The exact description of NS5-branes in the Penrose limit

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Abstract

We construct plane wave backgrounds with time-dependent profiles corresponding to Penrose limits of NS5-branes with transverse space symmetry group broken from $SO(4)$ to $SO(2) \times Z_N$. We identify the corresponding exact theory as the five-dimensional Logarithmic Conformal Field Theory (CFT) arising from the contraction of the $SU(2)_N/U(1) \times SL(2,R)_{-N}$ exact CFT, times $\mathbb{R}^5$. We study several general aspects and construct the free field representation in this theory. String propagation and spectra are also considered and explicitly solved in the light-cone gauge.
1 Introduction

The purpose of the present paper is to provide and analyze the first example of a Logarithmic CFT arising as the exact description of certain supergravity brane configurations in the Penrose limit [1] and to also study string propagation in this background. Novel Logarithmic CFTs arose recently in the large level $N$ limit of coset conformal field theories for compact groups combined with a free time-like boson [2] and provided the exact description of various PP-waves configurations (with no brane interpretation however). It was shown quite generally in [2] that, at the level of the chiral algebra involving the compact parafermions [3, 4] and the $U(1)$-current corresponding to the free time-like boson, the Penrose limiting procedure gives rise to a Saletan-type contraction (for mathematical details in the case of Lie groups, see [5]) and to a Logarithmic CFT theory [6], [7] (for earlier work see [8]; for a review and more references see [9]).

This type of Penrose limits and contractions have already appeared in string theory some years ago where they were used to construct plane wave solutions starting from WZW model current algebra theories or gauged WZW model coset theories. The prototype example is the plane wave solution of [10], corresponding to a WZW model for the non-semisimple group $H_4$, which is the Penrose limit of the background corresponding to the $SU(2)_N \times U(1)_{-N}$ current algebra theory [11] (for construction of plane waves either directly as WZW and gauged WZW models or by contractions see [12]-[18]). The interest in plane wave solutions arising in string and M-theory has been recently revived following the construction of a maximally supersymmetric plane wave solution of type-IIB supergravity [19], the fact that, it can be obtained as a Penrose limit of the maximally supersymmetric vacuum solution, $AdS_5 \times S^5$ [20] and that, string theory is exactly solvable in this plane wave background [21], which allows to extend the AdS/CFT correspondence in a non-trivial way to include the effect of highly massive string states [22]. In accordance with experience and general expectations, the superalgebra for the $AdS_5 \times S^5$ background contracts to the corresponding plane wave superalgebra, as it has been explicitly demonstrated in [23]. These more recent developments are in accordance with the fact that, the Penrose limiting procedure can be straightforwardly generalized to supergravity theories [24].

The organization of this paper is as follows: In section 2 we review the relevant aspects of certain NS5-brane configurations which will serve as the starting point in our constructions. In the supersymmetric case they have the distinct feature that they preserve an $SO(2) \times Z_N$ subgroup of the $SO(4)$ symmetry group of the transverse to the branes space $\mathbb{R}^4$. We also review the non-extremal extension of this solution, representing the most general non-extremal rotating NS5-brane solution. In section 3 we systematically
construct plane wave backgrounds with time dependent profiles by performing several different Penrose limits. In section 4 we identify the exact CFT corresponding to these plane waves as the contraction of the $SU(2)_N'/U(1) \times SL(2, \mathbb{R})_{-N}$ theory. The result is a one parameter family of five-dimensional Logarithmic CFTs. For a particular member of this family the theory degenerates into the four-dimensional current algebra for the non-semisimple group $H_4$ times a $U(1)$ factor. In section 5 we show that it is possible to explicitly solve string theory in the light-cone gauge, even though our plane wave backgrounds have time dependent profiles. We end our paper in section 6 with concluding remarks and directions for feature research.

2 The background

The most general solution of either type-II or Heterotic string theory representing the gravitational field and flux of $N$ NS5-branes is given by [25]

$$ds^2 = \sum_{a=1}^{5} dy_a^2 - dt^2 + H \sum_{i=1}^{4} dx_i^2 ,$$

$$H_{ijk} = \epsilon_{ijkl} \partial_l H ,$$

$$e^{2\Phi} = H ,$$

(2.1)

where the coordinates $y_a$’s and $t$ parametrize the world-volume of the branes and the $x_i$’s the directions transverse to the branes space which is $\mathbb{R}^4$. The function $H$ is harmonic in $\mathbb{R}^4$. Demanding that asymptotically the space is flat we have in general

$$H = 1 + \sum_{i=0}^{N-1} \frac{\alpha'}{|\mathbf{x} - \mathbf{x}_i|^2} ,$$

(2.2)

where $\mathbf{x}_i$ denote the locations of the branes in $\mathbb{R}^4$. Such configurations preserve sixteen supercharges. A generic choice for the centers breaks the $SO(4)$ symmetry completely. In this paper we will consider NS5-brane configurations preserving an $SO(2) \times Z_N$ subgroup of $SO(4)$. In particular, we will consider backgrounds corresponding to NS5-branes with centers distributed at the circumference of a circle with radius $r_0$ [26]. In the rest of this section we review the relevant to this paper results of [26] where the reader is referred for further details. The centers of the NS5-branes are located at

$$\mathbf{x}_i = r_0(0, 0, \cos \phi_i, \sin \phi_i) , \quad \phi_i = 2\pi \frac{i}{N} , \quad i = 0, 1, 2, \ldots, N - 1 .$$

(2.3)

In our presentation we will strictly restrict to the decoupling, near horizon limit, where the unit in (2.2) become unimportant. We also keep finite the energies of strings with
their ends attached between different centers. Then the harmonic function is computed to be

\[ H = N \Lambda_N \frac{1}{\sqrt{(r^2 + r_0^2)^2 - 4r_0^2 \rho^2}} , \]  

(2.4)

where \( \rho^2 = x_3^2 + x_4^2 \) and \( r^2 = x_1^2 + x_2^2 + \rho^2 \) and

\[ \Lambda_N = \frac{\sinh N \chi}{\cosh N \chi - \cos N \psi} , \]  

(2.5)

with the angular variable \( \psi \) defined as \((x_3, x_4) = \rho (\cos \psi, \sin \psi)\) and the auxiliary variable \( \chi \) being given by

\[ e^\chi = \frac{r^2 + r_0^2}{2r_0 \rho} + \sqrt{\left(\frac{r^2 + \rho_0^2}{2r_0 \rho}\right)^2 - 1} . \]  

(2.6)

When \( N \gg 1 \) and \( r/r_0 - 1 \gg 1/N \) the branes are practically continuously distributed in the circumference of the circle. Since then \( \Lambda_N \approx 1 \), the \( Z_N \) symmetry becomes a continuous \( SO(2) \) isometry. In the rest we will restrict to the continuum limit. It turns out that this is enough for the construction of our plane wave solutions in the following section.

