PERTURBATION THEORY FOR LYAPUNOV EXPONENTS
OF A TORAL MAP: EXTENSION OF A RESULT OF
SHUB AND WILKINSON

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PERTURBATION THEORY FOR LYAPUNOV EXONENTS OF A TORAL MAP: EXTENSION OF A RESULT OF SHUB AND WILKINSON.

by David Ruelle*.

Abstract. Starting from a hyperbolic toral automorphism times a rotation of the circle, we obtain, for a small volume preserving perturbation, an exact and rigorous second order perturbation expansion of the Lyapunov exponents.

Keywords: Lyapunov exponent, toral automorphism, hyperbolicity.

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We consider volume preserving perturbations $F$ of a diffeomorphism $F_0 = (\Phi,J)$ of $T^{m+1} = T^m \times T$, where $\Phi$ is a hyperbolic automorphism of $T^m$, and $J$ is a translation of $T$. Writing $F = F_0 + aF'$, we shall show that the Lyapunov exponents for $(F, \text{volume})$ can be expanded to second order in $a$ (Theorem 1). In particular, the central Lyapunov exponent $\lambda^c$ of $(F, \text{volume})$, to second order in $a$, is generally $\neq 0$ (Corollary 11). For a special family of perturbations one obtains particularly simple formulae, first noted by Shub and Wilkinson [17]. We recover their result in Theorem 12. We deviate from [17] mostly in that we don’t have differentiability of $\lambda^c$, only a second order expansion around $a = 0$. The ideas used here are largely those in Shub and Wilkinson [17], and can be appreciated in the background provided by Hirsh, Pugh and Shub [9], Burns and Wilkinson [5], Ruelle and Wilkinson [16], Niteica and Török [12], Pugh, Shub and Wilkinson [14]. Among older regularity results let us mention Katok, Knieper and Weiss [11], Flaminio [9], Ruelle [15]. For recent work concerning Lyapunov exponents, see Bonatti, Gómez-Mont and Viana [3], Avila and Bochi [2]. Closely related to the subject of the present paper are the references [4] and [6].

After completing the writing of this paper, the author received a preprint by D. Dolgopyat [7], which develops similar ideas in a more general setting, but without the specific formulas we obtain here.

1. Theorem.

Let $\Phi$ be a hyperbolic automorphism of $T^m$, and $J : y \mapsto y + \alpha \pmod{1}$ a translation of $T$. Define $F_0 = (\Phi,J)$, and let $F = F_0 + aF'(\text{+ higher order in } a)$ be a $C^2$ perturbation of $F_0$, volume preserving to first order in $a$. (We take $F' : T^{m+1} \to R^{m+1}$ and $F_0 \xi + aF'(\xi)$ has to be understood $\pmod{1}$ in each component). Let $\lambda_1 < \lambda_2 < \ldots$ be the Lyapunov exponents of $(F_0, \text{volume})$ and $m_1, m_2, \ldots$ their multiplicities (the exponent $= 0$ occurs with multiplicity 1). Also let $\lambda^{(1)}_a \leq \lambda^{(2)}_a \leq \ldots$ be the Lyapunov exponents of $(F, \text{volume})$ repeated according to multiplicity. Then we have the second order expansion

$$\sum_{\ell = m_1 + \ldots + m_{r-1} + 1}^{m_1 + \ldots + m_r} \lambda^{(\ell)}_a = m_r \lambda_r + a^2 L_r + o(a^2)$$

If $m_r = 1$, and writing $\lambda_r = \lambda^{(r)}_0$, we have

$$\lambda^{(\ell)}_a = \lambda^{(\ell)}_0 + a^2 L^{(\ell)} + o(a^2)$$

(this applies in particular to $\lambda^c = \lambda^{(0)}_0$ for $\lambda^{(0)}_0 = 0$).

An explicit expression for $L_r$ can be obtained (see Proposition 9). We do not assume ergodicity of $(F, \text{volume})$, and therefore we use integrated Lyapunov exponents (averaged over the volume), see however Remark 15(a).

Because the perturbation $+aF'(\text{+ higher order in } a)$ to $F_0$ gives only a quadratic contribution in the above formulas, the higher order terms do not contribute to order $a^2$. Since the higher order terms do not change our results, these terms will be omitted in what follows.
2. Normal hyperbolicity.

As in [17], we invoke the theory of normal hyperbolicity of [10]. We start from the fact that $F_0$ is normally hyperbolic to the smooth fibration of $\mathbb{T}^{m+1}$ by circles $\{x\} \times \mathbb{T}$. Taking some $k \geq 2$ we apply [10] Theorems (7.1), (7.2). Thus we obtain a $C^1$ neighborhood $U$ of $F_0$ in the $C^k$ diffeomorphisms of $\mathbb{T}^{m+1}$ such that, for $F \in U$, there is an equivariant fibration $\pi : \mathbb{T}^{m+1} \to \mathbb{T}^m$ with

$$\pi F = \Phi \pi$$

The fibers $\pi^{-1}\{x\}$ are $C^k$ circles forming a continuous fibration of $\mathbb{T}^{m+1}$ (this fibration is in general not smooth). Furthermore there is a $TF$-invariant continuous splitting of $TT^{m+1}$ into three subbundles:

$$TT^{m+1} = E^s + E^u + E^c$$

such that $E^c$ is 1-dimensional tangent to the circles $\pi^{-1}\{x\}$, $E^s$ is $m^s$-dimensional contracting and $E^u$ is $m^u$-dimensional expanding for $TF$.

