i g \Phi(x, \theta). \tag{3.14}
\]

We used the formula

\[
\int \frac{d^2 k}{(2\pi)^2} \exp[-k^2/4] = \frac{i}{\pi} \tag{3.15}
\]

where in the last integral we Wick-rotated to Euclidean space by \( d^2 k \rightarrow id^2 k \). We thus obtain the following result for the Jacobian in Minkowski space

\[
\ln J = i \int d^2 x d^2 \theta \omega'(x, \theta) \left[ g \Phi(x, \theta) \right]. \tag{3.16}
\]

Note that this calculation remains valid for general (non-derivative) interactions depending on \( \Phi \); if one has a potential \( V(\Phi) \) instead of \( g \Phi \) in (2.19), one makes the same replacement in (3.16).

In the spirit of the background field method, one may replace the variable \( \Phi(x, \theta) \) by the full variable \( \phi(x, \theta) \) to the accuracy of the one-loop approximation. The final result for the result of the Jacobian of the path integral in Minkowski space is thus given by

\[
\ln J = i \int d^2 x d^2 \theta \omega'(x, \theta) \left[ g \phi(x, \theta) \right]. \tag{3.17}
\]
For example, for the class of transformations
\[\delta \phi(x, \theta) = \bar{\Omega} D \phi + c(D \Omega) \phi = \bar{\Omega} D \phi + \frac{1}{2} (D \Omega) \phi + (c - \frac{1}{2})(D \Omega) \phi \quad (3.18)\]
with a constant \(c\), one obtains for the Jacobian
\[i \frac{g}{\pi} \int d^2 x d^2 \theta \left[ \frac{1}{2} \bar{D} (\Omega \Phi(x, \theta)) + (c - \frac{1}{2}) \left( D \Omega(x, \theta) \right) \right] \phi(x, \theta)
= -i(c - \frac{1}{2}) \frac{g}{\pi} \int d^2 x d^2 \theta \Omega(x, \theta) D \phi(x, \theta). \quad (3.19)\]
We emphasize that this is a one-loop result.

4 Superconformal anomalies from the measure

Our analysis of superconformal anomalies starts with the generalized supersymmetry transformation
\[\delta \phi(x, \theta) = \bar{\Omega}(x, \theta) Q \phi(x, \theta) \quad (4.1)\]
which gives rise to a change of the action
\[\delta S = \int d^2 x d^2 \theta \left[ \frac{1}{2} (D^\alpha \bar{\Omega}_\beta)(D_\alpha \phi) Q^\beta \phi + \bar{\Omega}_\alpha Q^\alpha \mathcal{L} \right] \quad (4.2)\]
where
\[\mathcal{L} = \frac{1}{4} \bar{D}_\alpha \phi D^\alpha \phi + \frac{1}{3} g \phi^3 - g v^2 \phi. \quad (4.3)\]
Any transformation of \(\phi\), whether it is a symmetry of the action or not, leads to a corresponding Ward identity, but using a local supersymmetry transformation has the advantage that one obtains a hierarchy of Ward identities in \(x\)-space which contain the Ward identities for ordinary and conformal supersymmetry. These are, of course, the Ward identities we are interested in, and we expect in this multiplet of Ward identities also to find a Ward identity for the central charge current.

For constant superfields \(\Omega^\alpha\), the action is invariant, but for local \(\Omega^\alpha\), the variation of \(S\) is proportional to the Noether current. One thus obtains the following Ward identity for correlation functions
\[\langle \frac{i}{\hbar} \int d^2 x d^2 \theta \left[ \frac{1}{2} (D^\alpha \bar{\Omega}_\beta)(D_\alpha \phi) Q^\beta \phi + \bar{\Omega}_\alpha Q^\alpha \mathcal{L} \right] \phi(x_1, \theta_1) \ldots \phi(x_n, \theta_n) \rangle
= -\langle i \frac{g}{2\pi} \int d^2 x d^2 \theta (\bar{\Omega}_\beta Q^\beta \phi(x, \theta)) \phi(x_1, \theta_1) \ldots \phi(x_n, \theta_n) \rangle
- \langle \delta \phi(x_1, \theta_1) \ldots \phi(x_n, \theta_n) \rangle \ldots - \langle \phi(x_1, \theta_1) \ldots \delta \phi(x_n, \theta_n) \rangle. \quad (4.4)\]
The first term on the right-hand side is the Jacobian; it contains all anomalies, and the extra factor of \(1/2\) arises from the rule (3.6). We shall from now on replace Ward identities such as (4.4) by the simplified operator expression
\[\frac{1}{2} D^\alpha [(\bar{D}_\alpha \phi) Q^\beta \phi] + Q^\beta \mathcal{L} = -\frac{\hbar g}{2\pi} Q^\beta \phi(x, \theta). \quad (4.5)\]
Here we write $h$ explicitly to indicate that we are working at the one-loop level.

Using $\frac{1}{2} D^\alpha ([\bar{D}_\alpha \phi] Q^\beta \phi) + Q^\beta \mathcal{L} = \frac{1}{2} (D^\alpha \bar{D}_\alpha \phi \phi - Q^\beta \phi V$, and $\frac{1}{2} D^\alpha \bar{D}_\alpha \phi = F + \theta \partial \psi + \frac{1}{2} \bar{\theta} \partial \bar{\psi}, \partial^\mu \phi$ and $Q^\beta \phi = \psi \beta + \bar{\phi} \theta \beta + \frac{1}{2} \theta \partial \psi \beta$, the Ward identity in superspace can be expanded as follows

$$
\frac{1}{2} (\gamma_{\mu} \bar{J}^\mu)(x) - \bar{T}^\mu_{\mu}(x) \theta + \frac{1}{2} (\bar{\psi} \gamma_5 \psi \gamma \gamma_\theta + \gamma_{\mu} \bar{\zeta}^\mu(x) \gamma_\theta + \frac{1}{2} (\bar{\psi} \gamma_\nu \psi \gamma_\nu \theta - \delta(\theta) \partial \mu \bar{\psi}(x) - \frac{h g}{2\pi} [\bar{\psi}(x) + F(x) \theta - (\gamma_{\mu} \theta) \partial \mu \bar{\psi}(x) + \delta(\theta) \partial \psi(x)] (4.6)
$$

or in component notation

$$
(\gamma_{\mu} \bar{J}^\mu)(x) = (\gamma_{\mu} \bar{J}^\mu)(x) - \frac{h g}{\pi} \psi(x),
\bar{T}^\mu_{\mu}(x) = (T^\mu_{\mu}(x) + \frac{h g}{2\pi} F(x),
\bar{\zeta}^\mu(x) - \frac{1}{2} (\bar{\psi} \gamma_\mu \gamma_\psi)(x) = (\zeta^\mu(x) + \frac{h g}{2\pi} \epsilon_\mu \sigma \partial_\sigma \psi(x),
(\bar{\psi} \gamma_5 \psi)(x) = 0,
\partial_\mu \bar{J}^\mu(x) = \frac{h g}{2\pi} \partial \psi(x). (4.7)
$$

All superconformal anomalies are included. The operators $\gamma_{\mu} \bar{J}^\mu$, $\bar{T}^\mu_{\mu}$ and $\zeta^\mu$ are constructed only from the part $\bar{D}_\phi \phi$ of the Lagrangian. This is the massless part which is superconformally invariant, and these terms appear on the left-hand side of (4.6) and (4.7). The terms coming from the superpotential depend on the dimensionful coupling constant $g$ and thus break superconformal invariance. These terms appear on the right-hand side of (4.6) and (4.7), and they are given by

$$
\frac{1}{2} (\gamma_{\mu} \bar{J}^\mu)(x) - (T^\mu_{\mu}(x) \theta + (\gamma_{\mu} \zeta^\mu(x) \gamma_\theta - \delta(\theta) \partial_\mu (U \gamma^\mu \psi)
\equiv QV(\bar{\psi}(x, \theta))
\equiv -U(\psi) \psi - [FU(\psi) - g \psi \psi] \theta + \partial_\mu \psi U(\psi) \gamma^\mu \theta - \delta(\theta) \partial_\mu (U \gamma^\mu \psi) (4.8)
$$

where $V = -[\frac{1}{2} g \phi^3(x, \theta) - g v^2 \psi(x, \theta)]$ is the superpotential. The term proportional to $\delta(\theta)$ in (4.8) has been moved to the left-hand side of (4.6) where it forms part of $-\delta(\theta) \partial_\mu j^\mu(x)$.

The Ward identity contains contractions of currents and does not specify the full expressions of the various operators involved. The full expressions of the supercurrent $j^\mu$ (and $\bar{J}^\mu$), energy-momentum tensor $T^\mu_{\nu}$ (and $\bar{T}^{\mu}_{\nu}$) and central charge current $\zeta_\mu$ (and $\bar{\zeta}_\mu$) are, respectively, given by

$$
\bar{J}^\mu(x) = -[\bar{\partial} \phi(x) + U(\phi(x))] \gamma^\mu \psi(x),
$$
\[\tilde{J}^\mu(x) = -[\partial_\varphi(x) - F(x)]\gamma^\mu\psi(x)\]
\[= j^\mu - \frac{\hbar g}{2\pi}\gamma^\mu\psi,\]
\[T_{\mu\nu}(x) = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}[(\partial^\rho\varphi)(\partial_\rho\varphi) - FU] + \frac{1}{4}\bar{\psi}\gamma_5\partial_\mu\gamma^\rho\gamma_5\partial_\rho\psi + 2g\bar{\psi}\psi,\]
\[\tilde{T}_{\mu\nu}(x) = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}[(\partial^\rho\varphi)(\partial_\rho\varphi) + F^2] + \frac{1}{4}\bar{\psi}[\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu]\psi\]
\[= T_{\mu\nu}(x) + \eta_{\mu\nu}\frac{\hbar g}{4\pi}F(x),\]
\[\zeta_\mu(x) = \epsilon_{\mu\nu}\partial_\nu\varphi(x)U(\varphi),\]
\[\bar{\zeta}_\mu(x) = -\epsilon_{\mu\nu}\partial_\nu\varphi(x)F(x)\]
\[= \zeta_\mu(x) + \frac{1}{2}\bar{\psi}\gamma_\mu\gamma_5\bar{\psi}\psi(x) + \frac{\hbar g}{2\pi}\epsilon_{\mu\nu}\partial_\nu\varphi(x)\]
\[(4.9)\]

with \(U(\varphi) = g(\varphi^2(x) - v_0^2)\) and \(\epsilon_{01} = -1\). These expressions for the various operators are derived later. At this point we only keep that they satisfy (4.7). One may already note that the currents \(j^\mu\) and \(\tilde{J}^\mu\), \(T_{\mu\nu}\) and \(\tilde{T}_{\mu\nu}\), and \(\zeta_\mu\) and \(\bar{\zeta}_\mu\) only differ by the \(F\) and \(\psi\) field equations. We repeat that these relations between \(\tilde{J}_\mu\) and \(j^\mu\), \(\tilde{T}_{\mu\nu}\) and \(T_{\mu\nu}\), and \(\zeta_\mu\) and \(\bar{\zeta}_\mu\), are only identities when used in the Ward identity (4.4).