In order to exhibit the \( SO(2) \times SO(2) \) symmetry explicitly we change variables as

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} = r_0 \sinh \rho \cos \left( \frac{\cos \phi}{\sin \phi} \right), \quad \begin{pmatrix}
  x_3 \\
  x_4
\end{pmatrix} = r_0 \cosh \rho \sin \left( \frac{\cos \psi}{\sin \psi} \right),
\]  

(2.7)

where the range of the various variables is

\[ 0 \leq r < \infty, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \psi \leq 2\pi, \quad 0 \leq \phi \leq 2\pi \]  

(2.8)

and also we rescale for future convenience the \( y_a \)'s and \( t \) by a factor of \( \sqrt{N} \). Then the background (2.1) takes the form

\[
\frac{1}{N} ds^2_{10} = \sum_{a=1}^{5} dy_a^2 - dt^2 + d\rho^2 + d\theta^2 + \frac{1}{1 + \tanh^2 \rho \tan^2 \theta} (\tan^2 \theta \, d\psi^2 + \tanh^2 \rho \, d\phi^2) ,
\]

\[
\frac{1}{N} B_{\phi\psi} = \frac{1}{1 + \tanh^2 \rho \tan^2 \theta} , \]

\[
e^{-2\Phi} = e^{-2\Phi_0} (\cos^2 \theta \cos^2 \rho + \sin^2 \theta \sinh^2 \rho) .
\]

(2.9)

Performing a T-duality transformation with respect to the Killing vector \( \frac{\partial}{\partial \phi} \) we can relate this background to the one corresponding to the \( SL(2, R)_{-N}/U(1) \times SU(2)_N/U(1) \times \mathbb{R}^{1,5} \) exact theory.

The metric in (2.9) is singular at the location of the branes at the ring with coordinates \( \rho = 0 \) and \( \theta = \pi/2 \). The reason for this is that near the branes the continuous approximation breaks down and one has to use the full solution (2.1) with harmonic function given by (2.4). Then one indeed verifies that the metric is non-singular [26].
2.1 The non-extremal NS5-brane rotating solution

The background (2.9) is the extremal supersymmetric limit of the most general non-extremal NS5-brane rotating solution constructed in [27] and further analyzed in [28]. In the field-theory limit this solution has a metric

\[
\frac{1}{N}ds^2 = -(1 - \frac{\mu^2}{\Delta_0})dt^2 + \sum_{a=1}^{5} dy_a^2 + \frac{dp^2}{\rho^2 + a_1^2 a_2^2 / \rho^2 + a_1^2 + a_2^2 - \mu^2} \\
+ d\theta^2 + \frac{1}{\Delta_0} ((\rho^2 + a_1^2) \sin^2 \theta d\phi_1^2 + (\rho^2 + a_2^2) \cos^2 \theta d\phi_2^2) \\
- \frac{2}{\Delta_0} \mu dt (a_1 \sin^2 \theta d\phi_1 + a_2 \cos^2 \theta d\phi_2) ,
\]

(2.10)
an antisymmetric tensor

\[
\frac{1}{N} B = \frac{2}{\Delta_0} (- (\rho^2 + a_1^2) \cos^2 \theta d\phi_1 \wedge d\phi_2 + \mu a_2 \sin^2 \theta dt \wedge d\phi_1 + \mu a_1 \cos^2 \theta \mu dt \wedge d\phi_2) ,
\]

(2.11)
and a dilaton

\[
e^{-2\Phi} = \Delta_0 ,
\]

(2.12)
where we have defined

\[
\Delta_0 = \rho^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta 
\]

(2.13)
and we have denoted by \(a_1, a_2\) the angular parameters and by \(\mu\) the non-extremal parameter. In was shown in [27] that for \( \mu = 0 \) this background becomes the multicenter continuous distribution background in (2.9) with \( \rho_0^2 = |a_1^2 - a_2^2| \). Moreover, it can be obtained by an \( O(3,3) \) duality transformation on the background corresponding to the \( SU(2) \times SL(2, \mathbb{R}) / U(1) \times \mathbb{R}^5 \) exact CFT. Thermodynamic properties of this solution can be found in [27, 28].

3 The Penrose limit

In this section we will construct plane waves by taking Penrose limits on the NS5 backgrounds (2.9) and (2.10).

In general, plane waves arise as classical solutions to theories of gravity and share the attractive feature that they depart from the flat space solution in the most controllable possible manner. This is essentially due to the existence of a covariantly constant null Killing vector, which, as it turns out, guarantees that curvature effects are kept to a minimum [29]-[35]. Plane waves, are relatively simple solutions which is reflected in the
simplicity of the equations of motions that they obey. For a background, with only NS-NS fields turned on, of the form

\[ ds^2 = 2dudv + F_{ij}(u)x_i x_j du^2 + dx_i^2 , \]
\[ B = B_{ij}(u)dx^i \wedge dx^j , \]
\[ \Phi = \Phi(u) , \]

the equations for 1-loop conformal invariance follow if the condition

\[-\text{Tr}(F) + \frac{1}{4} \text{Tr}(S^2) + 2 \frac{d^2 \phi}{du^2} = 0 , \quad S_{ij} = \frac{dB_{ij}}{du} , \]

is satisfied. It can be readily checked that this is indeed the case for all plane waves we construct below. The conditions for conformal invariance at 1-loop are sufficient to prove conformal invariance to all orders in perturbation theory [30, 31].