If $\lambda_r < 0$ (and $F$ is in a suitable $C^1$-small neighborhood $U$ of $F_0$), we can introduce a continuous vector subbundle $E^r$ of $TT^{m+1}$ which consists of vectors contracting under $TF^n$ faster than $(\lambda_r + \epsilon)^n$ where $\epsilon > 0$ and $\lambda_r + \epsilon < \lambda_{r+1}$. In fact $E^r$ is a hyperbolic (attracting) fixed point for the action induced by $TF^{-1}$ on the bundle of $m_1 + \ldots + m_r$ dimensional linear subspaces of $TT^{m+1}$ (over $F^{-1}$ acting on $T^{m+1}$).

If $\lambda_r > 0$, replacement of $F$ by $F^{-1}$ similarly yields a continuous subbundle $\tilde{E}^r$ of $m_r + \ldots$ dimensional subspaces.

3. Proposition.

Assume that $F$ is of class $C^k$, $k \geq 2$, and that $F$ is $C^k$ close to $F_0$. The bundles $E^r$, $\tilde{E}^r$, when restricted to a circle $\pi^{-1}\{x\}$ are of class $C^{k-1}$, continuously in $x$.

If $G$ denotes the (Grassmannian) manifold of $m_1 + \ldots + m_r$ dimensional linear subspaces of $\mathbb{R}^{m+1}$, we may identify the bundle of $m_1 + \ldots + m_r$ dimensional linear subspaces of $TT^{m+1}$ with $T^{m+1} \times G$. We denote by $E \in G$ the spectral subspace of the matrix defining $\Phi$ corresponding to the smallest $m_1 + \ldots + m_r$ eigenvalues (in absolute value, and repeated according to multiplicity).

If $F_0$ is the action defined by $TF_0$ on $TT^{m+1} \times G$, the circles $\{x\} \times T \times \{E\}$ form an $F_0$ invariant fibration of $T^{m+1} \times \{E\}$, to which $F_0$ is normally hyperbolic. If $F$ is $C^k$ close to $F_0$, the corresponding $C^{k-1}$ action $F$ is normally hyperbolic to a perturbed fibration where $\{x\} \times T \times \{E\}$ is replaced by $E^r|\pi^{-1}\{x\}$. According to [10] Theorem 7.4, Corollary (8.3) and the following Remark 2, we find that the $C^{k-1}$ circle $E^r|\pi^{-1}\{x\} \subset T^{m+1} \times G$ depends continuously on $x \in T^{m+1}$. Similarly for $\tilde{E}$. $\square$

Note that in [17], the $C^r$ section theorem is used in a similar situation, giving estimates uniform in $x$. However, continuity in $x$ (not just uniformity) will be essential for us in what follows.

The splitting $TT^m+1 = E^s + E^u + E^c$ when restricted to a circle $\pi^{-1}\{x\}$ is of class $C^{k-1}$, continuously in $x$.

It is clear that $E^c|\pi^{-1}\{x\}$ is of class $C^{k-1}$ because it is the tangent bundle to the $C^k$ circle $\pi^{-1}\{x\}$. As to $E^s$, $E^u$, they are special cases of $E^r$, $\tilde{E}^r$. []

Notation.

Remember that $F = F_0 + aF'$, and fix $F'$. We shall use the notation $\pi_a$, $E^r_a$, ... to indicate the $a$-dependence of $\pi$, $E^r$, ...

5. Proposition.

For small $\epsilon > 0$ there is a continuous function $x \mapsto \gamma_x$ from $T^m$ to $C^k(T \times (-\epsilon, \epsilon) \to T^m)$ such that $\gamma_x(y, 0) = 0$ and $\pi^{-1}_a\{x\} = \{(x + \gamma_x(y, a), y) : y \in T\}$.

To see this define $\tilde{F} : T^{m+1} \times (-\epsilon, \epsilon) \to T^{m+1} \times (-\epsilon, \epsilon)$ by $\tilde{F}(\xi, a) = ((F_0 + aF')(\xi), a)$ and observe that $\tilde{F}$ is normally hyperbolic to the 2-dimensional manifolds

$$\cup_{a \in (-\epsilon, \epsilon)}(\pi^{-1}_a\{x\}, a)$$

and these are thus $C^k$ 2-dimensional submanifolds of $T^{m+1} \times (-\epsilon, \epsilon)$. []

We may in the same manner replace $\pi^{-1}_a\{x\}$ by $\cup_{a \in (-\epsilon, \epsilon)}(\pi^{-1}_a\{x\}, a)$ in Proposition 3 and Corollary 4. Writing $E_a$ for $E^r_a$, $\tilde{E}^r_a$, $E^s_a$, $E^u_a$, $E^c_a$, we obtain that $(\cdot, a) \mapsto E_a(\cdot)$, when restricted from $T^{m+1} \times (-\epsilon, \epsilon)$ to $\cup_{a \in (-\epsilon, \epsilon)}(\pi^{-1}_a\{x\}, a)$ is of class $C^{k-1}$. We rephrase this as follows:

6. Proposition.

The map

$$x \mapsto \{(y, a) \mapsto E_a(x + \gamma_x(y, a), y)\}$$

where $E_a$ stands for $E^r_a$, $\tilde{E}^r_a$, $E^s_a$, $E^u_a$, $E^c_a$, is continuous $T^m \to C^{k-1}(T \times (-\epsilon, \epsilon) \to \text{Grassmannian of } R^{m+1})$ where we have used the identification $TT^{m+1} = T^{m+1} \times R^{m+1}$. []

Notation.