We now give some discussion of (4.7) and (4.9). One may expand the spinor parameter \(\Omega^a\) in (4.1) as follows

\[\Omega^a(x, \theta) = s^a(x) + w(x)\theta^a + l(x)(\gamma_5\theta)^a + c_\mu(x)(\gamma^\mu\gamma_5\theta)^a + t^a(x)\delta(\theta).\]
\[(4.10)\]

The parameter \(s^a(x)\) generates local ordinary supersymmetry and thus yields a Ward identity for the divergence of the supercurrent, see the terms with \(\delta(\theta)\) in (4.6). The parameter \(w(x)\) generates Weyl transformations in (4.1), namely \(\delta\phi = w(x)\theta^a\frac{\partial}{\partial\theta^a}\phi\), and thus yields the trace anomaly. The parameter \(l(x)\) generates local Lorentz transformations (or chiral transformations since \(\gamma_5\) is the Lorentz generator), but the Lorentz transformation is, of course, anomaly-free for our vector-like model ⁵ and its generator vanishes identically

\[\bar{\psi}\gamma^\mu\gamma_5\psi \equiv 0.\]
\[(4.11)\]

The parameter \(t^a(x)\) generates the transformation \(\delta F = \bar{\psi},\psi\), and it leads to the Ward identity containing the gamma-trace of the supercurrent.

The most interesting case are the transformations with \(c_\mu(x)\). The parameter \(c_\mu(x)\) generates the transformations \(\delta F = -\epsilon^{\mu\nu}c_\mu\partial_\nu\varphi\) and \(\delta\psi = c_\mu\gamma^\mu\gamma_5\psi\), and these transformations yield the Ward identity

\[c_\mu\tilde{\zeta}_\mu(x) - \frac{1}{2}c_\mu\epsilon^{\mu\nu}(\bar{\psi}\gamma_\nu\partial_\mu\varphi) = c_\mu\zeta_\mu(x) + c_\mu\frac{\hbar g}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi(x).\]
\[(4.12)\]

⁵The fact that the transformations \(\delta\psi = l(x)\gamma_5\psi\) and \(\delta\psi = c_\mu\gamma^\mu\gamma_5\psi\) are anomaly-free follows in superspace from rewriting these transformations as \(\delta\phi(x, \theta) = l(x)\theta^a\gamma_5\partial_\theta^a\phi(x, \theta)\) and \(\delta\phi(x, \theta) = c_\mu(x)\theta^a\gamma^\mu\gamma_5\partial_\theta^a\phi(x, \theta)\) respectively, and applying (3.17). The \(d^2\theta\) integral projects out only the contributions \(\theta\gamma_5\theta\) and \(\theta\gamma^\mu\gamma_5\theta\), which vanish.
The last term in this Ward identity is an anomaly, and this anomaly constitutes the anomalous part in the central charge current itself. This is an unusual point that may lead to confusion: the anomaly is proportional to the central charge current itself. In this respect it resembles neither the trace anomaly nor the chiral anomaly: the trace anomaly contains a contraction of the current while the chiral anomaly contains a divergence of the current. In fact, it has been shown in [10] that the central charge current is the anomaly in the divergence of the conformal central charge current (which is explicitly $x$-dependent, just like the dilation current). Actually, the anomaly comes only from the $\delta F$ variation and not from the $\delta \psi$ variation, see footnote 4. We shall later explicitly compute the anomaly from the $\delta F$ variation separately, see (4.19). In that case one finds the relation in (4.12) without the $\bar{\psi} \gamma_{\nu} \not\partial \psi$ term

\[ \tilde{\zeta}^\mu(x) = \zeta^\mu(x) + \frac{\hbar g}{2\pi} \epsilon^\mu_{\nu} \partial_\nu \varphi(x). \] (4.13)

There is no contradiction: (4.12) should be used for correlation functions with the variations $\delta F$ and $\delta \psi$, while (4.19) should be used for correlation functions with only $\delta F$. Both relations are valid and different because the transformation rules are different.

From (4.9) we obtain the relation

\[ \tilde{T}_{\mu\nu}(x) - T_{\mu\nu}(x) = \frac{1}{2} \eta_{\mu\nu} [-F^2 - FU + \frac{1}{2} \bar{\psi} \not\partial \psi + g\varphi \bar{\psi} \psi] \]

\[ = \eta_{\mu\nu} \frac{\hbar g}{4\pi} F. \] (4.14)

This relation, and others in (4.9), was obtained by extracting contracted currents from the Ward identity, and by generalizing the contracted currents and the anomalies in the contracted currents to the currents and the anomalies in the currents themselves. This does not determine the current completely. We now prove these uncontracted identities. We begin with (4.14) and claim the following relation

\[ \langle -F^2 - FU + \frac{1}{2} \bar{\psi} \not\partial \psi + g\varphi \bar{\psi} \psi \rangle = \langle -F(x) \frac{\delta S}{\delta F(x)} - \frac{1}{2} \bar{\psi}(x) \frac{\delta S}{\delta \psi(x)} \rangle \]

\[ = \langle \frac{\hbar g}{2\pi} F \rangle. \] (4.15)

Again we stress that this relation, and others to follow, holds in Ward identities for correlation functions. To prove this relation, consider

\[ \delta \phi(x, \theta) = \frac{1}{2} w(x) \bar{\theta} \frac{\partial}{\partial \theta} \phi(x, \theta) = \frac{1}{2} w(x) \bar{\theta} Q \phi(x, \theta) \]

\[ = \frac{1}{2} w(x) \bar{\psi}(x) + w(x) \delta(\theta) F(x) \] (4.16)

which is the “R-symmetry” in the present context and physically corresponds to a Weyl transformation. Equating the variation of the action to that of the Jacobian in the path
integral in the form of (4.4) gives the identity \(^6 (4.15)\). This analysis fixes the magnitude of the Weyl anomaly. At the end of Section 5 we show that \(T_\mu^\mu\) does not contribute to the trace anomaly, so the trace anomaly comes only from the conserved tensor.

The last relation in (4.9) to be proven is the one with \(\bar{J}^{\mu}(x)\) and \(j^\mu(x)\). We can show that

\[
\langle \bar{J}^{\mu}(x) - j^{\mu}(x) \rangle = \langle \gamma^{\mu} \psi(x)(F(x) + U(\varphi)) \rangle = \langle \gamma^{\mu} \psi(x) \frac{\delta S}{\delta F(x)} \rangle = \langle -\frac{h}{2\pi} \gamma^{\mu} \psi(x) \rangle
\]

by considering the variation

\[
\delta \phi(x, \theta) = \delta(\theta) \epsilon^{\mu}(x) \gamma^{\mu} \frac{\partial}{\partial \theta} \phi(x, \theta)
\]

which agrees with (4.9)\(^7\).

Although we already obtained the relations between \(\bar{\zeta}_\mu\) and \(\zeta_\mu\) from (4.7), we can apply the same techniques as used for \(\bar{T}_{\mu\nu}\) and \(\bar{J}^{\mu}\), and obtain then

\[
\langle \bar{\zeta}_\mu(x) - \zeta_\mu(x) \rangle = \langle -\epsilon_{\mu\nu} \partial^\nu \varphi(x)(F(x) + U) \rangle = \langle -\epsilon_{\mu\nu} \partial^\nu \varphi(x) \frac{\delta S}{\delta F(x)} \rangle = \langle \frac{h}{2\pi} \epsilon_{\mu\nu} \partial^\nu \varphi(x) \rangle
\]

by considering the variation

\[
\delta \phi(x, \theta) = -\delta(\theta) \nu^{\mu}(x) \epsilon_{\mu\nu} \partial^\nu \phi(x, \theta)
\]

which is indeed consistent with (4.9). As we already discussed, the transformation (4.19) generates only the central charge current without inducing a variation of the fermion, and it generates the Ward identities with the central charge current itself, not its divergence.

\(^6\)The Noether current \(T_{\mu\nu}^N(x)\) generated by the variation \(\delta \phi(x, \theta) = \xi^{\mu}(x) \partial_\mu \phi(x, \theta)\) is given by \(\delta S = \int \left(\partial^\mu \xi^\nu\right) T_{\mu\nu}^N\). It reads

\[
T_{\mu\nu}^N(x) = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \eta_{\mu\nu} \left[ \left( \partial^\rho \varphi \right) \left( \partial_\rho \varphi \right) - F^2 - 2FU \right] + \frac{1}{4} \tilde{\psi} \left[ \gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu \right] \psi + \frac{1}{4} \tilde{\psi} \left[ \gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu \right] \psi - \frac{1}{2} \eta_{\mu\nu} \left[ \tilde{\psi} \gamma^\mu \partial_\mu \psi + 2 \gamma^{\nu} \varphi \tilde{\psi} \right],
\]

\[
= \bar{T}_{\mu\nu}(x) - \eta_{\mu\nu} \left[ -F^2 - FU + \frac{1}{2} \tilde{\psi} \partial_\mu \varphi + g \varphi \tilde{\psi} \right] + \frac{1}{4} \tilde{\psi} \left[ \gamma_\nu \partial_\mu - \gamma_\mu \partial_\nu \right] \psi
\]

where we used the Weyl anomaly in (4.15). The last term is manifestly antisymmetric, and it can be written as \(\tilde{\psi} \left[ \gamma_\nu \partial_\mu - \gamma_\mu \partial_\nu \right] \psi = -\epsilon_{\mu\nu} \tilde{\psi} \epsilon^{\rho\sigma} \gamma_\rho \partial_\sigma \psi = -\epsilon_{\mu\nu} \tilde{\psi} \gamma^{\rho} \partial_\rho \psi\). It yields the antisymmetric part of the stress tensor, and is proportional to the divergence of the Lorentz current. One may show that \(\partial^{\mu} T_{\mu\nu}^N\) by evaluating the Jacobian for \(\delta \phi(x, \theta) = \xi^{\mu}(x) \partial_\mu \phi(x, \theta)\). This Jacobian equals \(\xi^{\mu}(\frac{h}{2\pi} \partial_\mu F(x))\), and this implies \(\partial^{\mu} \bar{T}_{\mu\nu}(x) = 0\). In our analysis, the conserved \(\bar{T}_{\mu\nu}\), and \(T_{\mu\nu}\) which is not conserved but free of a trace anomaly, play a basic role. Note that both \(\bar{T}_{\mu\nu}\) and \(T_{\mu\nu}\) are manifestly symmetric.