3.1 Plane waves from continuously distributed NS5-branes

We start with the Penrose limit for the background (2.9). According to the general prescription Penrose had suggested, we are supposed to magnify the space-time around a null geodesic. There are three natural such geodesics dictated essentially by symmetry. The first geodesic goes through the center of the ring corresponding to \( \theta = 0 \) in our coordinate parametrization. The second geodesic corresponds to choosing \( \theta = \pi/2 \), therefore laying on the plane of the ring which hits perpendicularly. Finally, the third geodesic corresponds to \( \rho = 0 \) and hence also lays on the plane of the ring. We recall that the background (2.9) has a singularity at \( \rho = 0 \) and \( \theta = \pi/2 \), corresponding to a breakdown of the continuous approximation of the NS5-brane distribution. Hence, the first geodesic never gets near the singularity of the background and will give rise to a completely non-singular plane wave. In contrast, the second and third geodesics hit the singularity and give rise to singular plane wave backgrounds. Nevertheless, the origin of this singularity is well understood as arising from the breakdown of the continuous approximation at the location of the branes. It is remarkable that, in some sense, this is captured by the solution giving rise to a seemingly well define string spectrum in the light-cone gauge, as we will see in section 5.

3.1.1 Geodesic at \( \theta = 0 \)

We first consider the geodesic with \( \theta = 0 \) that goes through the center of the ring and has constant values for the space-like world-volume directions \( y_a \). In this case we have the effective 3-dim metric

\[ ds_3^2 = -dt^2 + d\rho^2 + \tanh^2 \rho d\phi^2 . \]
We search for solutions to geodesic equations with \( \rho, t, \phi \) being functions of the proper time \( \tau \). Then, energy and \( U(1) \)-charge conservation imply that

\[
i = E, \quad \tanh^2 \rho \dot{\phi} = l. \tag{3.4}\]

Using these conservation laws and the nullness of the geodesic we find that

\[
\dot{\rho}^2 = E^2 - l^2 \coth^2 \rho. \tag{3.5}
\]

A real solution of this obviously exists, provided that the dimensionless ratio \( J = l/E \) obeys the condition \(|J| \leq 1\).

Having such a null geodesic we define a new variable \( u \) to be essentially the proper time \( \tau \). Specifically, we choose \( u = E\tau \) and therefore the meaningful parameter will be \( J \). The other two variables parametrizing the directions normal to the null geodesic in the \((t, \rho, \phi)\) space will be denoted by \( v \) and \( x \). The appropriate change of variables in terms of exact differentials is explicitly given by

\[
\begin{align*}
d\rho &= \sqrt{1 - J^2 \coth^2 \rho} \, du, \\
dt &= du - dv/N + Jdx/\sqrt{N}, \\
d\phi &= J \coth^2 \rho \, du + dx/\sqrt{N}. \\
\end{align*} \tag{3.6}
\]

We note that the above change of variables requires that

\[
\rho \geq \frac{1}{2} \ln \left( \frac{1 + J}{1 - J} \right), \quad |J| \leq 1. \tag{3.7}
\]

Also let

\[
\theta = \frac{z}{\sqrt{N}}, \quad y_a \rightarrow y_a/\sqrt{N} \tag{3.8}
\]

and define \( dz^2 + z^2 d\psi^2 = d\vec{z}^2 \), so that \( z^2 = \vec{z}^2 \). The relation between the variables \( \rho \) and \( u \) is

\[
\cosh \rho(u) = \frac{1}{\sqrt{1 - J^2}} \cosh \sqrt{1 - J^2} u. \tag{3.9}
\]

Then the background in the limit \( N \rightarrow \infty \) takes the form

\[
\begin{align*}
ds^2 &= 2dudv + \sum_{i=1}^{5} dx_i^2 + d\vec{z}^2 + (1 - J^2) \tanh^2 \sqrt{1 - J^2} u dx^2 - J^2 \vec{z}^2 du^2, \\
B_{12} &= 2Ju, \\
e^{-2\Phi} &= e^{-2\Phi_0} \cosh^2 \sqrt{1 - J^2} u. \tag{3.10}
\end{align*}
\]
Going to the Brinkman coordinates\(^1\) we find

\[
ds^2 = 2dudv + \sum_{i=1}^{5} dx_i^2 + dx_2^2 + dz^2 - \left(2 \frac{1 - J^2}{\cosh^2 \sqrt{1 - J^2}u} x^2 + J^2 z^2 \right) du^2 ,
\]

\[
B_{12} = 2Ju ,
\]

\[
e^{-2\Phi} = e^{-2\Phi_0} \cosh^2 \sqrt{1 - J^2u} .
\]

The parameter \(J\) takes values in the interval \([-1, 1]\). At the end points of this interval we recognize two known cases. For \(J = 0\) the non-trivial part of our background is the three-dimensional plane wave obtained in [14, 2] as the Penrose limit of the background for the \(SU(2, \mathbb{R})_N / U(1) \times U(1)_{-N}\) exact CFT. At the endpoints with \(|J| = 1\) we obtain as a non-trivial part of our background the four-dimensional plane wave of [10] constructed as a WZW model based on the semi-simple group \(H_4\). In our context, this background arises as the Penrose limit of the background for the \(SU(2)_N \times U(1)_{-N}\) exact CFT [11]. For general values of \(J\) we have a non-trivial five-dimensional theory which, as we will show, it can be obtained as a contraction of the \(SU(2)_N \times SL(2, \mathbb{R})_{-N}/U(1)\) exact CFT.

### 3.1.2 Geodesic at \(\rho = 0\)

In this case as well the geodesic lays on the plane of the ring and can be considered in parallel to the previous one. Skipping some of the details, we first perform the change of variables

\[
d\theta = \sqrt{1 - J^2 \cot^2 \theta} du ,
\]

\[
dt = du - dv/N + Jdx/\sqrt{N} ,
\]

\[
d\psi = J \cot^2 \theta du + dx/\sqrt{N} .
\]

Then we let

\[
\rho = \frac{z}{\sqrt{N}} , \quad y_a \rightarrow y_a/\sqrt{N} \quad (3.15)
\]

\(^1\)If the metric has the block-diagonal form

\[
ds^2 = 2dudv + \sum_i g_i(u) dx_i^2 , \quad (3.11)
\]

we obtain a metric in the Brinkman form (3.1) with \(F_{ij} = F_i(u)\delta_{ij}\) after performing the transformation

\[
u \rightarrow u , \quad v \rightarrow v + \frac{1}{4} \sum_i \frac{\dot{g}_i}{g_i} , \quad F_i = \frac{1}{4} \frac{\dot{g}_i^2}{g_i} + \frac{1}{2} \frac{d}{du} \left(\frac{\dot{g}_i}{g_i}\right) . \quad (3.12)
\]
and define $dz^2 + z^2 d\phi^2 = dz^2$, so that $z^2 = \bar{z}^2$. The explicit relation between the variables $u$ and $\theta$ is

$$\cos \theta = \frac{1}{\sqrt{1 + J^2}} \cos \sqrt{1 + J^2} u .$$

(3.16)