From now on we write $E_a$ for $E^r_a$, $\tilde{E}^r_a$, $E^s_a$, $E^u_a$, $E^c_a$. When $a = 0$, $E_0$ is a trivial subbundle of $TT^{m+1} = T^{m+1} \times R^{m+1}$, and we shall write $E_0 = T^{m+1} \times \mathcal{E}$, denoting thus by $\mathcal{E}$ a spectral subspace of the matrix on $R^{m+1}$ defining $(\Phi, 1)$. We denote by $\mathcal{E}^\perp$ the complementary spectral subspace.

Taking $k = 2$ we have then:

7. Corollary.

There are linear maps $G(x, y), R(x, y, a) : \mathcal{E} \to \mathcal{E}^\perp$ such that $G(x, y)$ depends continuously on $(x, y) \in T^m \times T$, $R(x, y, a)$ on $(x, y, a) \in T^m \times T \times (-\epsilon, \epsilon)$,

$$E_a(x + \gamma_x(y, a), y) = \{X + aG(x, y)X + R(x, y, a)X : X \in \mathcal{E}\}$$

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and \( ||R(x, y, a)|| \) is \( o(a) \) uniformly in \( x, y \). \[ \]

Notice now that, if \( \tilde{x} = \pi_a(x, y) \), then \( x = \tilde{x} + \gamma(\tilde{x}, y, a) \), where \( \gamma(\tilde{x}, y, a) = O(a) \). Now

\[
E_a(x, y) = E_a(\tilde{x} + \gamma(\tilde{x}, y, a), y) = \{ X + aG(\tilde{x}, y)X + R(\tilde{x}, y, a)X : X \in \mathcal{E} \}
\]
differs from

\[
E_a(x + \gamma(x, y, a), y) = \{ X + aG(x, y)X + R(x, y, a)X : X \in \mathcal{E} \}
\]
by the replacement \( \tilde{x} \to x \) in the right-hand side, and since \( \operatorname{dist}(\tilde{x}, x) = O(a) \), we find that \( \operatorname{dist}(E_a(x, y), E_a(x + \gamma(x, y, a), y)) = o(a) \). Therefore, changing the definition of \( R \), we can again write:

8. Corollary.

There are linear maps \( G(x, y), R(x, y, a) : \mathcal{E} \to \mathcal{E}^\perp \), depending continuously on their arguments, such that

\[
E_a(x, y) = \{ X + aG(x, y)X + R(x, y, a)X : X \in \mathcal{E} \}
\]
and \( ||R(x, y, a)|| \) is \( o(a) \) uniformly in \( x, y \). \[ \]

We may write \( T\xi F = T\xi(F_0 + aF') = D_0 + aD'(\xi) \) where \( D_0 \) does not depend on \( \xi \) and preserves the decomposition \( T\xi M = \mathcal{E} + \mathcal{E}^\perp \). If we apply \( TF \) to an element \( X + aGX + RX \) of \( E_a \) (as in Corollary 8) we obtain \( X_1 + \) element of \( \mathcal{E}^\perp \in E_a \), with \( X_1 \in \mathcal{E} \):

\[
X_1 = D_0X + aD'X + a^2D'GX + aD'RX \quad \text{projected on } \mathcal{E}
\]

Under \((TF)^\wedge\), the volume element \( \theta \) in \( E_a(\xi) \) is multiplied by a factor \( M(\xi, a) \), and the projection in \( \mathcal{E} \) of \((TF)^\wedge\theta\) is equal to the projection in \( \mathcal{E} \) of \( \theta \) multiplied by a factor \( N(\xi, a) \) such that

\[
M(\xi, a) = N(\xi, a) + \ell_a(\xi) - \ell_a(F\xi)
\]
for suitable \( \ell_a \). We may compute \( N \) from (1):

\[
N(\xi, a) = N_{(0)} + aN_{(1)}(\xi) + a^2N_{(2)}(\xi) + o(a^2)
\]

To proceed we take now \( E_a = E_a^r \), and assume \( \lambda_r < 0 \). We have then, writing \( d\xi \) for the volume element in \( T^{m+1}_a \),

\[
L_a = \sum_{\ell=1}^{m_1 + \ldots + m_r} \lambda_\ell = \int d\xi \log M(\xi, a) = \int d\xi \log N(\xi, a)
\]

\[
= L_{(0)} + aL_{(1)}(\xi) + a^2L_{(2)}(\xi) + o(a^2)
\]

More precisely, we shall prove

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If \( \lambda_r < 0 \), we have

\[
\sum_{\ell=1}^{m_1+\ldots+m_r} \lambda_{a}^{(\ell)} = \sum_{k=1}^{r} m_k \lambda_k + a^2 L + o(a^2)
\]

where

\[
L = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}_E \left( D_0^{-1} D'(\xi) \right) \text{Tr}_E \left( D_0^{-1} D'(F_0^n \xi) \right) \geq 0
\]

and \( \text{Tr}_E \) is defined as follows. Let \( E \) be the spectral subspace of the matrix \( D_0 \) (defining \( (\Phi,1) \) in \( \mathbb{R}^{n+1} \)) corresponding to the smallest \( m_1 + \ldots + m_r \) eigenvalues (in absolute value, and repeated according to multiplicity). Also let \( E^\perp \) be the complementary spectral subspace. We define \( P \) to be the projection on \( E \) parallel to \( E^\perp \), and write \( \text{Tr}_E \ldots = \text{Tr}_{\mathbb{R}^{n+1}} P \ldots P \).

The convergence of the series defining \( L \) is exponential as will result from the proof. We postpone showing that \( L \geq 0 \) until Remark 15(b).

The proposition is obtained by comparing formula (2) with the formula (5) below, which we shall obtain by a second order perturbation calculation.

