Casimir densities for parallel plate in the Domain Wall background

M.R. Setare *
P. O. Box 19395-5531, Tehran, IRAN
and
Department of Science, Physics group, Kordestan University, Sanandeg, Iran
and
Department of Physics, Sharif University of Technology, Tehran, Iran

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Abstract

The Casimir forces on two parallel plates in conformally flat domain wall background due to conformally coupled massless scalar field satisfying mixed boundary conditions on the plates is investigated. In the general case of mixed boundary conditions formulae are derived for the vacuum expectation values of the energy-momentum tensor and vacuum forces acting on boundaries.

*E-mail: rezakord@yahoo.com
1 Introduction

The Casimir effect is one of the most interesting manifestations of nontrivial properties of the vacuum state in quantum field theory [1,2]. Since its first prediction by Casimir in 1948[2] this effect has been investigated for different fields having different boundary geometries[3-7]. The Casimir effect can be viewed as the polarization of vacuum by boundary conditions or geometry. Therefore, vacuum polarization induced by a gravitational field is also considered as Casimir effect.

Casimir effect may have interesting implications for the early universe, in [8] the Casimir effect of a massless scalar field with Dirichlet boundary condition in spherical shell having different vacua inside and outside which represents a bubble in early universe with false/true vacuum inside/outside have been investigated. The Casimir stress on two concentric spherical shell with constant comoving radius having different vacua inside and outside in de Sitter space which is corresponding to a spherical symmetry domain wall with thickness have been calculated in [9].

In the context of hot big bang cosmology, the unified theories of the fundamental interactions predict that the universe passes through a sequence of phase transitions. Different types of topological objects may have been formed during these phase transitions, these include domain walls, cosmic strings and monopoles [10, 11, 12]. These topological defects appear as a consequence of breakdown of local or global gauge symmetries of a system composed by self-coupling iso-scalar Higgs fields \( \Phi^a \).

It has been shown in [13] and [14], that the gravitational field of the vacuum domain wall with a source of the form

\[
T^{\nu}_{\mu} = \sigma \delta(x) \text{diag}(1, 0, 1, 1)
\]  

(1)

does not correspond to any exact static solution of Einstein equations (on domain wall solutions of Einstein-scalar-field equations see [15]). However the static solutions can be constructed in presence of an additional background energy-momentum tensor. Such a type solution has been fond in [16].

In the present paper we will investigate the vacuum expectation values of the energy–momentum tensor of the conformally coupled scalar field on background of the conformally flat domain wall geometries. We will consider the general plane–symmetric solutions of the gravitational field equations and boundary conditions of the Robin type on the plates. The latter includes the Dirichlet and Neumann boundary conditions as special cases. The Casimir energy-momentum tensor for these geometries can be generated from the corresponding flat spacetime results by using the standard transformation formula. Previously this method has been used in [17, 18] to derive the vacuum characteristics of the Casimir configuration on background of the static domain wall geometry for a scalar field with Dirichlet boundary condition on plates. Also this method has been used in [19] to derive the vacuum characteristics of the Casimir configuration on background of conformally flat brane-world geometries for massless scalar field with Robin boundary condition on plates. For Neumann or more general mixed boundary conditions we need to have the Casimir energy-momentum tensor for the flat spacetime counterpart in the case of the Robin boundary conditions with coefficients related to the metric components of the domain wall geometry. The Casimir effect for the general Robin boundary conditions on background of the Minkowski spacetime was investigated in Ref. [20] for flat boundaries, here we use the results of Ref. [20] to generate vacuum energy–momentum tensor for the conformally flat background.
2 Vacuum expectation values for the energy-momentum tensor

In this paper we shall consider the conformally coupled real scalar field \( \phi \), which satisfies

\[
(\Box + \frac{1}{6} R)\phi = 0, \quad \Box = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu),
\]

and propagates on background of gravitational field generated by domain wall solution from [13]. In [13] Vilenkin has found the gravitational of domain walls in the linear approximation of general relativity. At first we review this work briefly. In linear approximation the metric is given by