Then the background in the limit $N \to \infty$ takes the form

$$ds^2 = 2dudv + \sum_{i=1}^{5} dx_i^2 + dx^2 + d\bar{z}^2 + (1 + J^2) \tan^2 \sqrt{1 + J^2} u dx^2 - J^2 \bar{z}^2 du^2 .$$

$$B_{12} = 2Ju ,$$

(3.17)

$$e^{-2\Phi} = e^{-2\Phi_0} \cos^2 \sqrt{1 + J^2} u .$$

In Brinkman coordinates it reads

$$ds^2 = 2dudv + \sum_{i=1}^{5} dx_i^2 + d\bar{z}^2 + \left( \frac{2}{\cos^2 \sqrt{1 + J^2} u} - J^2 \bar{z}^2 \right) du^2 ,$$

$$B_{12} = 2Ju ,$$

(3.18)

$$e^{-2\Phi} = e^{-2\Phi_0} \cos^2 \sqrt{1 + J^2} u .$$

We also note that (3.17) and (3.18) can be obtained from (3.10) and (3.13) by analytically continuing as $J \to iJ, u \to iu, v \to -iv$ and $x \to ix$. As we have mentioned, the singularity at $\sqrt{1 + J^2} u = \pi/2$ originates from the breakdown of the continuous approximation near the location of the NS5-branes.

The plane wave, corresponding to the remaining geodesic with $\theta = \pi/2$ will not be presented in this subsection since it is a particular case of the plane waves constructed in subsection 3.2 below.

### 3.2 PP-wave limit of the non-extremal rotating NS5-brane

It turns out that, if we start with the most general rotating NS5-brane solution in the field theory limit (2.10)-(2.12), it is enough to consider the null geodesic corresponding to $\theta = \pi/2$. For instance, a the null geodesics with $\theta = 0$ and $\theta = \pi/2$ are related by the symmetry transformation of the background under $\theta \to \pi/2 - \theta$, followed by the interchange of $a_1$ with $a_2$ and $\phi_1$ with $\phi_2$. The 3-dim effective metric is then

$$ds_3^2 = -G_{tt} dt^2 + G_{\rho\rho} d\rho^2 + G_{\phi_1\phi_1} d\phi_1^2 - 2G_{t\phi_1} dtd\phi_1 ,$$

(3.19)

where the various functions are given by

$$G_{tt} = 1 - \frac{\mu^2}{\rho^2 + a_2^2} , \quad G_{\rho\rho}^{-1} = \rho^2 + a_1^2 a_2^2/\rho^2 + a_1^2 + a_2^2 - \mu^2 ,$$

$$G_{\phi_1\phi_1} = \frac{\rho^2 + a_1^2}{\rho^2 + a_2^2} , \quad G_{t\phi_1} = \frac{\mu - a_1}{\rho^2 + a_2^2} .$$

(3.20)
The energy and $U(1)$-change conservation laws give
\begin{align}
G_{tt} \dot{t} + G_{t\phi_1} \dot{\phi}_1 &= 1 , \\
G_{\phi_1\phi_1} \dot{\phi}_1 - G_{t\phi_1} \dot{t} &= J ,
\end{align}
with solution
\begin{align}
\dot{t} &= \frac{G_{\phi_1\phi_1} - JG_{t\phi_1}}{G_{t\phi_1}^2 + G_{tt}G_{\phi_1\phi_1}} = \frac{(\rho^2 + a_2^2)(\rho^2 + a_1^2 - J\mu a_1)}{\rho^4 + (a_1^2 + a_2^2 - \mu^2)\rho^2 + a_1^2a_2^2} , \\
\dot{\phi}_1 &= \frac{G_{t\phi_1} + JG_{tt}}{G_{t\phi_1}^2 + G_{tt}G_{\phi_1\phi_1}} = \frac{(\rho^2 + a_2^2)(\mu a_1 + J(\rho^2 + a_2^2 - \mu^2))}{\rho^4 + (a_1^2 + a_2^2 - \mu^2)\rho^2 + a_1^2a_2^2} .
\end{align}
In the above we understand that all of the expressions when are given in terms of general functions $G_{tt}$ etc. hold in general. Then, the nullness of the geodesic leads to
\begin{align}
\dot{\rho}^2 &= \frac{G_{\phi_1\phi_1} - 2JG_{t\phi_1} - J^2G_{tt}}{G_{\rho\rho}(G_{t\phi_1}^2 + G_{tt}G_{\phi_1\phi_1})} = \left(1 + \frac{a_2^2}{\rho^2}\right) \left[(J\mu - a_1)^2 - J^2a_2^2 + (1 - J^2)\rho^2\right] .
\end{align}
We then perform the change of variables
\begin{align}
d\rho &= \dot{\rho}du , \\
dt &= \dot{t}du - dv/N + Jdx/\sqrt{N} , \\
d\phi_1 &= \dot{\phi}_1du + dx/\sqrt{N} ,
\end{align}
where $\dot{\rho}$, $\dot{t}$ and $\dot{\phi}_1$ denote the functions in (3.23) and (3.22) and also let
\begin{align}
\theta &= \frac{\pi}{2} - \frac{z}{\sqrt{N}} , \\
x_i &\rightarrow x_i/\sqrt{N} .
\end{align}
Then, in the limit $N \rightarrow \infty$ we obtain the background
\begin{align}
ds^2 &= 2dudv + \sum_{a=1}^{5} dy_a^2 + dz^2 + \left(1 - J^2 + \frac{(a_1 - J\mu)^2 - a_2^2}{\rho^2 + a_2^2}\right) dx^2 - J^2z^2du^2 , \\
B_{12} &= -2Ju , \\
e^{-2\Phi} &= \rho^2 + a_2^2 ,
\end{align}
where for the antisymmetric tensor we have used the gauge freedom in its definition.