\(^7\)Clearly, the naive equation of motion \(F(x) + U(\varphi) = 0\) cannot be used in this derivation.
If one sets \( v^\mu(x) = \partial^\mu v(x) \) in (4.20), one generates the divergence of the central charge current but the procedure gives no information for a topological current. In analogy with \( U(1) \) gauge theory, we are considering the change of variable \( A_\mu \to A_\mu + a_\mu \) instead of \( A_\mu \to A_\mu + \partial_\mu a \) to generate the current.

It is important to recognize that all operators appearing on the left-hand sides of the relations in (4.7) have higher mass dimensions than those of the corresponding operators on the right-hand sides. For example, \( \tilde{\zeta}^\mu(x) \) and \( \zeta^\mu(x) \) are, respectively, dimension 2 and 1 operators since the coupling constant \( g \) carries a unit mass dimension. Similarly, \( \gamma_\mu \tilde{J}^\mu \) and \( \gamma_\mu j^\mu \) are, respectively, dimension 3/2 and 1/2 operators, though both of \( \tilde{J}^\mu \) and \( j^\mu \) are dimension 3/2 operators. Also, \( \tilde{T}^\mu_\nu \) and \( T^\mu_\nu \) are, respectively, dimension 2 and 1 operators, though both of \( \tilde{T}^\mu_\nu \) and \( T^\mu_\nu \) are dimension 2 operators. In this sense all the composite operators on the right-hand sides of (4.7) are soft operators. This suggests that only the “hard” operators generate anomalies. In the next section we prove this statement.

5 Supersymmetry algebra of the quantum operators

In the previous section we gave a direct derivation of the anomalies based on path integrals, but we already mentioned in the introduction that one can also obtain the anomalies from the \( \gamma_\mu \tilde{J}^\mu \) anomaly by making successive susy transformations. In this section we implement this second approach. Since this involves commutators of currents, we convert the path integral relations into operator relations by following the BJL method.

We begin by considering the variation

\[
\delta \phi(x, \theta) = \bar{\epsilon}(x) Q \phi(x, \theta).
\]  (5.1)

The change of the action defines the Noether current

\[
\delta S = \int d^2 x (\partial_\mu \bar{\epsilon}(x)) j^\mu(x)
\]  (5.2)

where

\[
j^\mu,\alpha(x) = -\{[\partial_\phi(x) + U(\phi(x))]\gamma^\mu \psi(x)\}^\alpha
\]  (5.3)

with \( U(\phi) = g(\phi^2 - v_0^2) \). The Jacobian factor for (5.1) gives the anomaly, and we obtain the identity

\[
\partial_\mu j^\mu(x) = \frac{\hbar g}{2\pi} \partial \psi(x)
\]  (5.4)

corresponding to the term proportional to \( \delta(\theta) \) in (4.6). Thus the current \( \tilde{J}^\mu \) defined in (4.9) is conserved

\[
\tilde{J}^\mu(x) \equiv j^\mu(x) - \frac{\hbar g}{2\pi} \gamma^\mu \psi(x), \quad \partial_\mu \tilde{J}^\mu(x) = 0.
\]  (5.5)

It contains the contributions from the action and Jacobian, appears in all the Ward identities, and this implies, as we shall see, that the relations among various Green’s functions obtained by global supersymmetry are not modified in form by non-trivial Jacobians.
If one considers the rigid conformal supersymmetry transformation generated by the parameter
\[ \bar{\epsilon}(x) = \bar{a}(x) \neq \]
the action transforms as follows
\[ \delta S = - \int d^2x [ (\partial_\mu \bar{a}(x)) \neq \partial_\mu j^\mu(x) + \bar{a}(x) \gamma_\mu j^\mu(x) ], \]
and one obtains the identity
\[ \partial_\mu (\neq \gamma_\mu j^\mu(x)) = \gamma_\mu j^\mu(x) - \frac{hg}{\pi} \psi(x) = \gamma_\mu \tilde{J}^\mu(x) \]

The right hand side does not vanish in general, since the superconformal symmetry is explicitly broken in the present model. If the action would have been invariant under the transformation \( \bar{\epsilon}(x) = \bar{a} \neq \) with \( x \)-independent parameter \( \bar{a} \), the term \( \gamma_\mu j^\mu \) would not have appeared in (5.7) and thus in the identity (5.8). This shows that \( \gamma_\mu j^\mu \) does not contain a superconformal anomaly.

We thus write the identity (5.8) as
\[ \partial_\mu (\neq \gamma_\mu j^\mu(x)) = (\gamma_\mu j^\mu(x))_{\exp} - \frac{hg}{\pi} \psi(x) = \gamma_\mu \tilde{J}^\mu(x) \]

where \( (\gamma_\mu j^\mu(x))_{\exp} \) stands for the terms which explicitly break superconformal symmetry. Using \( \gamma_\mu \gamma_\nu \gamma_\mu = 0 \) in strictly \( d = 2 \) one obtains
\[ (\gamma_\mu \tilde{J}^\mu(x))_{\exp} = (\gamma_\mu j^\mu(x))_{\exp} = \gamma_\mu j^\mu = -2U(\varphi)\psi(x) \]

and
\[ \begin{align*}
(\gamma_\mu \tilde{J}^\mu(x))_{\text{anomaly}} &= \frac{-hg}{\pi} \psi(x), \\
(\gamma_\mu j^\mu(x))_{\text{anomaly}} &= 0.
\end{align*} \]

We thus have two kinds of supersymmetry currents, \( \tilde{J}^\mu \) and the Noether current \( j^\mu(x) \). The current \( \tilde{J}^\mu(x) \) is conserved but contains the superconformal anomaly, whereas \( j^\mu \) is not conserved but free from a superconformal anomaly. Both contain explicit conformal supersymmetry breaking terms.

The conserved current \( \tilde{J}^\mu \) generates supersymmetry for all the components of \( \phi(x, \theta) \). This can be shown by starting with
\[ \langle \phi(y, \theta) \rangle = \int \mathcal{D} \phi(y, \theta) \exp[iS] \]
(recall that we suppress writing further fields \( \phi_1(x_1)\ldots\phi_n(x_n) \)) and considering the change of variables (5.1). One obtains then
\[ -i\partial_\mu \langle T^a \tilde{J}^{\mu, a}(x) \phi(y, \theta) \rangle + \langle \delta_{\text{susy}} \phi(y, \theta) \rangle = 0 \]

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where \( \delta_{\text{susy}}\phi(y, \theta) \) stands for the variation of \( \phi(y, \theta) \)

\[
\delta_{\text{susy}}\phi(y, \theta) = \delta^2(x - y)Q^\alpha \phi(x, \theta).
\]  \( \text{(5.14)} \)

We then apply the Bjorken-Johnson-Low (BJL) analysis to replace the \( T^* \) product by the \( T \) product\(^8\)

\[
-i \partial_\mu \langle T_\mu J^{\mu, \alpha}(x) \phi(y, \theta) \rangle + \langle \delta^2(x - y)Q^\alpha \phi(x, \theta) \rangle = 0
\]  \( \text{(5.15)} \)

and obtain in the limit \( k_0 \to \infty \) the equal time commutator (see appendix)

\[
i[\tilde{J}^{0, \alpha}(x), \phi(y, \theta)]\delta(x^0 - y^0) = \delta^2(x - y)[\frac{\partial}{\partial \theta_\alpha} - (\gamma^\mu \theta)\partial_\mu] \phi(x, \theta).
\]  \( \text{(5.16)} \)

This proves the conservation equation \( \langle T_\mu \partial_\mu \tilde{J}^{\mu, \alpha}(x) \phi(y, \theta) \rangle = 0 \), and shows that \( \tilde{J}^{0, \alpha} \) generates ordinary supersymmetry transformations.

We next specify the energy-momentum tensor \( T^{\mu \nu} \) and the central charge current \( \zeta_\mu(x) \).

The charge \( \tilde{Q}^\alpha \)

\[
\tilde{Q}^\alpha = \int dx J^{0, \alpha}(x)
\]  \( \text{(5.17)} \)

generates supersymmetry, as we have shown above, and thus we have

\[
i[\bar{\epsilon} \tilde{Q}, j^{\nu, \beta}(y)] = -2T^{\mu \nu}(y)(\gamma^\mu)_\beta^\alpha \epsilon^\alpha - 2\zeta_\nu(y)(\gamma_5)_\beta^\alpha \epsilon^\alpha - (F + U)(\partial_\nu \epsilon)\gamma^\beta
\]

\[
= -2T^{\mu \nu}(y)(\gamma^\mu)_\beta^\alpha \epsilon^\alpha - 2\zeta_\nu(y)(\gamma_5)_\beta^\alpha \epsilon^\alpha + \frac{\hbar g}{2\pi}(\partial_\nu \epsilon)\gamma^\alpha \gamma^\beta
\]  \( \text{(5.18)} \)

where we used (4.19). In the path integral framework, this relation is derived by starting with \( \langle j^{\nu}(y) \rangle = \int D\phi j^{\nu}(y) e^{iS} \) and considering the change of variables corresponding to (local) supersymmetry

\[
i \partial_\mu \langle T^* \tilde{J}^{\mu}(x) j^{\nu}(y) \rangle = \delta(x - y) \langle \delta_{\text{susy}}j^{\nu}(y) \rangle.
\]  \( \text{(5.19)} \)

The local variations of the action and the measure give together the left-hand side, just as in (5.13). The BJL analysis then gives rise to the commutator. The operators appearing here are given by\(^9\)

\[
\zeta_\mu(x) = \epsilon_\mu \partial_\nu \varphi(x) U(\varphi), \quad \epsilon_{01} = -1,
\]

\(^8\)The crucial point in the BJL analysis in the present case is the treatment of the anomaly of the term \( \partial^\mu \langle T_{\mu} T_{\nu} \gamma_\nu \phi(x, \theta) \rangle \). It should be replaced by \( \partial^\mu \langle T_{\mu} T_{\nu} \gamma_\nu \phi(x, \theta) \rangle \) by noting that the \( T \)-product should satisfy the condition \( \lim_{k_0 \to -\infty} \int d^2k \exp[ik(x - y)] \langle T_{\mu} T_{\nu} \gamma_\nu \phi(x, \theta) \rangle = 0 \). If one would keep the derivative operator inside the \( T \)-product, this condition is spoiled. See the appendix for an account of the BJL prescription.