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},
\]

where \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \), and \(|h_{\mu\nu}| \ll 1\), Einstein equations are as following

\[
(\nabla^2 - \partial_t^2) h_{\mu\nu} = 16\pi G (T_{\mu\nu} - 1/2 \eta_{\mu\nu} T),
\]

with the harmonic coordinate conditions

\[
\partial_\nu (h_\nu^\nu - 1/2 \delta_\nu^\nu h) = 0.
\]

The remaining coordinate freedom is restricted to the transformations

\[
h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}.
\]

where

\[
(\nabla^2 - \partial_t^2) \xi_\mu = 0.
\]

The solution of Eqs. (4) and (5) for a vacuum domain wall with energy-momentum tensor given by Eq.(1) (For a vacuum domain wall we have \(-p = \sigma\) where \(-p\) is the surface tension and \(\sigma\) is the surface energy density) is easily found where

\[
h_{00} = -h_{22} = -h_{33} = -4\pi G \sigma |x|, \quad h_{11} = 12\pi G \sigma |x|.
\]

If we consider the coordinate transformation Eq.(6) with \(\xi_1 = 2\pi G \sigma x^2 \text{sgn} x\) and \(\xi_2 = \xi_3 = \xi_0 = 0\) brings the metric Eq.(8) to a conformally flat form

\[
ds^2 = (1 - 4\pi G \sigma |x|)(dt^2 - dx^2 - dy^2 - dz^2).
\]

In Eq. (2) \(R\) is the Ricci scalar for the metric \(g_{\mu\nu}\). Note that for the metric tensor from Eq. (9) one has

\[
R = -4\pi G \sigma \frac{-(\text{sgn}(x))^2 + 4(\text{sgn}(x))^3 \pi G \sigma |x| - 2\pi G \sigma (\text{sgn}(x))^4}{(1 - 4\pi G \sigma |x|)^3},
\]

In what follows as a boundary configuration we shall consider two plates parallel to each other and to domain wall, with \(x\) coordinates equal to \(x_1\) and \(x_2\) (to be definit we shall consider right half space of domain wall geometry \(x_1, x_2 > 0\) ). We will assume that the field satisfies the mixed boundary condition

\[
(a_j + b_j n^\mu \nabla_\mu) \varphi(x) = 0, \quad x = x_j, \quad j = 1, 2.
\]
on the plate $x = x_1$ and $x = x_2$, $x_1 < x_2$, $n^\mu$ is the normal to these surfaces, $n_\mu n^\mu = -1$, and $a_j$, $b_j$ are constants. The results in the following will depend on the ratio of these coefficients only. However, to keep the transition to the Dirichlet and Neumann cases transparent we will use the form (11). For the case of plane boundaries under consideration introducing a new coordinate $u$ in accordance with

\[ du = \sqrt{1 - 4\pi G \sigma |x|} dx \]

conditions (11) take the form

\[
\left( a_j + (-1)^{j-1} b_j \frac{1}{\sqrt{1 - 4\pi G \sigma |x|}} \partial_x \right) \varphi(x) = 0, \quad u = u_j, \quad j = 1, 2.
\]

Note that the Dirichlet and Neumann boundary conditions are obtained from Eq. (11) as special cases corresponding to $(a_j, b_j) = (1, 0)$ and $(a_j, b_j) = (0, 1)$ respectively. The Robin boundary condition may be interpreted as the boundary condition on a thick plate [21]. Rewriting Eq.(13) in the following form

\[
\varphi(x) = -(-1)^{j-1} \frac{b_j}{a_j} \partial u \varphi(x),
\]

where $\frac{a_j}{b_j}$, having the dimension of a length, may be called skin-depth parameter. This is similar to the case of penetration of an electromagnetic field into a real metal, where the tangential component of the electric field is proportional to the skin-depth parameter.