Let us first note that if the relation between the various parameters $(a_1 - J\mu)^2 = a_2^2$ holds, then we have the case of the four-dimensional plane wave of $[10]$ times $\mathbb{R}^6$. Also, if we shift $a_1 \rightarrow a_1 + J\mu$ the effect of non-extremality disappears from the above background as well as from the differential eq. (3.23). Solving the latter we obtain a metric of the form (3.1) with diagonal $F_{ij}$. In all cases $F_1 = F_2 = -J^2$, whereas the function $F_x$ and the dilaton depend on the range of various parameters. We have three different cases (we use the shifted $a_1$ below):
\[ G_{xx} = (1 - J^2) \coth^2 \sqrt{1 - J^2} u, \quad e^{-2\Phi} = e^{-2\Phi_0} \sinh^2 \sqrt{1 - J^2} u. \] (3.27)

\[ a_1^2 < a_2^2 \text{ and } J^2 < 1: \quad G_{xx} = (1 - J^2) \tanh^2 \sqrt{1 - J^2} u, \quad e^{-2\Phi} = e^{-2\Phi_0} \cosh^2 \sqrt{1 - J^2} u. \] (3.28)

\[ a_1^2 > a_2^2 \text{ and } J^2 > 1: \quad G_{xx} = (J^2 - 1) \cot^2 \sqrt{J^2 - 1} u, \quad e^{-2\Phi} = e^{-2\Phi_0} \sin^2 \sqrt{J^2 - 1} u. \] (3.29)

All cases are related by analytic continuations of the various parameters and variables.

Passing to the Brinkman coordinates we obtain a metric of the form

\[ ds^2 = 2dudv + \sum_{a=1}^{5} dy_a^2 + dx^2 + dz^2 + (F_{xx} - J^2 z^2) du^2, \] (3.30)

with

\[ F_x = 2 \frac{1 - J^2}{\sinh^2 \sqrt{1 - J^2} u}, \]
\[ F_x = -2 \frac{1 - J^2}{\cosh^2 \sqrt{1 - J^2} u}, \] (3.31)
\[ F_x = 2 \frac{J^2 - 1}{\sin^2 \sqrt{J^2 - 1} u}, \]

corresponding to the three cases above, respectively. Note that the last two cases correspond to plane waves identical to the ones we obtained in subsection 3.1 (up to a trivial redefinition of \( u \) in the trigonometric case). In addition, it can be checked that the first case is identical to the plane wave corresponding to the geodesic with \( \theta = \pi/2 \) in the background (2.9).

We started with the non-supersymmetric, non-extremal rotating NS5-background (2.10)-(2.12) and obtained after the Penrose limit was taken, the same plane waves as those we obtained from the supersymmetric extremal background (2.9) in the Penrose limit. The physical reason for this is related to the fact that global information about a space-time is lost after the Penrose limit is taken, since essentially we focus and expand the space-time seen by a particular null geodesic. Hence, there is no notion of a horizon in the plane wave geometry and of the Hawking temperature associated with it. This is related to the apparent loss of holography in the Penrose limit (for discussion and some work in these directions see, for instance, [36, 37]). We believe that holography is
present, but encoded in a string theoretical way beyond the supergravity approximation. It can be explicitly checked, using results of [27, 28], that the Hawking temperature for the metric (2.10) goes to zero in the Penrose limit. Hence, states in the theory whose mass degeneracy is broken due to thermal effects, acquire again the same mass, resulting into a restoration of supersymmetry. Having an exact CFT description in our case must be very important in working out the details of such a scenario.

4 The relation to exact CFT

In this section we relate the plane waves (3.13) and (3.18) to backgrounds corresponding to exact CFTs through a Penrose limit. We also construct the corresponding symmetry algebra, via the associated with the Penrose limit, contraction.

We start with the backgrounds corresponding to the $SU(2)_N \times SL(2, \mathbb{R})_{N'}/U(1) \times \mathbb{R}^5$ exact CFT

$$ds^2 = N (d\theta^2 + d\phi^2 + d\psi^2 + 2 \cos \theta d\phi d\psi) + N'(-dt^2 + \tanh^2 td\omega^2) + \sum_{i=1}^{5} dx_i^2,$$

$$B_{\phi\psi} = N \cos \theta,$$

$$e^{-2\phi} = e^{-2\Phi_0} \cosh^2 t.$$  \hfill (4.1)

Consider the change of variables

$$\theta = \frac{\psi}{2JN} + 2Ju,$$

$$t = \sqrt{1 - J^2 u},$$

$$\phi = \frac{1}{2\sqrt{N}}(x_1 + x_2), \quad \psi = \frac{1}{2\sqrt{N}}(x_1 - x_2), \quad \omega = \sqrt{\frac{N/N'}{N + N'}} x$$  \hfill (4.2)

and take the limit

$$N, N' \rightarrow \infty, \quad J^2 = \frac{N'}{N + N'} = \text{finite}. \quad (4.3)$$

We obtain

$$ds^2 = 2dudv + \sum_{i=1}^{5} dx_i^2 + dx^2 + \cos^2 Ju dx_1^2 + \sin^2 Ju dx_2^2 + (1 - J^2) \tanh^2 \sqrt{1 - J^2} u,$$

$$B_{12} = \cos 2Ju,$$

$$e^{-2\Phi} = e^{-2\Phi_0} \cosh^2 \sqrt{1 - J^2} u.$$  \hfill (4.4)

Passing to the Brinkman coordinates we obtain (3.13). We also note that, had we started with the background for $SU(2)_{N'}/U(1) \times SL(2, \mathbb{R})_{-N} \times \mathbb{R}^5$ exact CFT

$$ds^2 = N(-dt + d\phi^2 + d\psi^2 + 2 \cosh td\phi d\psi) + N'(d\theta^2 + \tan^2 \theta d\omega^2) + \sum_{i=1}^{5} dx_i^2,$$
\[ B_{\phi \psi} = N \cosh t , \quad (4.5) \]
\[ e^{-2\phi} = e^{-2\Phi_0} \cos^2 \theta , \]
we would have obtained in a Penrose limit, similar to (4.2), (4.3), the plane wave (3.18).