\(^9\)By noting the completeness of \( (1, \gamma_5, \gamma^\mu) \), the supersymmetry variation of the current \( j^{\nu}(y) \) is expanded as

\[
\delta j^{\nu, \beta}(y) = -2T^{\mu \nu}(y)(\gamma^\mu)_\beta^\alpha \epsilon^\alpha - 2\zeta_\nu(y)(\gamma_5)_\beta^\alpha \epsilon^\alpha - 2\nu_\nu(y) \epsilon^\beta
\]

By multiplying this relation by \( \bar{\epsilon} \gamma_\mu, \bar{\epsilon} \gamma_5 \) and \( \bar{\epsilon} \), respectively, we can project out the 3 components above by noting \( \bar{\epsilon} \gamma^\mu \epsilon = \bar{\epsilon} \gamma_5 \epsilon = 0 \). The vector component \( \nu^\nu \) is shown to vanish on-shell by using safe (i.e., anomaly-free) relations such as \( \bar{\psi}(x)\gamma^\mu \frac{\delta S}{\delta \psi(x)} = \bar{\psi}(x)\gamma_5 \gamma^\mu \frac{\delta S}{\delta \psi(x)} = 0 \), except for the term explicitly written in (5.18).
\[
T_{\mu\nu}(x) = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \eta_{\mu\nu}[(\partial^\rho \varphi)(\partial_\rho \varphi) - UF] + \frac{1}{4} \bar{\psi} \gamma_\mu \partial_\nu \gamma^\mu \psi - \frac{1}{4} \eta_{\mu\nu}[\bar{\psi} \gamma^\mu \partial_\mu \psi + 2g \varphi \bar{\psi} \psi] \tag{5.20}
\]

which agrees with the operators in (4.9).

We thus obtain
\[
i[\bar{\epsilon} \tilde{Q}, \gamma_\nu j_\mu^\nu(y)] = -2T_{\mu\nu}(y) \epsilon - 2\gamma_\nu \zeta_\nu(y) \gamma_5 \epsilon = -2(FU - g \varphi \bar{\psi} \psi)e - 2(\gamma_\nu \epsilon^\nu \partial_\mu \varphi(x)U) \gamma_5 \epsilon \tag{5.21}
\]

Since the operator \(\gamma_\mu j^\mu\) does not contain an anomaly, the operators on the right-hand side are expected not to contain superconformal anomalies either
\[
T_{\mu\mu}(x)_{\text{anomaly}} = 0, \quad \gamma_\mu \zeta^\mu(x)_{\text{anomaly}} = 0. \tag{5.22}
\]

The absence of an anomaly in \(T_{\mu\mu}\) is consistent with the analysis in the previous section. If \(\gamma_\mu j^\mu = 0\), these operators also vanish. The operator \(T_{\mu\nu}\) is not conserved
\[
\partial_\nu T_{\mu\nu}(x) \neq 0 \tag{5.23}
\]
due to the effects of the anomaly, in accord with the non-conservation of the Noether current \(j^\nu\). We emphasize that \(T_{\mu\mu}(x)\) and \(T_{\nu\mu}(x)\) should be clearly distinguished, and in fact one cannot reproduce \(T_{\nu\mu}(x)\) from the knowledge of \(T_{\mu\mu}(x)\) alone. For this reason, it is possible that \(T_{\mu\mu}(x)\) does not contain anomaly while \(T_{\nu\mu}(x)\) does contain anomalies. Either the conservation condition or the trace is influenced by the anomaly. For \(T_{\mu\mu}(x)\), we have no conservation condition.

We next examine the supersymmetry algebra for the conserved supercurrent
\[
i[\bar{\epsilon} \tilde{Q}, \tilde{J}^\nu(y)] = -2T_{\mu\nu}(y) \gamma^\mu J_\alpha^\nu \gamma_\alpha^\nu - 2\tilde{\zeta}^\nu(y) \gamma_5 \epsilon \tag{5.24}
\]
where
\[
\partial_\nu \tilde{T}_{\mu\nu}(y) = 0, \quad \partial_\nu \tilde{\zeta}(y) = 0 \tag{5.25}
\]
because of the conservation \(\partial_\nu \tilde{J}^\nu = 0\). Again this relation is obtained by making a local supersymmetry transformation as in (5.13) and (5.19). If one uses the expression
\[
\tilde{J}^\mu(x) = -[\bar{\delta} \varphi(x) - F] \gamma^\mu \psi(x) \tag{5.26}
\]
in the path integral
\[
\int \mathcal{D} \phi \tilde{J}^\nu(y) \exp[iS] \tag{5.27}
\]
and considers the change of variables corresponding to a supersymmetry transformation, one obtains
\[
i\partial_\mu \langle T^* \tilde{J}^\mu(x) \tilde{J}^\nu(y) \rangle = \delta(x - y) \langle \delta_{\text{susy}} \tilde{J}^\nu(y) \rangle.
\]

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(To be precise, the current $\tilde{J}^\mu(x)$ on the left-hand side is defined by (5.5).) The operator commutation relation $i[\bar{\epsilon}\tilde{Q}, \tilde{J}^\nu(y)]$ is then obtained by the BJL analysis. In this way one obtains the expressions for $\tilde{T}_{\mu\nu}(y)$ and $\tilde{\zeta}^\nu(y)$ in (4.9).

Instead, if one uses the form of $J^\mu = j^\mu - \frac{\hbar g}{2\pi} \gamma^\mu \psi$ and (5.18) and repeats the same procedure, one obtains (in this case, we have two identical $\tilde{J}^\mu$)

$$
\tilde{T}_{\mu\nu}(y) = T_{\mu\nu}(y) + \eta_{\mu\nu} \frac{\hbar g}{4\pi} F,
$$

$$
\tilde{\zeta}^\nu(y) = \zeta^\nu(y) + \frac{\hbar g}{2\pi} \epsilon^{\mu\nu} \partial_\mu \varphi(y).
$$

(5.28)

Note that the extra term in (5.18) combines with a part of the variation of $-\bar{\epsilon} \tilde{Q} \gamma_5 \tilde{J}^\nu(y)$ to give rise to the central charge anomaly in the algebra for conserved current.

We next note that

$$
i[\bar{\epsilon}\tilde{Q}, \gamma_\nu \tilde{J}^\nu(y)] = -2\tilde{T}^\mu_{\mu}(y)\epsilon - 2\gamma_\nu \tilde{\zeta}^\nu(y)\gamma_5 \epsilon
= -2[T^\mu_{\mu}(y) + \frac{\hbar g}{2\pi} F(y)]\epsilon
-2[\gamma_\nu \tilde{\zeta}^\nu(y)\gamma_5 \epsilon + \frac{\hbar g}{2\pi} \partial_\mu \varphi(y) \gamma^\mu \gamma_5 \epsilon]
$$

(5.29)

where the second expression is obtained by the super transformation generated by $\tilde{Q}$ of the expression

$$
\gamma_\nu \tilde{J}^\nu(y) = \gamma_\nu j^\nu(y) - \frac{\hbar g}{\pi} \psi(y) = -2U(\varphi)\psi(y) - \frac{\hbar g}{\pi} \psi(y).
$$

(5.30)

We thus have

$$
\tilde{T}^\mu_{\mu}(x) = T^\mu_{\mu}(x) + \frac{\hbar g}{2\pi} F(x),
$$

$$
\gamma_\nu \tilde{\zeta}^\nu(x) = \gamma_\nu \zeta^\nu(x) + \frac{\hbar g}{2\pi} \partial_\mu \varphi(x) \gamma^\mu \gamma_5.
$$

(5.31)

We would like to explain that the separation of anomaly terms from the explicit (soft) symmetry breaking terms makes sense already in the trivial vacuum and to order $\hbar g$. We start with the renormalization of the vacuum value by a tadpole contribution. We adopt the following minimal renormalization convention [6, 17]

$$
U = g(\varphi^2 - v_0^2) = g(\varphi^2 - v^2) - g \delta v^2 = U_{\text{ren}} + \Delta U
$$

(5.32)

at the one-loop level accuracy in the trivial vacuum. We then analyze the explicit breaking term in the central charge current

$$
\zeta_\mu(x) = \epsilon_{\mu\nu} \partial^\nu \varphi(x) U(\varphi)
= g \epsilon_{\mu\nu} \partial^\nu \varphi(x) (\varphi^2 - v_0^2).
$$

(5.33)
Our regularization of the Jacobian by the heat kernel in (2.19) and (2.30) corresponds to the following regularization of the propagator

\[ \langle T^* \eta(x, \theta) \eta(y, \theta') \rangle = \frac{i}{\Gamma} \exp \left[ \frac{\Gamma^2}{M^2} \right] \delta(x-y) \delta(\theta - \theta') \]  

(5.34)

The factor \( \exp[\Gamma^2/M^2] \) gives an exponential cut-off whose leading term is \( \exp[-\frac{1}{4}k^2/M^2] \). Since there is a coupling constant \( g \) in front of \( \zeta_\mu \), only the tadpole diagrams are relevant to order \( h g \). The tadpole \( \langle \varphi(x)\varphi(x) \rangle \) is absorbed into the renormalization of the vacuum value. The remaining tadpole has the form

\[ \langle \partial_\mu \varphi(x) \varphi(x) \rangle = 0 \]  

(5.35)

since, to this order, the propagator becomes the free one and our regulator becomes \( \exp[-\frac{1}{4}p^2/M^2] \). Lorentz symmetry thus gives a vanishing result: the operator \( \zeta_\mu \) does not induce an anomaly term\(^{10}\) to order \( h g \).