To first order in \( a \) we have

\[
F^n = (F_0 + aF')^n = F_0^n + a \sum_{j=1}^{n} F_0^{n-j} \circ F' \circ F_0^{j-1}
\]

hence

\[
T\xi F^n = D_0^n + a \sum_{j=1}^{n} D_0^{n-j} D'(F^{j-1}\xi) D_0^{j-1}
\]

If we apply \( TF^n \) to \( X + aGX + RX \in E_a \) we obtain \( X_n + \) element of \( E^\perp \in E_a \), with \( X_n \in E \).

To zero-th order in \( a \), \( X_n = D_0^n X \), so we may write to first order \( X_n = D_0^n X + aY_n(\xi) \).

Therefore, to first order in \( a \),

\[
D_0^n X + aY_n(\xi) + aG(F^n\xi)D_0^n X = D_0^n X + a \sum_{j=1}^{n} D_0^{n-j} D'(F^{j-1}\xi) D_0^{j-1} X + aD_0^n G(\xi) X
\]

and, taking the components along \( E^\perp \),

\[
G(F^n\xi)D_0^n X = \sum_{j=1}^{n} D_0^{n-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X + D_0^n G(\xi) X
\]

where \( D'_\perp(\cdot) \) is \( D'(\cdot) \) followed by taking the component along \( E^\perp \), or

\[
\sum_{j=1}^{n} D_0^{-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X + G(\xi) X = D_0^n G(F^n\xi)D_0^n X
\]
When \( n \to \infty \), the right-hand side tends to zero (exponentially fast, remember that \( X \in \mathcal{E} \), \( GX \in \mathcal{E}^\perp \)). Therefore (to order 0 in \( a \))

\[
G(\xi)X = -\sum_{j=1}^{\infty} D_0^{-j} D_0^{j-1} (F^{j-1} \xi) D_0^{j-1} X
\]

which we shall use in the form

\[
G(\xi)X = -\sum_{n=0}^{\infty} D_0^{-n-1} D_0^{n} (F^n \xi) D_0^{n} X
\]  \hspace{1cm} (3)

where we have written \( F^n_0 \) instead of \( F^n \) since \( G \) is evaluated to order 0 in \( a \). (The right-hand side is an exponentially convergent series).

Returning to (1) we see that, to second order in \( a \),

\[
X_1 = D_0 X + a D'(\xi)X + a^2 D'(\xi)G(\xi)X \quad \text{projected on } \mathcal{E}
\]

\[
= D_0 (1 + a D_0^{-1} D'(\xi) + a^2 D_0^{-1} D'(\xi)G(\xi))X \quad \text{projected on } \mathcal{E}
\]

Let now \((u^{(i)})\) and \((u^{(i)\perp})\) be conjugate bases of \( \mathcal{E} \). Also let \( \delta^{(i)} \) for \( i = 1, \ldots, m_1 + \ldots + m_r \) be the eigenvalues of \( D_0 \) restricted to \( \mathcal{E} \). Then, to second order in \( a \),

\[
N(\xi, a) \chi_1^{m_1 + \ldots + m_r} u^{(i)}
\]

is, up to a factor of absolute value 1,

\[
\left( \prod_{\ell=1}^{m_1 + \ldots + m_r} \delta^{(\ell)} \right) [1 + a \sum_{i=1}^{m_1 + \ldots + m_r} (u^{(i)\perp}, D_0^{-1} D'(\xi)u^{(i)})
\]

\[
+ a^2 \sum_{i<j} ((u^{(i)\perp}, D_0^{-1} D'(\xi)u^{(i)})(u^{(j)\perp}, D_0^{-1} D'(\xi)u^{(j)}))
\]

\[-(u^{(i)\perp}, D_0^{-1} D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-1} D'(\xi)u^{(i)})) + a^2 \sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi)G(\xi)u^{(i)})] \chi_1 u^{(i)}
\]

so that

\[
N(\xi, a) = \left( \prod_{\ell=1}^{m_1 + \ldots + m_r} |\delta^{(\ell)}| \right)[1 + \left\{ a \sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi)u^{(i)})
\right.
\]

\[
+ a^2 \sum_{i<j} ((u^{(i)\perp}, D_0^{-1} D'(\xi)u^{(i)})(u^{(j)\perp}, D_0^{-1} D'(\xi)u^{(j)}))
\]

\[-(u^{(i)\perp}, D_0^{-1} D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-1} D'(\xi)u^{(i)})) + a^2 \sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi)G(\xi)u^{(i)}) \}
\]

\[= \text{order } a^3\]
Since \( \log |\delta^{(t)}| = \lambda_0^{(t)} \) we obtain, to second order in \( a \),

\[
L_a = \int d\xi \log N(\xi, a) = m_1 \lambda_1 + \ldots + m_r \lambda_r + \int d\xi \{ \ldots \} - \frac{a^2}{2} \left( \sum_i (u^{(i) \perp}, D_0^{-1} D'(\xi) u^{(i)})^2 \right)
\]

where \( \{ \ldots \} \) has the same meaning as above. Write

\[
\Psi_i \left( \sum_{\ell} \xi \epsilon u^{(\ell)} \right) = (u^{(i) \perp}, D_0^{-1} F' \left( \sum_{\ell} \xi \epsilon u^{(\ell)} \right))
\]