Our main interest in the present paper is to investigate the vacuum expectation values (VEV’s) of the energy–momentum tensor for the field $\varphi(x)$ in the region $x_1 < x < x_2$. The presence of boundaries modifies the spectrum of the zero-point fluctuations compared to the case without boundaries. This results in the shift in the VEV’s of the physical quantities, such as vacuum energy density and stresses. This is the well known Casimir effect. It can be shown that for a conformally coupled scalar by using field equation (2) the expression for the energy–momentum tensor can be presented in the form

\[
T_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{6} \left[ \frac{4 \sigma}{x} \nabla_\rho \nabla^\rho + \nabla_\mu \nabla_\nu + R_{\mu\nu} \right] \varphi^2,
\]

where $R_{\mu\nu}$ is the Ricci tensor. The quantization of a scalar filed on background of metric Eq.(9) is standard. Let $\{ \varphi_\alpha(x), \varphi_\alpha^*(x) \}$ be a complete set of orthonormalized positive and negative frequency solutions to the field equation (2), obeying boundary condition (11). By expanding the field operator over these eigenfunctions, using the standard commutation rules and the definition of the vacuum state for the vacuum expectation values of the energy-momentum tensor one obtains

\[
\langle 0 | T_{\mu\nu}(x) | 0 \rangle = \sum_\alpha T_{\mu\nu} \{ \varphi_\alpha, \varphi_\alpha^* \},
\]

where $| 0 \rangle$ is the amplitude for the corresponding vacuum state, and the bilinear form $T_{\mu\nu} \{ \varphi, \psi \}$ on the right is determined by the classical energy-momentum tensor (15). In the problem under consideration we have a conformally trivial situation: conformally
invariant field on background of the conformally flat spacetime. Instead of evaluating Eq. (16) directly on background of the curved metric, the vacuum expectation values can be obtained from the corresponding flat spacetime results for a scalar field $\bar{\phi}$ by using the conformal properties of the problem under consideration. Under the conformal transformation $g_{\mu \nu} = \Omega^2 \eta_{\mu \nu}$ the $\bar{\phi}$ field will change by the rule

$$\varphi(x) = \Omega^{-1}\bar{\varphi}(x),$$

where for metric Eq.(9) the conformal factor is given by $\Omega = \sqrt{1 - 4\pi G\sigma|x|}$. The boundary conditions for the field $\bar{\varphi}(x)$ we will write in form similar to Eq. (13)

$$\left(\bar{a}_j + (-1)^j \bar{b}_j \partial_x\right) \bar{\varphi} = 0, \quad x = x_j, \quad j = 1, 2,$$

with constant Robin coefficients $\bar{a}_j$ and $\bar{b}_j$. Comparing to the boundary conditions (11) and taking into account transformation rule (17) we obtain the following relations between the corresponding Robin coefficients

$$\bar{a}_j = a_j + (-1)^j 2\pi G\sigma \text{sgn}(x) \frac{b_j}{1 - 4\pi G\sigma|x|^{3/2}}, \quad \bar{b}_j = b_j \frac{1}{\sqrt{1 - 4\pi G\sigma|x|}}$$

Note that as Dirichlet boundary conditions are conformally invariant the Dirichlet scalar in the curved bulk corresponds to the Dirichlet scalar in a flat spacetime. However, for the case of Neumann scalar the flat spacetime counterpart is a Robin scalar with

$$\bar{a}_j = (-1)^j \frac{2\pi G\sigma \text{sgn}(x)}{1 - 4\pi G\sigma|x|}, \quad \bar{b}_j = 1$$