On the other hand, the “hard” operator \( \tilde{\zeta}_\mu(x) = -\epsilon_{\mu\nu} \partial^\nu \varphi(x) F(x) \), which is of zeroth order in \( g \), produces an anomaly to order \( h g \), in addition to the soft breaking term \( \zeta_\mu \). In this case we need to use the full expression of the propagator in (5.34) and this gives rise to the one loop anomaly as was shown by a superfield calculation in Section 4.

We finally examine the soft breaking term of conformal symmetry

\[ T_\mu^\mu(x) = FU - g \varphi \overline{\psi} \psi = gF(\varphi^2 - v_0^2) - g \varphi \overline{\psi} \psi \]  

(5.36)

This operator does not generate a trace anomaly in the trivial vacuum to order \( h g \), since this combination of composite operators appears inside the action, where the renormalization prescription of these composite operators is precisely specified. One can also confirm this statement by an analysis of one-loop tadpole diagrams; the tadpole \( \langle \varphi(x)\varphi(x) \rangle \) is absorbed by the renormalization of the vacuum value, and \( 2 \langle \varphi(x)F(x) \rangle \) and \( \langle \overline{\psi}(x)\psi(x) \rangle \) precisely cancel each other\(^{11}\). We prove this statement by a manifestly supersymmetric calculation

\[
T_\mu^\mu(x) = gF(x)(\varphi^2(x) - v_0^2) - g\varphi(x)\overline{\psi}(x)\psi(x) \\
= \int d^2\theta g[\frac{1}{3} \phi^3(x, \theta) - v_0^2 \phi(x, \theta)]
\]

which gives rise to the following \( h g \) contribution in the trivial vacuum

\[
\delta T_\mu^\mu(x) = \int d^2\theta g[\phi(x, \theta)(T^*\phi(x, \theta)\phi(x, \theta)) - h\delta v^2 \phi(x, \theta)] \\
= \int d^2\theta g[\phi(x, \theta)(\langle T^*\eta(x)\eta(x) \rangle) - h\delta v^2] = 0. 
\]

\(^{10}\)This conclusion also holds in some other regularization schemes such as the dimensional regularization and the dimensional reduction \([10]\). On the other hand, using point splitting, the operator \( \int \delta(x-y)(\partial_\sigma \varphi(x)U(y) + \partial_\sigma \varphi(y)U(x))dx \) does contain the anomaly \([18]\). Setting naively \( x = y \) in their expressions, one loses the anomaly and obtains \( \zeta_\sigma(y) \).

\(^{11}\)The same conclusions are reached for the kink vacuum: the operator \( T_\mu^\mu \) does not contribute to the anomaly.
In the first line we encounter both of a $\langle \psi \psi \rangle$ and a $\langle \varphi F \rangle$ terms, and in the second line we used (2.22) for $\theta_1 = \theta_2$. Hence $2\langle \varphi(x) F(x) \rangle$ is indeed equal to $\langle \psi(x) \psi(x) \rangle$.

When one analyzes the kink vacuum, one obtains non-anomalous one-loop corrections from these soft operators (though these corrections vanish after spatial integration).

6 The BPS bound

We have identified the quantum operators in the supersymmetry algebra and deduced the presence of superconformal anomalies from a manifestly supersymmetric superfield calculation. We next examine how the anomalies modify the BPS bound for a kink solution.

The algebra of supersymmetry charges follows from (5.24) and reads

$$i\{\bar{Q}^{\alpha}, \bar{Q}^{\beta}\} = -2(\gamma^\mu)^{\alpha\beta} \bar{P}_\mu - 2\bar{Z} (\gamma_5)^{\alpha\beta}$$

(6.1)

where we defined

$$\bar{P}_\mu = \int dx \bar{T}_{0\mu}(x), \quad \bar{H} = \bar{P}_0,$$

$$\bar{Z} = \int dx \bar{\xi}_0(x) = -\int dx \bar{\xi}(x)$$

(6.2)

We can now evaluate the modified energy and central charge in the vacuum of a time independent kink solution $\varphi_K(x)$, whose center is located at the origin, (to the accuracy of $O(h)$)

$$\langle 0 | \varphi(x) | 0 \rangle = \varphi_K(x) + h \varphi_1(x) \equiv v \tanh(gvx) + h \varphi_1(x)$$

(6.3)

where $h \varphi_1(x)$ is a quantum correction to the kink solution [8, 18]. The results are

$$\langle 0 | \bar{H} | 0 \rangle = -\langle 0 | \int dx \bar{T}_{00}(x) | 0 \rangle = -\langle 0 | \int dx [\bar{T}_{00}(x) + \bar{T}_{11}(x)] | 0 \rangle$$

$$= -\langle 0 | \int dx [T_{\mu}^\mu(x) + \frac{h}{2\pi} F(x)] | 0 \rangle$$

$$= -\langle 0 | \int dx [(F_U - g \varphi \bar{\psi} \psi) + \frac{h}{2\pi} F(x)] | 0 \rangle$$

$$= M - \frac{hm}{2\pi}$$

(6.4)

where $M$ is the classical mass of the kink solution, and $m = 2gv$ stands for the fermion mass at spatial infinity. Furthermore

$$\langle 0 | \bar{Z} | 0 \rangle = -\langle 0 | \int dx \bar{\xi}_0(x) | 0 \rangle = -\langle 0 | \frac{1}{2} tr \int d x \gamma^0 [\gamma_0 \bar{\xi}_0 + \gamma_1 \bar{\xi}_1] | 0 \rangle$$

$$= -\langle 0 | \frac{1}{2} tr \int d x \gamma^0 [\gamma_\mu \bar{\xi}_\mu + \frac{h}{2\pi} \partial_\mu \varphi(x) \gamma^\mu \gamma_5] | 0 \rangle$$

$$= -\langle 0 | \frac{1}{2} tr \int d x \gamma^0 [(\gamma_\nu e^\nu_\mu \partial_\mu \varphi(x) U) + \frac{h}{2\pi} \partial_\mu \varphi(x) \gamma^\mu \gamma_5] | 0 \rangle$$

$$= -\langle 0 | \int d x \{ g \partial_1 \varphi(x) [\varphi^2(x) - v_0^2] + \frac{h}{2\pi} \partial_1 \varphi(x) \} | 0 \rangle$$

$$= M - \frac{hm}{2\pi}.$$
Here we used (6.10) below in the last step.

In the first step of both of (6.4) and (6.5) we used

$$
\langle 0 \mid \int dx \tilde{\zeta}^1(x) \mid 0 \rangle = 0, \quad \langle 0 \mid \int dx \tilde{T}^1_1 \mid 0 \rangle = 0
$$

(6.6)

which are the consequences of the conservation condition of $\tilde{\zeta}^\mu(x)$ and $\tilde{T}_\mu^\nu$

$$
\langle 0 \mid [\partial_\mu \tilde{\zeta}^0(x) + \partial_1 \tilde{\zeta}^1(x)] \mid 0 \rangle = \partial_1 \langle 0 \mid \tilde{\zeta}^1(x) \mid 0 \rangle = 0, \\
\langle 0 \mid [\partial_\mu \tilde{T}^0_1(x) + \partial_1 \tilde{T}^1_1(x)] \mid 0 \rangle = \partial_1 \langle 0 \mid \tilde{T}^1_1(x) \mid 0 \rangle = 0.
$$

(6.7)

Namely, $\langle 0 \mid \tilde{\zeta}^1(x) \mid 0 \rangle$ and $\langle 0 \mid \tilde{T}^1_1(x) \mid 0 \rangle$ are independent of $x$, which may be fixed at the values at spatial infinity $\langle 0 \mid \tilde{\zeta}^1(\infty) \mid 0 \rangle = 0$ and $\langle 0 \mid \tilde{T}^1_1(\infty) \mid 0 \rangle = 0$ (or $\langle 0 \mid \tilde{\zeta}^1(0) \mid 0 \rangle = 0$ due to parity invariance of the kink vacuum). We note that the particle spectrum at spatial infinity has a well-defined mass gap, and in the asymptotic region of the kink vacuum there is no flow of central charge, namely, $\langle 0 \mid \tilde{\zeta}^1(\infty) \mid 0 \rangle = 0$, and no pressure, namely, $\langle 0 \mid \tilde{T}^1_1(\infty) \mid 0 \rangle = 0$.

We now explain that the BPS bound is generally satisfied for our kink solution and for our formulas of $\langle 0 \mid \tilde{H} \mid 0 \rangle$ and $\langle 0 \mid \tilde{Z} \mid 0 \rangle$ in (6.4) and (6.5). We start with the analysis of

$$
\langle 0 \mid Q^\alpha \frac{\hbar g}{2\pi} \phi(x, \theta) \mid 0 \rangle = \frac{\hbar g}{2\pi} \langle 0 \mid \theta^\alpha F(x) - (\gamma^1 \theta)^\alpha \partial_1 \varphi(x) \mid 0 \rangle
$$

(6.8)

where $Q^\alpha$ is the differential operator in (2.8). We also use the fact that $\partial_\mu \langle 0 \mid \tilde{O}(t, x) \mid 0 \rangle = 0$ for a general operator $\tilde{O}(t, x)$ and that the expectation values of the fermionic components vanish in (6.8). When one chooses $\alpha = +$ in the above relation, one obtains

$$
\langle 0 \mid Q^+ \frac{\hbar g}{2\pi} \phi(x, \theta) \mid 0 \rangle = \frac{\hbar g}{2\pi} [\tilde{F}(x) - \partial_1 \varphi(x)] \mid 0 \rangle = 0
$$

(6.9)

since $i[Q^+, \phi] = Q^+ \phi$ according to (5.16), while $\tilde{Q}^+$ preserves the kink vacuum. This identity shows that the anomalous contributions to $\langle 0 \mid \tilde{H} \mid 0 \rangle$ in (6.4) and $\langle 0 \mid \tilde{Z} \mid 0 \rangle$ in (6.5) are identical.