In order to construct the symmetry algebra for our plane waves (3.13) and (3.18) we should start with the symmetry algebras of the original backgrounds before the Penrose limit was taken. We do that explicitly for the plane wave in (3.18) and start with the symmetry algebra corresponding to the background (4.5). The \( IR^5 \) factor corresponds to five free currents and therefore will be ignored in our discussion. For the WZW model factor \( SL(2, IR)_{-N} \) we have a current algebra theory generated by the three currents \( J_0, J_\pm \) of conformal dimension \( \Delta = 1 \) which obey the Operator Product Expansions (OPE's)

\[ J_0(z) J_0(w) = -\frac{N/2}{(z-w)^2} + \text{regular} , \]
\[ J_0(z) J_\pm(w) = \frac{\pm J_\pm(w)}{z-w} + \text{regular} , \quad (4.6) \]
\[ J_+(z) J_-(w) = -\frac{2J_0(w)}{z-w} + \frac{N}{(z-w)^2} + \text{regular} . \quad (4.7) \]

The symmetry algebra corresponding to the \( SU(2)_{N'/U(1)} \) factor is the parafermionic algebra of \([3]\). This algebra is generated by chiral operators \( \psi_l, l = 0, 1, \ldots N'-1 \), with the complex conjugation acting as \( \psi^\dagger = \psi_{N'-l} \). Their conformal dimension is \( \Delta_l = l(N' - l)/N' \) and hence it differs from the classical result \( \Delta_l^{\text{clas}} = l \) (for finite \( l \)). This will have very important consequences for the nature of the exact CFT that follow from the contraction in the \( N, N' \to \infty \) limit. Here we are particularly interested for the two parafermionic operators \( \psi_1 \) and its complex conjugate \( \psi_1^\dagger \), with the lowest conformal dimension \( \Delta_1 = 1 - 1/N' \). Their OPE's are

\[ \psi_1(z) \psi_1(w) = \frac{\sqrt{2(1 - 1/N')}}{(z-w)^{2/N'}} (\psi_2(w) + O(z-w)) , \]
\[ \psi_1(z) \psi_1^\dagger(w) = \frac{1}{(z-w)^{2\Delta_1}} \left( 1 + \left( 1 + \frac{2}{N'} \right) (z-w)^2 T_{\text{par}}(w) + O(z-w)^2 \right) . \quad (4.8) \]

where \( T_{\text{par}} \) is the Virasoro stress energy tensor of the parafermionic theory.

### 4.1 The contraction

The quantum OPE’s that we have exhibited, have their classical counterparts in terms of Poisson brackets. These are obeyed by the classical versions of the currents \( J_0, J_\pm \) and of the parafermions \( \psi_1 \) and \( \psi_1^\dagger \) which are realized in terms of the spacetime variables
\( t, \phi, \psi, \theta \) and \( \omega \). Hence, the Penrose limit for the background (3.18) will give rise to a limiting procedure for the corresponding symmetry generating classical objects. This is a quite straightforward, but lengthly, procedure and, as it turns out, it gives rise to a contraction of the Poisson bracket algebra that the symmetry generators obey. At this more abstract level the result can be taken over and be applied to the quantum case, where OPE’s replace the classical Poisson brackets. For the special case of the plane wave obtained as a Penrose limit of the background for \( SU(2)_N/U(1) \times U(1) \) exact CFT this procedure was carried out in full detail in [2]. Here we skip all details and only quote the end result dealing directly with the quantum case. We define new operators \( \Phi, \Psi, P, P_\pm \) as

\[
(1 - J^2)^{1/2} \Phi = \frac{i}{2\sqrt{N'}}(\psi_1 - \psi_1^\dagger) - \frac{1}{N'} \frac{J}{\sqrt{1 - J^2}} J_0, \\
(1 - J^2)^{-1/2} \Psi = \frac{i\sqrt{N'}}{2}(\psi_1 + \psi_1^\dagger) + \frac{J}{\sqrt{1 - J^2}} J_0, \\
P = \frac{\psi_1 + \psi_1^\dagger}{2}, \\
P_\pm = \sqrt{N}J_\pm.
\]

In the limit (4.3), all fields are primaries of the total Virasoro stress tensor with conformal dimension equal to one, except for \( \Psi \) for which we find

\[
T(z) \Psi(w) = \frac{\Psi(w) - \frac{1}{2}(1 - J^2) \Psi(w)}{(z - w)^2} + \frac{\partial \Psi(w)}{z - w} + \text{regular}.
\]

Hence, \( \Psi \) is not a primary field unless \( J^2 = 1 \). In addition, we find that

\[
\Psi(z) \Psi(w) = \frac{1}{(z - w)^2}, \quad \Phi(z) \Phi(w) = \text{regular}.
\]

These are the defining relations of a Logarithmic CFT [6]. The essential reason that a logarithmic structure for the CFT arose is that the original theory contains as basic chiral operators the lowest parafermions with conformal dimension \( 1 - 1/N' \) as well as the \( SL(2, \mathbb{R})_{-N} \) currents with dimension exactly 1 for all \( N \). Hence, although classically all operators have dimension 1, quantum mechanically this dimension is protected only in the case of the \( SL(2, \mathbb{R})_{-N} \) currents. The remnant of this, in the limit \( N', N \to \infty \), is the fact that \( \Psi \) is not a primary field. This phenomenon did not occur in the similar contraction of current algebra theories [12, 13] since the combinations of currents that one forms in order to take the limit \( N \to \infty \), have conformal dimension 1 for all \( N \).

The algebra of the logarithmic partners \( (\Phi, \Psi) \) in (4.11) is enhanced by the presence of non-trivial OPE’s corresponding to the other fields in the theory. We find

\[
P_+(z) P_-(w) = \frac{1}{(z - w)^2} + J \frac{\Phi(w)}{z - w} + \text{regular},
\]
\begin{align}
\Psi(z)P_\pm(w) &= \pm J P_\pm(w) + \text{regular}, \\
\Psi(z)P(w) &= -(1 - J^2) \ln(z - w)(P\Phi)(w) + \text{regular}, \quad (4.12) \\
P(w)P(w) &= \frac{1}{2(z-w)^2} + \frac{1-J^2}{2} \ln(z-w)\Phi^2(w) + \text{regular}, \\
\Psi(z)\Psi(w) &= (1 - J^2) \left[ \frac{\ln(z-w)}{(z-w)^2} + 2 \ln(z-w)(P^2)(w) \right] \\
&\quad + \frac{(1-J^2)^2}{2} \ln^2(z-w)\Phi^2(w) + \text{regular}.
\end{align}

For \( J = 0 \) the operators \( P_\pm \) become abelian currents and decouple from \( \Phi, \Psi \) and \( P \). The later obey the three-dimensional Logarithmic CFT found in [2]. For \( |J| = 1 \) the operator \( P \) becomes an abelian current and decouples from \( \Phi, \Psi \) and \( P_\pm \) which, then, obey the current algebra based on the non-semisimple group \( H_4 \) [10]. In this case we also note that \( \Psi \) is a primary field. This limiting behaviour is in agreement with the behaviour we have already noted for the background (3.13).