The Casimir effect with boundary conditions (18) on two parallel plates on background of the Minkowski spacetime is investigated in Ref. [20] for a scalar field with a general conformal coupling parameter. In the case of a conformally coupled scalar the corresponding regularized VEV’s for the energy-momentum tensor are uniform in the region between the plates and have the form

$$\langle \tilde{T}_\mu^\nu [\eta_{\alpha \beta}] \rangle_{\text{ren}} = -\frac{J_3(B_1, B_2)}{8\pi^{3/2}a^4\Gamma(5/2)} \text{diag}(1, 1, 1, -3), \quad x_1 < x < x_2,$$

where

$$J_3(B_1, B_2) = \text{p.v.} \int_0^\infty \frac{t^3 dt}{(B_1 t)^{B_1+1}(B_2 t)^{B_2+1}},$$

and we use the notations

$$B_j = \frac{\bar{b}_j}{\bar{a}_ja}, \quad j = 1, 2, \quad a = x_2 - x_1.$$

For the Dirichlet scalar $B_1 = B_2 = 0$ and one has $J_D(0, 0) = 2^{-4}\Gamma(4)\zeta_R(4)$, with the Riemann zeta function $\zeta_R(s)$. Note that in the regions $x < x_1$ and $x > x_2$ the Casimir densities vanish [17, 20]:

$$\langle \tilde{T}_\mu^\nu [\eta_{\alpha \beta}] \rangle_{\text{ren}} = 0, \quad x < x_1, x > x_2.$$
This can be also obtained directly from Eq. (21) taking the limits \( x_1 \to -\infty \) or \( x_2 \to +\infty \). The values of the coefficients \( B_1 \) and \( B_2 \) for which the denominator in the subintegrand of Eq. (21) has zeros are specified in [20]. The vacuum energy-momentum tensor on domain wall background Eq.(9) is obtained by the standard transformation law between conformally related problems (see, for instance, [22]) and has the form

\[
\langle T_\nu^\mu [g_{\alpha\beta}] \rangle_{\text{ren}}^{(b)} = \langle T_\nu^\mu [g_{\alpha\beta}] \rangle_{\text{ren}}^{(0)} + \langle T_\nu^\mu [g_{\alpha\beta}] \rangle_{\text{ren}}^{(b)}.
\] (25)

Here the first term on the right is the vacuum energy–momentum tensor for the situation without boundaries (gravitational part), and the second one is due to the presence of boundaries. As the quantum field is conformally coupled and the background spacetime is conformally flat the gravitational part of the energy–momentum tensor is completely determined by the trace anomaly and is related to the divergent part of the corresponding effective action by the relation [22]

\[
\langle T_\nu^\mu [g_{\alpha\beta}] \rangle_{\text{ren}}^{(0)} = 2g^{\mu\sigma}(x) \frac{\delta}{\delta g^{\mu\sigma}(x)} W_{\text{div}}[g_{\alpha\beta}].
\] (26)

The boundary part in Eq. (25) is related to the corresponding flat spacetime counterpart (21),(24) by the relation [22]

\[
\langle T_\nu^\mu [g_{\alpha\beta}] \rangle_{\text{ren}}^{(b)} = \frac{1}{\sqrt{|g|}} \langle T_\nu^\mu [\eta_{\alpha\beta}] \rangle_{\text{ren}}.
\] (27)

By taking into account Eq. (21) from here we obtain

\[
\langle T_\nu^\mu [g_{\alpha\beta}] \rangle_{\text{ren}}^{(b)} = -\frac{J_3(B_1, B_2)}{8\pi^{3/2}a^4 \Gamma(5/2)(1 - 4\pi G\sigma|x|)^2} \text{diag}(1, 1, 1, -3),
\] (28)

for \( x_1 < x < x_2 \), and

\[
\langle T_\nu^\mu [g_{\alpha\beta}] \rangle_{\text{ren}}^{(b)} = 0, \quad \text{for} \quad x < x_1, x > x_2.
\] (29)

In Eq. (28) the constants \( B_j \) are related to the Robin coefficients in boundary condition (11) by formulae (23),(19) and are functions on \( x_j \). In particular, for Neumann boundary conditions \( B_j^{(N)} = \frac{(-1)^{j-1}}{a} \frac{1 - 4\pi G\sigma|x_j|}{2\pi G\sigma \text{sgn}(x_j)} \).