Similarly, one can show that the nonanomalous contributions to $\langle 0 \mid \tilde{H} \mid 0 \rangle$ and $\langle 0 \mid \tilde{Z} \mid 0 \rangle$ are equal. We use the identity

$$
\langle 0 \mid Q^+ \frac{\hbar g}{3} \phi^3(x, \theta) - gv_0^2 \phi(x, \theta) \mid 0 \rangle = \theta^+ \frac{\hbar g}{2\pi} \langle 0 \mid [\tilde{F}(x) - \partial_1 \varphi(x)] - \partial_1 \varphi(x)[\varphi^2(x) - v_0^2] \mid 0 \rangle = 0
$$

(6.10)

which shows that the contributions from explicit superconformal symmetry breaking terms to both of $\langle 0 \mid \tilde{H} \mid 0 \rangle$ and $\langle 0 \mid \tilde{Z} \mid 0 \rangle$ (and also to their densities) are identical in the kink vacuum which preserves $Q^+$ symmetry. This statement holds to all orders in quantum corrections as far as the kink vacuum preserves $Q^+$ symmetry.

\footnote{The classical kink solution which is specified by $\varphi_K(x)$ and $F_K(x)$, $F_K(x) = \partial_1 \varphi_K(x)$, preserves $\tilde{Q}^+$ symmetry and $Q^+$ has no ordinary- supersymmetry anomaly ($\tilde{J}_\mu$ is conserved), while the action is fully supersymmetric. Thus the $\tilde{Q}^+$ symmetry is preserved to all orders in quantum corrections.}
We next prove that higher order nonanomalous corrections to the mass $M$ of the kink solution are absent. We define the exact solution which includes higher order quantum corrections to the classical kink solution

$$
(0|\varphi(x)|0) = \varphi_c(x)
$$

(6.11)

with the boundary condition (see (6.3) to the order $O(h)$)

$$
\varphi_c(\pm \infty) = \varphi_K(\pm \infty)
$$

(6.12)

and set

$$
\varphi(x) = \varphi_c(x) + \eta(x).
$$

(6.13)

We evaluate the central charge (6.5) arising from the explicit superconformal symmetry breaking

$$
-g \int dx\langle 0|\partial_1 \varphi(x)[\varphi^2(x) - v^2]|0\rangle = -g \int dx\partial_1 \varphi_c(x)[\varphi^2_c(x) - v^2] - g \int dx\partial_1 \varphi_c(x)\langle 0|\eta(x)\eta(x) - \delta v^2]|0\rangle
$$

$$
-2g \int dx\varphi_c(x)\langle 0|\partial_1 \eta(x)\eta(x)|0\rangle - g \int dx\langle 0|\partial_1 \eta(x)\eta(x)\eta(x)\eta(x)|0\rangle
$$

(6.14)

where $\delta v^2$ in this expression includes all the higher order quantum corrections. The first term in (6.14) gives the classical kink mass

$$
-g \int dx\partial_1 \varphi_c(x)[\varphi^2_c(x) - v^2] = \frac{4}{3}gv^3 = M
$$

(6.15)

The second and the third terms in (6.14) together give

$$
-2g \int dx\varphi_c(x)\langle 0|\partial_1 \eta(x)\eta(x)|0\rangle - 2g \int dx\varphi_c(x)\langle 0|\partial_1 \eta(x)\eta(x)|0\rangle
$$

$$
= -g \int dx\partial_1 \{\varphi_c(x)\langle 0|\eta(x)\eta(x) - \delta v^2]|0\rangle = 0
$$

(6.16)

since (nonperturbatively)

$$
\langle 0|\eta(x)\eta(x) - \delta v^2]|0\rangle|_{x=\pm \infty} = 0
$$

(6.17)

because of the renormalization condition in the trivial vacuum. The last term in (6.14) gives

$$
-g \int dx\langle 0|\partial_1 \eta(x)\eta(x)\eta(x)|0\rangle = -g\frac{1}{3} \int dx\partial_1 \langle 0|\eta(x)\eta(x)\eta(x)|0\rangle = 0
$$

(6.18)

by noting

$$
\langle 0|\eta(x)\eta(x)\eta(x)|0\rangle|_{x=\infty} = \langle 0|\eta(x)\eta(x)\eta(x)|0\rangle|_{x=-\infty}.
$$

(6.19)

This last relation arises from the fact that we have in the trivial vacuum $|v\rangle$

$$
\langle v|\eta(x)\eta(x)\eta(x)|v\rangle|_{x=\infty} = \langle v|\eta(x)\eta(x)\eta(x)|v\rangle|_{x=-\infty}
$$

(6.20)
because of the translational invariance. The theory in the trivial vacuum is invariant under the replacement

\[ v \rightarrow -v, \quad g \rightarrow -g, \quad \eta(x) \rightarrow -\eta(x), \quad F(x) \rightarrow -F(x) \]  

(6.21)

and thus

\[ g \langle v|\eta(x)\eta(x)|v\rangle|_{x=-\infty} = g \langle -v|\eta(x)\eta(x)| -v\rangle|_{x=-\infty} \]  

(6.22)

which leads to (6.19). Note that the operator

\[ \eta(x)\eta(x) - 3\delta v^2 \eta(x) \]  

(6.23)

is finite since only the tadpole of \( \eta \) is divergent even in the kink vacuum. Thus \( \langle 0|\eta(x)\eta(x)|x\rangle|_{x=-\infty} = 0 \).

To the accuracy of \( O(\hbar) \), the energy and central charge densities receive the same amount of correction

\[ -g \partial_1 \left\{ \varphi_K(x)|0|[\eta(x)\eta(x) - \hbar\delta v^2]|0\right\} \]  

(6.24)

which vanishes after spatial integration. The explicit evaluation of \( \langle 0|[\eta(x)\eta(x) - \hbar\delta v^2]|0\rangle \) has been performed in [8, 18] and reads\(^\text{13}\)

\[ \langle 0|[\eta(x)\eta(x) - \hbar\delta v^2]|0\rangle = \int \frac{dk}{2\pi} \frac{1}{2\omega} (|\varphi(k,x)|^2 - 1) + \frac{1}{\omega_B} \varphi_B^2(x) \]  

\[ = -\frac{3}{8\pi} \left( \frac{1}{\cosh^4 \frac{m x}{2}} + \frac{1}{4\sqrt{3}} \sinh^4 \frac{m x}{2} \right) \]  

(6.25)

which obviously satisfies the boundary conditions in (6.17).

The anomaly term gives rise to

\[ -\frac{\hbar g}{2\pi} \int dx \partial_1 \langle |\varphi(x)|0 \rangle = -\frac{\hbar g}{2\pi} \int dx \partial_1 \varphi_c(x) = -\frac{2\hbar g v}{2\pi} = -\frac{\h m}{2\pi} \]  

(6.26)

We note that the total central charge can be written as (to the accuracy of \( O(\hbar) \))

\[ M - \frac{\h m}{2\pi} = \frac{4}{3} g (v^2 - \frac{\hbar}{2\pi})^{3/2} \]  

(6.27)

which shows that the superconformal anomalies could be effectively represented as a well-defined finite shift of the renormalized vacuum value.

We thus find that the BPS bound is maintained as a result of the uniform shifts in energy and central charge in the kink vacuum, to the accuracy of the present approximation. Our analysis of the ground state with a kink solution is consistent with previous analyses on the basis of various other regularization methods [7, 8, 9, 10, 11, 17, 18].

\(^\text{13}\)The term with \(-1\) is due to the counter term for the vacuum value renormalization, and the bound state has \( \omega_B = \frac{1}{2}\sqrt{3} m \) and \( \varphi_B(x) = \frac{3m}{4} \frac{\sinh(mx/2)}{\cosh^4(mx/2)} \). The continuum solutions read

\[ \varphi(k, x) = e^{ikx} [-3 \tanh^2 \frac{m x}{2} + 4(\frac{m}{3})^2 + 6i \frac{m}{3} \tanh \frac{m x}{2}] / N \]  

with \( \omega = \sqrt{k^2 + m^2} \) and \( N^2 = 16 \frac{m^2}{N}(\omega^2 - \frac{m^2}{3}) \).
It is instructive to consider the problem from the point of view of a quantum deforma-
tion of the supersymmetry algebra by anomalies. This deformation appears if one ex-
presses the right-hand side of the supersymmetry algebra in terms of the (non-conserved)
operators $T_{\nu}^\mu$ and $\zeta^\mu$ in (5.28).

The supersymmetry charge algebra is then written as

$$i\{\tilde{Q}^\alpha, \tilde{Q}^\beta\} = -2(\gamma^\mu)^{\alpha\beta} P_\mu + \int dx \frac{hg}{2\pi} F(y)(\gamma^0)^{\alpha\beta}$$

$$-2Z(\gamma_5)^{\alpha\beta} + \int dx \frac{hg}{\pi} \partial_1 \varphi(y)(\gamma_5)^{\alpha\beta}$$

(6.28)

where we recall

$$P_\mu = \int dx T_{0\mu}(x), \quad H = P_0,$$

$$Z = \int dx \zeta_0(x) = -\int dx \zeta^0(x).$$

(6.29)

We thus find that the naive supersymmetry algebra is deformed by the effects of the
anomaly, and the net effects are the replacement

$$H \rightarrow \tilde{H} = H - \int dx \frac{hg}{4\pi} F(x),$$

$$Z \rightarrow \tilde{Z} = Z - \int dx \frac{hg}{2\pi} \partial_1 \varphi(x).$$

(6.30)

The modification of $H$ is half of the trace anomaly because $\eta_{\mu\nu} F$ contributes half as much
to $\tilde{T}_{00}$ as to $\tilde{T}_\mu^\nu$. On the other hand, the modification to $Z$ is the full central charge
anomaly, so, as we have argued before, $Z$ contains no effects of anomalies. This shows
that $H$ itself still contains the effects of the anomaly.