A quite straightforward generalization of our results to higher dimensional cases follows by replacing the coset factor \( SU(2)_{N'}/U(1) \) with any compact coset CFT. For the particular case of \( SO(D+1)_{N'}/SO(D) \), this is immediate using the results of the second ref. in [2].

### 4.2 Free fields

In order to reproduce the operator algebra (4.12) using free fields, we utilize the free field realizations of the coset model \( SU(2)_{N'}/U(1) \) and of the \( SL(2,\mathbb{R})_{-N} \) current algebra at finite \( N, N' \) which are known [38]. We start with the former model and introduce two real free bosons,

\[ \langle \phi_i(z)\phi_j(w) \rangle = -\delta_{ij} \ln(z-w), \quad i, j = 1, 2. \quad (4.13) \]

One may represent the elementary parafermion currents as\(^2\)

\[ \psi_1 = \frac{1}{\sqrt{2}} \left( -\sqrt{1+2/N'} \partial \phi_1 + i \partial \phi_2 \right) e^{+\sqrt{2/N'} \phi_2}, \]
\[ \psi_1^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{1+2/N'} \partial \phi_1 + i \partial \phi_2 \right) e^{-\sqrt{2/N'} \phi_2}. \quad (4.14) \]

For the free field realization of the current algebra for \( SL(2,\mathbb{R})_{-N} \) we need a time-like free boson \( \phi_0 \) and two spacelike free bosons \( \varphi_i, i = 1, 2 \). They obey

\[ \langle \varphi_i(z)\varphi_j(w) \rangle = -\delta_{ij} \ln(z-w), \quad i, j = 1, 2, \]

\(^2\)All expressions appearing in the rest of this paper to involve products of free bosons and their derivatives, are understood as being properly normal ordered.
\begin{align}
\langle \phi_0(z)\phi_0(w) \rangle &= \ln(z - w). 
\end{align}

Then the currents are given by

\begin{align}
J_\pm &= \sqrt{\frac{N}{2}} \left( \mp \sqrt{1 - \frac{2}{N}} \partial \varphi_1 + i \partial \varphi_2 \right) e^{\mp i \sqrt{2/N} (\phi_0 - \varphi_2)}, \\
J_0 &= i \sqrt{\frac{N}{2}} \partial \phi_0. 
\end{align}

The stress-energy tensor of the entire five-dimensional theory is

\begin{align}
T(z) &= -\frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{i}{\sqrt{2(N' + 2)}} \partial^2 \phi_1 \\
&\quad + \frac{1}{2} (\partial \phi_0)^2 - \frac{1}{2} (\partial \varphi_1)^2 - \frac{1}{2} (\partial \varphi_2)^2 + \frac{1}{\sqrt{2(N - 2)}} \partial^2 \varphi_1 
\end{align}

and corresponds to a central charge $c = 3N'/(N' + 2) - 1 + 3N/(N - 2)$ theory, as it should be. Let us now consider the scalar field redefinition

\begin{align}
(1 - J^2)^{1/2} \phi_+ &= \frac{1}{\sqrt{2N'}} (\phi_0 + \phi_1), \quad (1 - J^2)^{-1/2} \varphi_- = \frac{\sqrt{N'}}{2} (\phi_0 - \phi_1), \\
\phi_2 &= \phi, \quad \varphi_\pm = \frac{1}{\sqrt{2}} (\mp \varphi_1 + i \varphi_2).
\end{align}

Then, the new set of scalars obey

\begin{align}
\langle \phi_+(z)\phi_-(w) \rangle = \langle \varphi_+(z)\varphi_-(w) \rangle = -\langle \phi(z)\phi(w) \rangle = \ln(z - w) 
\end{align}

and have zero correlators otherwise. In order to take the $N, N' \to \infty$ limit, the expansion

\begin{align}
\psi_1 &= -\frac{\sqrt{N'}}{2} \partial \phi_+ + \frac{1}{\sqrt{2}} (i \partial \phi - \phi \partial \phi_+) + \frac{1}{\sqrt{N'}} \left( \frac{1}{2} \partial \varphi_- + i \phi \partial \phi - \frac{1}{2} (\phi^2 + 1) \partial \phi_+ \right) + O \left( \frac{1}{N'} \right), 
\end{align}

and its conjugate one for $\psi_1^\dagger$ are useful. After some algebra we find the following free field representation for the basic operators of our five-dimensional Logarithmic CFT

\begin{align}
\Phi &= -i \partial \phi_+ , \\
P_\pm &= \partial \varphi_\pm e^{\mp J \phi_+} , \\
P &= \frac{1}{\sqrt{2}} \left( i \partial \phi - \sqrt{1 - J^2} \phi \partial \phi_+ \right) , \\
\Psi &= i \partial \phi_- - \frac{i}{2} (1 - J^2) (\phi^2 + 1) \partial \phi_+ - \sqrt{1 - J^2} \phi \partial \phi.
\end{align}

\textsuperscript{3}These are of course nothing but the free field realization of the non-compact parafermions [39] dressed with the extra boson $\phi_0$. 
The Virasoro stress energy tensor expressed in terms of free fields becomes

\[ T(z) = \partial \phi_+ \partial \phi_- - \frac{1}{2} (\partial \phi)^2 + \partial \varphi_+ \partial \varphi_- - \frac{i}{2} \sqrt{1 - J^2} \partial^2 \varphi_+ \]  

and corresponds to a central change \( c = 5 \) theory. It is straightforward to check that these obey the OPE in \((4.12)\). We also note that for \( J = 0 \) we obtain the free field representation of the three-dimensional Logarithmic CFT of [2] and for \(|J| = 1\) that of the current algebra for the non-semisimple group \( H_4 \) found in [40].

Using \((4.9)\), the known correlation functions for the currents and of the parafermions and then taking the contraction limit, we may compute correlation functions for the basic operators of our theory \( \Phi, \Psi, P \) and \( P_\pm \). Alternatively, we may use the free field representation we have derived. These computations will not be presented here, but the interested reader may get an idea of their form from the corresponding correlators in [2], where the parameter \( J = 0 \).