The first term of \( \int d\xi \{ \ldots \} \) is

\[
a \sum_i \int d\xi \left( (u^{(i) \perp}, D_0^{-1} T F'(\xi) u^{(i)}) = a \sum_i \int d\xi \frac{\partial}{\partial \xi_i} \Psi_i
\]

which vanishes because \( \int d\xi \frac{\partial}{\partial \xi_i} \ldots = 0 \). The next term in \( \int d\xi \{ \ldots \} \) is

\[
a^2 \sum_{i<j} \int d\xi \left( \left( \frac{\partial}{\partial \xi_i} \Psi_i \right) \frac{\partial}{\partial \xi_j} \Psi_j \right) - \left( \frac{\partial}{\partial \xi_j} \Psi_j \right) \frac{\partial}{\partial \xi_i} \Psi_i \right) = a^2 \sum_{i<j} \int d\xi \left( \frac{\partial}{\partial \xi_i} \Psi_i \frac{\partial}{\partial \xi_j} \Psi_j - \frac{\partial}{\partial \xi_j} \Psi_j \frac{\partial}{\partial \xi_i} \Psi_i \right)
\]

which vanishes as above. Thus we are left with

\[
L_a = (m_1 \lambda_1 + \ldots + m_r \lambda_r)
\]

\[
= a^2 \int d\xi \left( \sum_i (u^{(i) \perp}, D_0^{-1} D'(\xi) G(\xi) u^{(i)}) - \frac{1}{2} \left( \sum_i (u^{(i) \perp}, D_0^{-1} D'(\xi) u^{(i)})^2 \right) \right)
\]

and we may write, using (3),

\[
\sum_i (u^{(i) \perp}, D_0^{-1} D'(\xi) G(\xi) u^{(i)}) = - \sum_{n=0}^{\infty} \sum_i (u^{(i) \perp}, D_0^{-1} D'(\xi) D_0^{-n-1} D_0^n u^{(i)})
\]

\[
= - \sum_{n=0}^{\infty} \sum_i \sum_{j}^* (u^{(i) \perp}, D_0^{-1} D'(\xi) u^{(j)}) (u^{(j) \perp}, D_0^{-n-1} D_0^n F_0^n \xi D_0^n u^{(i)})
\]

where we have introduced conjugate bases \( (u^{(j)}) \), \( (u^{(j) \perp}) \) of \( \mathcal{E} \), indexed by \( j = m_1 + \ldots + m_r + 1, \ldots, m + 1 \), and \( \sum_i \) is over \( i \leq m_1 + \ldots + m_r \), \( \sum_j^* \) is over \( j \geq m_1 + \ldots + m_r + 1 \). The above expression is also

\[
= - \sum_{n=0}^{\infty} \sum_i \sum_{j}^* \frac{\partial}{\partial \xi_j} (u^{(i) \perp}, D_0^{-1} F' \left( \sum_{\ell} \xi \epsilon u^{(\ell)} \right)) \frac{\partial}{\partial \xi_i} (u^{(j) \perp}, D_0^{-n-1} F' \left( F_0^n \sum_{\ell} \xi \epsilon u^{(\ell)} \right))
\]

and integration by part gives thus

\[
\int d\xi \sum_i (u^{(i) \perp}, D_0^{-1} D'(\xi) G(\xi) u^{(i)})
\]
\[
= - \sum_{n=0}^{\infty} \int d\xi \sum_{i} \frac{\partial}{\partial \xi_i} (u^{(i)\perp}, D_0^{-1}F' (\sum_{\ell} \xi_\ell u^{(\ell)})) \sum_{j} \frac{\partial}{\partial \xi_j} (u^{(j)\perp}, D_0^{-n-1}F' (F_0^n \sum_{\ell} \xi_\ell u^{(\ell)}))
\]
\[
= - \sum_{n=0}^{\infty} \int d\xi \text{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)) \text{Tr}_{\mathcal{E}^\perp}(D_0^{-n-1}D'(F_0^n \xi) D_0^n)
\]
\[
= - \sum_{n=0}^{\infty} \int d\xi \text{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)) \text{Tr}_{\mathcal{E}^\perp}(D_0^{-1}D'(F_0^n \xi))
\]

(Here \(\text{Tr}_{\mathcal{E}^\perp} = \text{Tr}_{\mathbb{R}^{m+1}} - \text{Tr}_{\mathcal{E}}\). The fact that \(F = F_0 + aF'\) is volume preserving (to first order in \(a\)) is expressed by \(\text{Tr}_{\mathbb{R}^{m+1}}(D_0^{-1}D'(\xi)) = 0\) hence
\[
\int d\xi \sum_{i} (u^{(i)\perp}, D_0^{-1}D'(\xi) G(\xi) u^{(i)})
\]
\[
= \sum_{n=0}^{\infty} \int d\xi \text{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)) \text{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n \xi))
\]
and introducing this in (4) yields
\[
L_a = (m_1 \lambda_1 + \ldots + m_r \lambda_r)
\]
\[
= a^2 \left[ \sum_{n=1}^{\infty} \int d\xi \text{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)) \text{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n \xi)) + \frac{1}{2} \int d\xi (\text{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)))^2 \right]
\]
\[
= \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi \text{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)) \text{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n \xi))
\] (5)

where the last step used the invariance of \(d\xi\) under \(F_0^n\). \(\square\)

10. **Proof of Theorem 1.**

We use Proposition 9, the corresponding result with \(F\) replaced by \(F^{-1}\), and the fact that \(\sum_{k=1}^{m} \lambda_{a}^{(k)} = 0\) (because \(F\) is volume preserving). This gives an estimate of all the sums of \(\lambda_{a}^{(k)}\) that occur in Theorem 1. \(\square\)

11. **Corollary.**

In the situation of Theorem 1, the central Lyapunov exponent is
\[
\lambda^c = \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi [\text{Tr}^u(D_0^{-1}D'(\xi)) \text{Tr}^u(D_0^{-1}D'(F_0^n \xi)) - \text{Tr}^s(D_0^{-1}D'(\xi)) \text{Tr}^s(D_0^{-1}D'(F_0^n \xi))]
\]
\[
= \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi [\text{Tr}^s(D_0^{-1}D'(\xi)) - \text{Tr}^u(D_0^{-1}D'(\xi))] \text{Tr}^c(D_0^{-1}D'(F_0^n \xi))
\]
where $\text{Tr}^s$, $\text{Tr}^u$, $\text{Tr}^c$ denote the traces over the spectral subspaces $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$ of $D_0$ corresponding to eigenvalues $<1$, $>1$, or $=1$ in absolute value ($\mathcal{E}^c$ is one dimensional).