By our construction, the above expression gives the same result as before for the
ground state in the presence of a time-independent kink solution. For example,

$$\langle \tilde{H} \rangle = -\langle \int dx T^{00}_0(x) + \int dx \frac{hg}{4\pi} F(x) \rangle$$

$$= -\langle \int dx \tilde{T}^{00}_0(x) \rangle$$

(6.31)

in terms of the conserved tensor $\tilde{T}_\nu^\mu(x)$ by noting (5.28). Similarly

$$\langle \tilde{Z} \rangle = -\langle \int dx \zeta^0(x) \rangle - \langle \int dx \frac{hg}{2\pi} \partial_1 \varphi(x) \rangle$$

$$= -\langle \int dx \tilde{\zeta}^0(x) \rangle$$

(6.32)

in terms of the conserved current $\tilde{\zeta}^\mu$.

\footnote{The use of non-conserved quantities in the presence of background metric is common in conformal
field theory, and the anomalous conservation of $T_{\nu}^\mu$ there gives rise to the central charge in the Virasoro
algebra [19].}
When one solves the classical kink solution, which satisfies the BPS bound, one does not distinguish between the operators \( \tilde{T}_\mu^\nu \) and \( \tilde{\zeta}^\mu \) or \( T_\mu^\nu \) and \( \zeta^\mu \). When fully quantized as in our formulation, these two sets of operators give rise to different forms of the supersymmetry algebra. This difference is a quantum effect which is not taken account by the semi-classical Dirac bracket analysis. Properly distinguishing these two sets of currents clarifies the role of superconformal anomalies in the saturation of the BPS bound.

7 Conclusions

We have presented a superspace description of the anomalies in the energy and central charge of a supersymmetric kink. In this superspace description these anomalies form with other anomalies a multiplet, and this allows an analysis which preserves at all steps rigid ordinary supersymmetry. The anomalies occur in Ward identities for various currents, but do not always correspond to symmetries of the action. The conformal symmetries (Weyl transformations and conformal supersymmetry transformations) leave only the free part of the action invariant, so that the Ward identities corresponding to these transformations are explicitly broken Ward identities. We derived them in section 4 by making an ordinary supersymmetry transformation in superspace with a parameter which was local both in \( x \) and in \( \theta \). We derived a lemma (section 3) for the Jacobian of an arbitrary transformation of the scalar superfield \( \phi \) (not necessary corresponding to a symmetry of the action) and these Jacobians produced the anomalies in the explicitly broken Ward identities. The variation of the action produced the currents in these Ward identities, and it turned out to be useful to distinguish between currents \( \tilde{K}^\mu = \{ \tilde{J}^\mu, \tilde{T}_\mu^\nu, \tilde{\zeta}_\mu \} \) from the conformal invariant kinetic part of the action, and currents \( K^\mu = \{ j^\mu, T_\mu^\nu, \zeta_\mu \} \) from the nonconformal interacting part of the action. The former depend also on the interactions through the field equations of the Heisenberg operators. The currents \( \tilde{K}^\mu \) are conserved at the quantum level, and these are thus the currents which yield time-independent charges of the supersymmetry algebra. The anomalies involved traces and divergences of the currents \( \tilde{K}^\mu \) and \( K^\mu \). Denoting these contractions by \( O\tilde{K} \) and \( OK \), the anomalies were of the form \( O\tilde{K} = OK + F + hA_n \), where \( O \) is an (algebraic or differential) operator, \( F \) a term proportional to the fermion field equation and \( A_n \) the anomaly. Since the \( OK \) had lower field dimension than the \( O\tilde{K} \), only the \( O\tilde{K} \) contained anomalies. Thus we obtained a clear separation in the Ward identities between explicit symmetry breaking terms (due to \( OK \)) and anomalies.

The general local supersymmetry variation in superspace

\[
\delta \phi(x, \theta) = \bar{\Omega}(x, \theta)Q \phi(x, \theta)
\]

(7.1)
generates all the anomalous broken Ward identities. It contains a term \( \delta F = -\epsilon^{\mu\nu}c_\mu \partial_\nu \phi \) which generates the central charge current and its anomaly, together with the variation of fermion variable \( \delta \psi = c_\mu \gamma^\mu \gamma_5 \psi \) which is anomaly-free. It was then shown that the transformation

\[
\delta \phi(x, \theta) = -\delta(\theta)v^\mu(x)c_\mu \partial^\nu \phi(x, \theta)
\]

(7.2)
contains only the term \( \delta F = -\epsilon^{\mu\nu}v_\mu \partial_\nu \phi \) which generates the central charge current and its anomaly. If one uses \( v^\mu = \partial^\mu \phi \) in this variation, one obtains the divergence of the central
charge current but this yields no information because a topological current is identically conserved. In analogy with $U(1)$ gauge theory, the above variation (7.2) corresponds to the variation

$$A_\mu \rightarrow A_\mu + v_\mu$$  \hspace{1cm} (7.3)

to generate the current instead of the variation $A_\mu \rightarrow A_\mu + \partial_\mu v$ which generates the ordinary Noether current.

Another characteristic feature of the present formulation is that the conserved supersymmetry current

$$\tilde{J}_\mu(x) \equiv j_\mu(x) - \frac{\hbar g}{2\pi} \gamma^\mu \psi(x)$$  \hspace{1cm} (7.4)

appears in all the local Ward identities for rigid supersymmetry. Here $j_\mu$ stands for the Noether current coming from the action and $-\frac{\hbar g}{2\pi} \gamma^\mu \psi(x)$ from the Jacobian. The current $j_\mu$ is not conserved but free of the superconformal gamma-trace anomaly, whereas the conserved current $\tilde{J}_\mu$ contains the gamma-trace anomaly $(\gamma_\mu \tilde{J}_\mu)_{\text{anomaly}} = -\frac{\hbar g}{2\pi} \gamma^\mu \psi$. This appearance of two different currents with clear anomaly properties makes the analysis of the BPS bound transparent.

We then obtained the supersymmetry algebra for the conserved charge $\tilde{Q} = \int dx \tilde{J}_0(x)$ starting with a supersymmetry Ward identity by using the BJL prescription. The algebra is generally modified by the effects of the trace and central charge anomalies. The deformation of the algebra is in such a way that the BPS bound remains saturated as a result of uniform shifts in energy and central charge in the presence of a kink solution. If one uses the conserved quantities $\tilde{T}_\mu^\nu$ and $\tilde{\zeta}_\mu$, the supersymmetry algebra retains the naive form it had before one incorporates the effects of anomalies. If one uses the non-conserved quantities $T_\mu^\nu$ and $\zeta_\mu$, these currents are deformed by anomalies, but the physical content remains the same. Note that the “topology” of the kink vacuum is not modified by the superconformal anomalies. An interesting (and subtle) aspect of the present problem is that both the explicit and the anomalous breakings of superconformal symmetry are proportional to the coupling constant $g$.

There are only one-loop anomalies in the present (super-renormalizable) theory [7] [8]. Thus our analysis of the supersymmetry algebra and the BPS bound is exact.

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A The Bjorken-Johnson-Low method

We summarize the essence of the BJL method [16]. We start with the correlation function in the path integral approach

\[ \langle T^* A(x) B(y) \rangle \]  \hspace{1cm} (A.1)

where the \( T^* \) product is defined for all the space-time points except for \( x^0 = y^0 \). In the path integral approach one has

\[ \partial_\mu \langle T^* A(x) B(y) \rangle = \langle T^* \partial_\mu A(x) B(y) \rangle. \]  \hspace{1cm} (A.2)

Of course this does not hold in general for the canonical \( T \) product.

The basic observation of the BJL prescription is that the Fourier transform

\[ \int dx e^{ik(x-y)} \langle T^* A(x) B(y) \rangle \]  \hspace{1cm} (A.3)

differs from

\[ \int dx e^{ik(x-y)} \langle TA(x) B(y) \rangle \]  \hspace{1cm} (A.4)

at most by polynomials in \( k_0 \). We thus fix this difference by the following condition: If

\[ \lim_{k_0 \to \infty} \int dx e^{ik(x-y)} \langle T^* A(x) B(y) \rangle = 0 \]  \hspace{1cm} (A.5)

then one can set

\[ \int dx e^{ik(x-y)} \langle TA(x) B(y) \rangle = \int dx e^{ik(x-y)} \langle T^* A(x) B(y) \rangle. \]  \hspace{1cm} (A.6)

On the other hand, if (A.5) does not vanish, one defines (A.4) by subtracting (A.5)

\[ \int dx e^{ik(x-y)} \langle TA(x) B(y) \rangle = \int dx e^{ik(x-y)} \langle T^* A(x) B(y) \rangle \]

\[ - \lim_{k_0 \to \infty} \int dx e^{ik(x-y)} \langle T^* A(x) B(y) \rangle \]  \hspace{1cm} (A.7)

By construction one has then always the condition

\[ \lim_{k_0 \to \infty} \int dx e^{ik(x-y)} \langle TA(x) B(y) \rangle = 0 \]  \hspace{1cm} (A.8)

which is a basic property of the \( T \) product. The physical picture behind this construction is that the expectation value with the \( T \)-product is a smooth function of \( x - y \) near \( x^0 \sim y^0 \). Then the Fourier transform is also smooth, and tends to zero for large \( k_0 \).