The total bulk vacuum energy per unit physical surface on the plates at \( x = x_j \) is obtained by integrating over the region between the plates

\[
E_j^{(b)} = (1 - 4\pi G\sigma|x_j|)^{-3/2} \int_{x_1}^{x_2} \langle T_0^0 \rangle_{\text{ren}}^{(b)}(1 - 4\pi G\sigma|x|)^{-3/2} dx = -\frac{J_3(B_1, B_2)(1 - 4\pi G\sigma|x_j|)^{-3/2}}{8\pi^{3/2} \Gamma(5/2)a^3}.
\] (30)

The resulting vacuum force per unit boundary area acting on the boundary at \( x = x_j \) is determined by the difference

\[
\langle T_3^3 [g_{\alpha\beta}] \rangle_{\text{ren}}|_{x=x_j+0} - \langle T_3^3 [g_{\alpha\beta}] \rangle_{\text{ren}}|_{x=x_j-0}.
\] (31)

The first term in Eq.(19) is the vacuum polarization due to the gravitational field, without any boundary conditions, which can be rewritten as following

\[
\langle T_\nu^\mu [g_{\alpha\beta}] \rangle_{\text{ren}}^{(0)} = -\frac{1}{2880} \left[ \frac{1}{6} \bar{H}_\nu^{(1)} - \bar{H}_\nu^{(3)} \right].
\] (32)
The functions $H^{(1,3)\mu}_\nu$ are some combinations of curvature tensor components (see [22]). Now we see that as gravitational part (32) is a continuous function on $x$ it does not contribute to the forces acting on the boundaries and the vacuum force per unit surface acting on the boundary at $x = x_j$ is determined by the boundary part of the vacuum pressure, $p_b = -\langle T^3_{3}[g_{\alpha\beta}]\rangle_{\text{ren}}$, taken at the point $x = x_j$:

$$p_b(x_1, x_2) = -\frac{(1 - 4\pi G\sigma|x_j|)^2 J_3(B_1, B_2)}{4\pi^{3/2}a^4\Gamma(3/2)}.$$  (33)

This corresponds to the attractive/repulsive force between the plates if $p_{bj} < 0$. The equilibrium points for the plates correspond to the zero values of Eq. (33): $p_{bj} = 0$. These points are zeros of the function $J_3(B_1, B_2)$ defined by Eq. (22) and are the same for both plates. Hence, we have an example for the stabilization of the distance between the plates due to the vacuum forces (To see an application to the Randall-Sundrum brane-world model refer to [19]). Note that at these points the VEV’s of the bulk energy-momentum tensor given by Eq. (28) and the total bulk energy also vanish.

3 Conclusion

The study of vacuum quantum effect on the background of topological defects such as domain walls, cosmic strings, or magnetic monopoles have been considered in many references [23, 24, 25, 26, 27, 28, 29, 30, 31]. In the present paper we have investigated the Casimir effect for a conformally coupled massless scalar field confined in the region between two parallel plate on background of the conformally-flat domain wall. The general case of the mixed(Robin) boundary conditions is considered. The vacuum expectation values of the energy-momentum tensor are derived from the corresponding flat spacetime results by using the conformal properties of the problem. This method has been used in [19] to derive the vacuum characteristics of the Casimir configuration on background of conformally flat brane-world geometries for massless scalar field with Robin boundary condition on plates, then as an application of the general formulae, there we have considered the important special case of the AdS$_5$ bulk, in odd spacetime dimensions the conformal anomaly is absent and the corresponding gravitational vacuum polarization vanishes. In the present paper, spacetime have even dimension, the vacuum polarization due to the gravitational field, without any boundary conditions is not zero and given by Eq.(32), the corresponding gravitational pressure part has the same from both sides of the plates, and hence leads to zero effective force. In the region between the plates the boundary induced part for the vacuum energy-momentum tensor is given by Eq.(28), and the corresponding vacuum forces acting per unit surface of the plates have the form Eq. (33). These forces vanish at the zeros of the function $J_3(B_1, B_2)$.

References