Since $F$ preserves the volume, the sum of all Lyapunov exponents vanishes. Therefore $\lambda^c$ is minus the sum of the negative Lyapunov exponents, given by (5), minus the sum of the positive Lyapunov exponents. Note that replacing $F$ by $F^{-1}$, $\mathcal{E}^s$ by $\mathcal{E}^u$ (and, to the order considered, $D'(\xi)$ by $-D'(\xi)$) replaces the sum of the negative Lyapunov exponents by minus the sum of the positive exponents. This gives the first formula for $\lambda^c$.

To obtain the second formula, express $\text{Tr}^u\text{Tr}^s - \text{Tr}^s\text{Tr}^u$ in terms of $\text{Tr}^u \pm \text{Tr}^s$, and remember that (because $F$ preserves the volume) $\text{Tr}^s + \text{Tr}^u + \text{Tr}^c = 0$ when applied to $D_0^{-1}D'(\xi)$. □

The above formula (5) takes a particularly simple form in a special case described in the next theorem.

**12. Theorem.**

Let $\Phi$ be a hyperbolic automorphism of $\mathbb{T}^m$, with stable and unstable dimensions $m^s$ and $m^u = m - m^s$, and with entropy $\lambda_0^u$. Let $J : y → y + \alpha \pmod{1}$ be a translation of $\mathbb{T}$, and $\phi : \mathbb{T}^m → \mathbb{T}$ a group homomorphism $\neq 0$. Finally let $\psi : \mathbb{T} → \mathbb{R}^m$ be a nullhomotopic $C^2$ function.

Define $h, g_a : \mathbb{T}^m × \mathbb{T} → \mathbb{T}^m × \mathbb{T}$ by

$$h(x) = (Jy + \phi x - \phi x), \quad g_a(x) = \left( x + \alpha \psi(y) \pmod{1}, y \right)$$

and let $f_a = g_a \circ h$.

Denote by $\lambda_a^s$ (resp. $\lambda_a^u$) the sum of the smallest $m^s$ (resp. the largest $m^u$) Lyapunov exponents for $(f_a, \text{volume})$. Also let $\lambda_a^c = -\lambda_a^s - \lambda_a^u$ be the "central exponent". Then $\lambda_a^s$, $\lambda_a^u$, $\lambda_a^c$ have expansions of order 2 in $\alpha$:

$$\lambda_a^s = -\lambda_0^u + \frac{\alpha^2}{2} \int_T dy (\nabla \psi^s(y))^2 + o(\alpha^2)$$

$$\lambda_a^u = \lambda_0^u - \frac{\alpha^2}{2} \int_T dy (\nabla \psi^u(y))^2 + o(\alpha^2)$$

$$\lambda_a^c = \frac{\alpha^2}{2} \int_T dy [((\nabla \psi^u(y))^2 - (\nabla \psi^s(y))^2)] + o(\alpha^2)$$

Here $\psi^s(y)$ and $\psi^u(y)$ are the components of the derivative $\psi'(y) \in \mathbb{R}^m$ in the stable and unstable subspaces $\mathcal{E}^s$ and $\mathcal{E}^u$ for $\Phi$. Also, we have used $\nabla \phi : \mathbb{R}^m → \mathbb{R}$ to denote the derivative of the map $\phi : \mathbb{T}^m → \mathbb{T}$ with the obvious identifications.

This theorem is a simple (but nontrivial) extension of the result proved by Shub and Wilkinson [17]. In the situation that they consider $\Phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $J =$identity, $\phi = (1,0)$,
\( \psi' = \psi^m. \) [Remark that, in the notation of [17], \( u_0 = ((1,1)v_0)/(m-1) = ((1,0)v_0) \) so that the formula given in Proposition II of [17] agrees with our result above].

Notation.

We shall henceforth omit the \((\text{mod } 1)\). We shall keep \( \nabla \) to denote the derivative in \( T^m \). With obvious abuses of notation, the reader may find it convenient to think of \( \Phi \) or \( \nabla \Phi \) as an \( m \times m \) matrix (with integer entries and determinant \( \pm 1 \)), and \( \phi \) or \( \nabla \phi \) as a row \( m \)-vector (with integer entries not all zero).

13. Reformulation of the problem.

Note that \( f_a^{-1} = h^{-1} \circ g_a^{-1} \) where \( h^{-1}, g_a^{-1} \) are obtained from \( h, g_a \) by the replacements \( \Phi, J, \phi, \psi \rightarrow \Phi^{-1}, J^{-1}, \phi, -\psi \). These replacements also interchange the stable and unstable subspaces for \( \Phi \) and replace \( \lambda^s, \lambda^u \) by \(-\lambda^u, -\lambda^s\). Therefore the formula for \( \lambda^u \) in the theorem follows from the formula for \( \lambda^s \). And the formula for \( \lambda^c = -\lambda^s - \lambda^u \) also follows. To complete the proof of the theorem we turn now to the formula for \( \lambda^s \).