We illustrate this procedure by applying it to the quantization of the free massive Wess-Zumino model in \( d = 2 \)

\[ \mathcal{L}(x) = \frac{1}{2} [ FF - \partial^\mu \varphi \partial_\mu \varphi - \bar{\psi} \gamma^\mu \partial_\mu \psi + 2 m F \varphi - m \bar{\psi} \psi]. \]  \hspace{1cm} (A.9)
The path integral gives the correlation functions

\[
\langle T^* \varphi(x) \varphi(y) \rangle = \frac{i}{\partial_\mu \partial^\mu - m^2} \delta(x - y),
\]

\[
\langle T^* \bar{\psi}(x) \bar{\psi}(y) \rangle = \frac{-i}{\bar{\vartheta} + m} \delta(x - y),
\]

\[
\langle T^* \varphi(x) F(y) \rangle = \frac{-im}{\partial_\mu \partial^\mu - m^2} \delta(x - y),
\]

\[
\langle T^* F(x) F(y) \rangle = i[1 + \frac{m^2}{\partial_\mu \partial^\mu - m^2}] \delta(x - y) \quad \text{(A.10)}
\]

Since (A.5) is satisfied for the \(\varphi\varphi\) propagator, we have

\[
\int dx e^{ik(x-y)} \langle T \varphi(x) \varphi(y) \rangle = \frac{-i}{k^2 + m^2 - i\epsilon} \quad \text{(A.11)}
\]

From this result we can derive the equal time canonical commutation relations. First of all, consider the commutator \([\varphi(x), \varphi(y)] = 0\) at equal time. One has, using (A.11),

\[
0 = \lim_{k_0 \to \infty} k_0 \int dx e^{ik(x-y)} \langle T \varphi(x) \varphi(y) \rangle
\]

\[
= \lim_{k_0 \to \infty} (i) \int dx e^{ik(x-y)} \frac{\partial}{\partial x^0} \langle T \varphi(x) \varphi(y) \rangle
\]

\[
= \lim_{k_0 \to \infty} (i) \left\{ \int dx e^{ik(x-y)} \langle [\varphi(x), \varphi(y)] \rangle \delta(x^0 - y^0)
\]

\[
+ \int dx e^{ik(x-y)} \langle T \frac{\partial}{\partial x^0} \varphi(x) \varphi(y) \rangle \right\}
\]

\[
= \lim_{k_0 \to \infty} (i) \int dx e^{ik(x-y)} \langle [\varphi(x), \varphi(y)] \rangle \delta(x^0 - y^0). \quad \text{(A.12)}
\]

We used in the last step that the \(T\) product vanishes in the limit \(k_0 \to \infty\). Because the last line is \(k_0\)-independent due to \(\delta(x^0 - y^0)\) we have

\[
\langle [\varphi(x), \varphi(y)] \rangle \delta(x^0 - y^0) = 0 \quad \text{(A.13)}
\]

Using this result in (A.12) without taking the limit \(k_0 \to \infty\) yields

\[
i \int dx e^{ik(x-y)} \langle T \frac{\partial}{\partial x^0} \varphi(x) \varphi(y) \rangle = \frac{-ik_0}{k^2 + m^2 - i\epsilon}. \quad \text{(A.14)}
\]

By multiplying the expression in (A.14) by \(k_0\) and performing the same analysis as before, we obtain

\[
- \int dx e^{ik(x-y)} \langle \frac{\partial}{\partial x^0} \varphi(x), \varphi(y) \rangle \delta(x^0 - y^0)
\]

\[
= \lim_{k_0 \to \infty} \frac{-ik_0^2}{k^2 + m^2 - i\epsilon} = i \quad \text{(A.15)}
\]
Since again this result is $k_0$-independent, we have a second identity

$$
\int dx e^{ik(x-y)} \langle T \partial_0^2 \varphi(x) \varphi(y) \rangle = \frac{-i(k_1^2 + m^2)}{k^2 + m^2 - i\epsilon} \quad (A.16)
$$

This can be rewritten as follows by bringing the right-hand side to the left-hand side

$$
\int dx e^{ik(x-y)} \langle T(\partial_\mu \partial^\mu + m^2) \varphi(x) \varphi(y) \rangle = 0 \quad (A.17)
$$

which is consistent with the field equation $(\partial_\mu \partial^\mu + m^2) \varphi(x) = 0$.

Summarizing, we have proven the following commutation relations

$$
[\frac{\partial}{\partial x^0} \varphi(x), \varphi(y)] \delta(x^0 - y^0) = -i\delta^2(x - y),
$$

$$
[\varphi(x), \varphi(y)] \delta(x^0 - y^0) = 0. \quad (A.18)
$$

The analysis of $\langle T^* \varphi(x) F(y) \rangle$ is essentially the same and we obtain

$$
[\frac{\partial}{\partial x^0} \varphi(x), F(y)] \delta(x^0 - y^0) = im\delta^2(x - y),
$$

$$
[F(x), F(y)] \delta(x^0 - y^0) = 0. \quad (A.19)
$$

These relations are consistent with the field equation $F(x) + m\varphi(x) = 0$.

As for $\langle T^* F(x) F(y) \rangle$, we have

$$
\int dx e^{ik(x-y)} \langle T^* F(x) F(y) \rangle = i[1 - \frac{m^2}{k^2 + m^2 - i\epsilon}] \quad (A.20)
$$

and thus the BJL method gives according to (A.7)

$$
\int dx e^{ik(x-y)} \langle TF(x) F(y) \rangle = -i\frac{m^2}{k^2 + m^2 - i\epsilon} \quad (A.21)
$$

Using the same steps as before, one finds

$$
[\frac{\partial}{\partial x^0} F(x), F(y)] \delta(x^0 - y^0) = -im^2\delta^2(x - y),
$$

$$
[F(x), F(y)] \delta(x^0 - y^0) = 0. \quad (A.22)
$$

Also, these quantization conditions are consistent with the field equation $F = -m\varphi(x)$.

We now come to the fermionic sector

$$
\int dx e^{ik(x-y)} \langle T^* \psi(x) \bar{\psi}(y) \rangle = \frac{-i}{-i \frac{k}{k + m}} \quad (A.23)
$$

and thus

$$
\int dx e^{ik(x-y)} \langle T \psi(x) \bar{\psi}(y) \rangle = \frac{-i}{-i \frac{k}{k + m}} = -i \frac{k}{k^2 + m^2} \quad (A.24)
$$
Proceeding as before we obtain

\[
\lim_{k_0 \to \infty} k_0 \int dx e^{ik(x-y)} \langle T \psi(x) \psi(y) \rangle = i \lim_{k_0 \to \infty} \int dx e^{ik(x-y)} \frac{\partial}{\partial x^0} \langle T \psi(x) \psi(y) \rangle
\]

\[
= i \lim_{k_0 \to \infty} \int dx e^{ik(x-y)} \{(\{\psi(x), \bar{\psi}(y)\}) \delta(x^0 - y^0) + \langle T \frac{\partial}{\partial x^0} \psi(x) \bar{\psi}(y) \rangle \}
\]

\[
= \lim_{k_0 \to \infty} -i k_0 \frac{i k + m}{k^2 + m^2} = -\gamma^0
\]

We thus obtain the quantization condition

\[
\{\psi(x), \bar{\psi}(y)\} \delta(x^0 - y^0) = i \gamma^0 \delta(x-y)
\]

(A.26)

and

\[
\int dx e^{ik(x-y)} \langle T (\not{\partial} + m) \psi(x) \bar{\psi}(y) \rangle = 0
\]

(A.27)

which is consistent with the field equation \((\not{\partial} + m) \psi(x) = 0\). Note that (A.26) gives the same result as obtained from Dirac quantization, where the conjugate momentum of \(\psi\) equals \(-\frac{i}{2} \bar{\psi}\), but the Dirac bracket removes this factor 1/2.

When one applies the BJL prescription to perturbation theory, one should define the correlation functions in terms of unrenormalized (bare) fields. (Recall that the canonical formalism is phrased in terms of Heisenberg fields which are unrenormalized.) Since the canonical equal-time commutator is based on the basic assumption that it is not modified by interactions, one should also use bare fields in perturbation theory, with a momentum cut-off in the loop, in order that one reproduces by the BJL approach the results of the canonical formalism. However, one can incorporate the effects of anomalies naturally into the BJL analysis.

We illustrate the BJL analysis in the presence of anomalies, which goes beyond the naive canonical formulation, by taking the identity (5.13) as an example (although the anomaly in this case does not have a physical significance for rigid supersymmetry). We started in (5.13) from

\[
-i \partial_\mu \langle T^* J^{\mu,\alpha}(x) \phi(y, \theta) \rangle + \langle \delta_{\text{susy}} \phi(y, \theta) \rangle = 0
\]

(A.28)

where

\[
J^{\mu} = j^{\mu} - \frac{\hbar g}{2\pi} \gamma^\mu \psi(x)
\]

(A.29)

and \(\delta_{\text{susy}} \phi(y, \theta) = \delta^2(x-y)Q^\alpha \phi(x, \theta)\). The BJL subtraction procedure leads to the result that the derivative is outside the T-product: One writes down all Feynman graphs with \(j^{\mu}\) and \(-\frac{\hbar g}{2\pi} \gamma^\mu \psi(x)\), uses a cut-off \(\Lambda\) in each loop and takes the limit \(k_0 \to \infty\). The result is

\[
-i \partial_\mu \langle T J^{\mu,\alpha}(x) \phi(y, \theta) \rangle + \langle \delta^2(x-y)Q^\alpha \phi(x, \theta) \rangle = 0.
\]

(A.30)

This implies

\[
i[J^{0,\alpha}(x), \phi(y, \theta)] \delta(x^0 - y^0) = \delta^2(x-y)[\frac{\partial}{\partial \theta^\alpha} - (\gamma^\mu \theta)^\alpha \partial_\mu] \phi(x, \theta
\]

(A.31)
and \( \langle T \partial_\mu J^{\mu,\alpha}(x) \phi(y, \theta) \rangle = 0 \) which in turn implies the current conservation \( \partial_\mu J^{\mu,\alpha}(x) = 0 \).

If one combines this commutation relation with the anomaly relation (A.29), one obtains the anomalous commutation relation for the current \( j^\mu \)

\[
i [j^{0,\alpha}(x), \phi(y, \theta)] \delta(x^0 - y^0) = i \frac{\hbar g}{2\pi} \langle \gamma^0 \psi(x) \rangle^\alpha, \phi(y, \theta) \rangle \delta(x^0 - y^0) \\
+ \delta^2(x - y) \left[ \frac{\partial}{\partial \theta^\alpha} - (\gamma^\mu \theta) \partial_\mu \right] \phi(x, \theta). \tag{A.32}
\]

The term with \( \hbar g \) is the operator counterpart of the non-trivial Jacobian in the path integral formulation.

A special property of the anomaly (unlike the induced effect such as the anomalous magnetic moment in QED) is that we have a well-defined (local) operator relation expressed by (A.29), which is established in some cases up to all orders in perturbation theory, and that the anomaly itself is finite and independent of the cut-off parameter. In the above BJL analysis of (A.32), we regarded both of (A.29) and (A.31) as exact (bare) operator relations (though we worked only to the one-loop accuracy in the present paper).

References