Define

\[
\hat{\phi}(x, y) = \left( \begin{array}{c} x \\ y + \phi x \end{array} \right)
\]

then

\[
F_0(x, y) = \hat{\phi}^{-1} h \hat{\phi}(x, y) = \left( \begin{array}{c} \Phi x \\ J y \end{array} \right)
\]

\[
\hat{g}_a(x, y) = \hat{\phi}^{-1} g_a \hat{\phi}(x, y) = \left( \begin{array}{c} x + a \psi(y + \phi x) \\ y - a(\nabla \phi)\psi(y + \phi x) \end{array} \right)
\]

so that

\[
F(x, y) = \hat{\phi}^{-1} f_a \hat{\phi}(x, y) = \hat{g}_a F_0(x, y) = \left( \begin{array}{c} \Phi x + a \psi(J y + \phi \Phi x) \\ J y - a(\nabla \phi)\psi(J y + \phi \Phi x) \end{array} \right)
\]

Finally, \( F = F_0 + a F' \) with

\[
F_0(x, y) = \left( \begin{array}{c} \Phi x \\ J y \end{array} \right), \quad F'(x, y) = \left( \begin{array}{c} \psi(J y + \phi \Phi x) \\ -(\nabla \phi)\psi(J y + \phi \Phi x) \end{array} \right)
\]

Since \( F \) is conjugate (linearly) to \( f_a \), we may compute \( \lambda^s \) from \( F \) instead of \( f_a \).


Write \( \mathbb{R}^{m+1} = \mathcal{E}^s + \mathcal{E}^u \). We shall apply Proposition 9 with \( \mathcal{E} = \mathcal{E}^s, \mathcal{E}^\perp = \mathcal{E}^u \) + \( \mathbb{R} \). Using \( \xi = (x, y) \) and \( X \in \mathcal{E}^s, Y \in \mathcal{E}^u, Z \in \mathbb{R} \) we may write

\[
D_0(X + Y, Z) = \left( \begin{array}{c} (\nabla \Phi)(X + Y) \\ Z \end{array} \right)
\]

\[
D'(\xi)(X + Y, Z) = \left( \begin{array}{c} \psi'(J y + \phi \Phi x)((\nabla \phi \Phi)(X + Y) + Z) \\ -(\nabla \phi)\psi'(J y + \phi \Phi x)((\nabla \phi \Phi)(X + Y) + Z) \end{array} \right)
\]

where \( \psi' \) denotes the derivative of \( \psi \). Therefore

\[
\text{Tr}_{\mathcal{E}}(D'(\xi) D_0^{-1}) = (\nabla \phi)^{\psi^s}(J y + \phi \Phi x)
\]
and (5) contains the integrals
\[
\int d\xi \operatorname{Tr}(D_0^{-1}D'(\xi)) \operatorname{Tr}(D_0^{-1}D'(F^n_0\xi))
\]
\[
= \int d\xi [(\nabla \phi)\psi'(Jy + \phi Fx)] [(\nabla \phi)\psi'(J^{n+1}y + \phi F^{n+1}x)]
\]
Performing a change of variables \(\bar{x} = \Phi x\), \(\bar{y} = Jy + \phi Fx\) we find that this is
\[
= \int d\bar{x} d\bar{y} [(\nabla \phi)\psi'(\bar{y})] [(\nabla \phi)\psi'(J^n\bar{y} + \phi F^n\bar{x} - \phi \bar{x})]
\]
We claim that this last integral vanishes unless \(n = 0\). This is because, if \(n \neq 0\),
\[
\int d\bar{x} \psi'(J^n\bar{y} + \phi F^n\bar{x} - \phi \bar{x}) = 0
\]
Indeed, \(\phi F^n\bar{x} - \phi \bar{x}\) is a linear combination with integer coefficients of the components \(\bar{x}_1, \ldots, \bar{x}_m\) of \(\bar{x}\), and the coefficients do not all vanish because \(\Phi^n = \phi\) is impossible (\(\Phi\) is hyperbolic and \(\phi \neq 0\)). Integrating the derivative \(\psi'\) with respect to a variable \(\bar{x}_j\) really occurring in \(\phi F^n\bar{x} - \phi \bar{x}\) gives zero as announced.

Returning to (5) we have thus
\[
\lambda^a_s + \lambda^u_0 = \frac{a^2}{2} \int d\xi (\operatorname{Tr}(D_0^{-1}D'(\xi)))^2
\]
\[
= \frac{a^2}{2} \int d\bar{y} ((\nabla \phi)\psi'(\bar{y}))^2
\]
which is the formula given for \(\lambda^a_s\) in Theorem 12. And according to Section 13 this completes our proof. \(\Box\)

**15. Final remarks.**

(a) Shub and Wilkinson [17] showed that close to a diffeomorphism (hyperbolic automorphism \(\Phi\) of \(T^2\) × (identity on \(T\)) there is a \(C^1\) open set of ergodic volume preserving \(C^2\) diffeomorphisms of \(T^3\) with central Lyapunov exponent \(\lambda^c > 0\). They remark that their result extends to the situation where \(\Phi\) is a hyperbolic automorphism of \(T^m\) with one-dimensional expanding eigenspace. More generally, if \(\Phi\) is any hyperbolic automorphism of \(T^m\), Theorem 12 gives close to (\(\Phi\), rotation of \(T\)) in \(C^2(\mathbb{T}^{m+1})\) a diffeomorphism \(F\) with \(\lambda^c > 0\). Since \(\lambda^c\) is given by an integral over the volume of a local "central" stretching exponent, we have \(\lambda^c > 0\) in a \(C^1\) neighborhood of \(F\). But by a result of Dolgopyat and Wilkinson [8] (Corollary 0.5), stable ergodicity is here \(C^1\) open and dense in the \(C^2\) volume preserving diffeomorphisms (\(C^1\) is improved to \(C^k\) in [12]): we have center bunching and stable dynamical coherence because we consider perturbations of (\(\Phi\), rotation of \(T\)) for which the center foliation is \(C^1\), see [10], [13]. In conclusion, close to (hyperbolic automorphism \(\Phi\) of \(T^m\)) × (rotation on \(T\)) there is a \(C^1\) open set \(V\) of ergodic volume preserving \(C^2\)
diffeomorphisms of $T^{m+1}$ with central Lyapunov exponent $\lambda^c > 0$ (or also with $\lambda^c < 0$). In particular, if $F \in V$, the conditional measures of the volume on the circles $\pi^{-1}\{x\}$ are atomic, as discussed in [16].

(b) The coefficient $L$ in Proposition 9 is $\geq 0$. Consider indeed the unitary operator $U$ defined by $U\psi = \psi \circ F$ on $L^2(T^{m+1}, \text{volume})$, and let $E(.)$ be the corresponding spectral measure, so that

$$U = \int_T e^{2\pi i \theta} E(d\theta)$$

If $\psi(\xi) = \text{Tr}_\xi(D_0^{-1}D'(\xi))$ we have a measure $\nu \geq 0$ on $T$ defined by $\nu(d\theta) = (\psi, E(d\theta)\psi)$ and the Fourier coefficients

$$c_n = \int e^{2\pi i n \theta} \nu(d\theta) = \int d\xi \text{Tr}_\xi(D_0^{-1}D'(\xi))(D_0^{-1}D'(F_0^n \xi))$$

of this measure tend to zero exponentially. Therefore $\nu(d\theta) = \rho(\theta)d\theta$ has a smooth density $\rho$ and

$$L = \frac{1}{2} \sum_{n=-\infty}^{\infty} c_n = \frac{1}{2} \rho(0) \geq 0$$

(c) Suppose now that $F$ is not necessarily a volume preserving perturbation of $F_0$. We may still hope that $F$ has an SRB measure $\rho_a$. If $F_0$ were hyperbolic, we would have an expansion

$$\rho_a = \rho_0 + a\delta + o(a)$$

(see [15]) with $\rho_0$ = Lebesgue measure and $\delta$ a distribution. For smooth $\Psi$, $\delta(\Psi)$ is given (because $\rho_0$ is Lebesgue measure) by the simple formula (see [15])

$$\delta(\Psi) = -\sum_{n=0}^{\infty} \rho_0((\Psi \circ F_0^n) . \text{div}(F' \circ F_0^{-1}))$$

Similarly (replacing $F$ by $F^{-1}$, hence $F_0$, $D_0^{-1}D'(\xi)$ by $F_0^{-1}$, $-D'(F_0^{-1}\xi)D_0^{-1}$ we see that the anti-SRB state has an expansion

$$\tilde{\rho}_a = \rho_0 + a\tilde{\delta} + o(a)$$

with

$$\tilde{\delta}(\Psi) = \sum_{n=1}^{\infty} \int d\xi \Psi(F_0^{-n}\xi) \text{Tr}_{R^{m+1}}(D'(F_0^{-1}\xi)D_0^{-1})$$

$$= \sum_{n=0}^{\infty} \int d\xi \Psi(F_0^{-n}\xi) \text{Tr}_{R^{m+1}}(D_0^{-1}D'(\xi))$$

We can now estimate the Lyapunov exponents for $(F, \rho_a)$ to second order in $a$ even though we are not sure of the existence of the SRB measure $\rho_a$. We simply assume that we
can use the formula for $\delta(\Psi)$. Going through the proof of Proposition 9 we have to replace
\[
\int d\xi \log N(\xi, a) = \rho_a(\log N(\xi, a))
\]
and (to second order in $a$) this adds to the right-hand side of (4) a term
\[
-a^2 \sum_{n=1}^{\infty} \int d\xi \, \text{Tr}_{(D_0^{-1}D'(\xi))} \text{Tr}^{R_{m+1}} (D_0^{-1}D'(\xi))
\]
Taking into account the integrations by part we obtain now instead of (5) the formula
\[
L_a = (m_1 \lambda_1 + \ldots + m_r \lambda_r) = \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}_{(D_0^{-1}D'(\xi))} \text{Tr}_{(D_0^{-1}D'(F_0^n\xi))}
\]
\[
- \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}^s_{(D_0^{-1}D'(\xi))} \text{Tr}^{R_{m+1}} (D_0^{-1}D'(F_0^n\xi))
\]
\[
L^s = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}^{s}_{(D_0^{-1}D'(\xi))} \text{Tr}^{s}_{(D_0^{-1}D'(F_0^n\xi))}
\]
\[
- \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}^{u}_{(D_0^{-1}D'(\xi))} \text{Tr}^{u}_{(D_0^{-1}D'(F_0^n\xi))}
\]
\[
L^u = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}^{u}_{(D_0^{-1}D'(\xi))} \text{Tr}^{u}_{(D_0^{-1}D'(F_0^n\xi))}
\]
\[
L^c = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}^{c}_{(D_0^{-1}D'(\xi))} \text{Tr}^{c}_{(D_0^{-1}D'(F_0^n\xi))}
\]
\[
- \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}^{c}_{(D_0^{-1}D'(\xi))} \text{Tr}^{u}_{(D_0^{-1}D'(F_0^n\xi))}
\]
\[
L^s + L^u + L^c = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \text{Tr}_{R_{m+1}} (D_0^{-1}D'(\xi)) \text{Tr}^{R_{m+1}} (D_0^{-1}D'(F_0^n\xi))
\]
which can be rewritten variously.

In view of recent work [4], [1], [6], it seems reasonable to conjecture that if the above $L^c$ is $\neq 0$, then there exists an SRB measure for (small) finite $a$.

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References.