5 Strings in light-cone gauge

The purpose of this section is to explicitly solve for the string spectrum in the light-cone gauge. This is possible even though our plane wave backgrounds have \( u \)-dependent profiles. String theory on plane wave backgrounds with only NS-NS fields turned on and time dependent profiles of the form \( 1/u^2 \) have been solved in the light-cone gauge in [41, 42]. For extensive discussions of string on other NS-NS backgrounds the reader is referred to [43, 44].

5.1 General formalism

Let us first review the light-cone approach to string propagation in plane backgrounds (see, for instance, [30, 31]) suitably fitted for our purposes. We will consider bosonic backgrounds of the form \((3.1)\) with diagonal \( F_{ij} = F_i \delta_{ij} \) and zero antisymmetric tensor. This contains the case with \( J = 0 \) for our backgrounds \((3.13)\) and \((3.18)\). The extension to cases with \( J \neq 0 \) is easily done and will be briefly mentioned at the end of subsection 5.2. We will consider the variable \( u \) to be non-compact and let, as usual, \( u = P \tau \). The light-cone action for the transverse coordinates is

\[ S = \frac{1}{4\pi} \int d\tau d\sigma \left( \partial_\tau X^i \partial_\tau X^i - \partial_\sigma X^i \partial_\sigma X^i + P^2 F_i X^i \right) . \]  

The classical equations of motion are

\[ \delta X^i : - \partial_\tau^2 X^i + \partial_\sigma^2 X^i + P^2 F_i X^i = 0 \]  

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\[ \Pi^i = \frac{\delta S}{\delta X^i} = \frac{1}{2\pi} \partial_{\tau} X^i , \]  
(5.3)
denotes the conjugate to \( X^i \) momenta. Using these the Hamiltonian takes the form
\[ H = \frac{1}{4\pi} \int_{0}^{2\pi} d\sigma \left( \partial_{\tau} X^i \partial_{\tau} X^i + \partial_{\sigma} X^i \partial_{\sigma} X^i - P^2 F_i X^i \right) . \]  
(5.4)

We expand the string coordinates in Fourier modes as
\[ X^i(\sigma, \tau) = \sum_{n \neq 0} \frac{1}{n} e^{i n \sigma} \left( a^i_n X^i_n(\tau) - \bar{a}^i_{-n} X^i_{n}(\tau) \right) + X^i_0(\tau) , \]  
(5.5)
where we have used \( a^i_n = a^i_{-n} \) and \( X^i_{n} = X^i_{-n} \), which follows from the reality condition for the \( X^i \)'s. It will be convenient to identify a complex basis for the zero mode solution as
\[ X^i_0(\tau) = a^i \chi^i(\tau) + a^i \bar{\chi}^i(\tau) . \]  
(5.6)
Then using (5.2) we find that the amplitudes obey the harmonic oscillator equations with \( \tau \)-dependent frequencies
\[ \ddot{X}^i_n + (\omega^i_n)^2 X^i_n = 0 , \quad \ddot{\chi}^i + (\omega^i_0)^2 \chi^i = 0 , \]  
\[ (\omega^i_n)^2 = n^2 - P^2 F_i , \]  
(5.7)
where dots will denote ordinary derivatives with respect to \( \tau \). It is particularly useful to think of (5.7) as a Schrödinger equation
\[ -\ddot{X}^i_n + V^i X^i_n = n^2 X^i_n , \]  
(5.8)
where the potential is given by \( V^i = P^2 F_i \). For each \( i \), the differential equation (5.7) has two independent solutions. Their Wronskian will be normalized as
\[ \dot{X}^i_n \dot{X}^{i*}_n - X^i_n \dot{X}^{i*}_n = i n , \quad n \neq 0 , \quad \chi^i \dot{\chi}^i - \dot{\chi}^i \chi^{i*} = i . \]  
(5.9)
In a canonical quantization, one promotes \( X^i, \Pi^i, a_n^i, \bar{a}_n^i \) and \( a^i, a^{i*} \) to operators and starts from the basic commutators

\[
[X^i(\sigma, \tau), X^j(\sigma', \tau)] = [\Pi^i(\sigma, \tau), \Pi^j(\sigma', \tau)] = 0 ,
\]

\[
[X^i(\sigma, \tau), \Pi^j(\sigma', \tau)] = i\delta^{ij}\delta(\sigma-\sigma') .
\] (5.10)

These give rise to

\[
[a_n^i, a_m^j] = n\delta^{ij}\delta_{n+m} , \quad [a^i, a^{j*}] = \delta^{ij} ,
\] (5.11)

provided that the normalization (5.9) is chosen. Replacing the expansion (5.5) into (5.4) we obtain for the Hamiltonian

\[
H = H_0 + H_{\text{string}} ,
\] (5.12)

where \( H_0 \) is the part of the Hamiltonian corresponding to the zero mode

\[
H_0 = \frac{1}{2} (P_0^i P_0^i + (\omega_0^i)^2 X_0^i X_0^i)
\]

\[
= \frac{1}{2} \left[ \dot{\chi}^i \chi^i + (\omega_0^i)^2 \chi^i \chi^i \right] a^2 + \frac{1}{2} \left[ \dot{\chi}^{i*} \chi^{i*} + (\omega_0^i)^2 \chi^{i*} \chi^{i*} \right] (a^i)^2
\]

\[
+ \left[ \dot{\chi}^i \chi^{i*} + (\omega_0^i)^2 \chi^i \chi^{i*} \right] (a^i a^i + \frac{1}{2}) .
\] (5.13)

The relation to position and momentum operators is

\[
X_0^i = a^i(\tau)\chi^i + a^{i*}(\tau)^\dagger \chi^{i*} ,
\]

\[
P_0^i = a^i(\tau)\dot{\chi}^i + a^{i*}(\tau)^\dagger \dot{\chi}^{i*} .
\] (5.14)

Since in the Schrödinger picture \( X_0^i \) and \( P_0^i \) are time-independent operators we have that

\[
\frac{da^i}{d\tau} = \frac{\partial a^i}{\partial \tau} + i[H_0, a^i] = 0 ,
\] (5.15)

which can be proved by using (5.14) and the equation of motion for \( \chi^i \). We have denoted by \( H_{\text{string}} \) the part of the Hamiltonian containing the string oscillation modes

\[
H_{\text{string}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \left( \dot{X}_n^i X_n^i + (\omega_n^i)^2 X_n^i X_n^i \right) (a_n^i a_n^{i*} + \bar{a}_n^i \bar{a}_n^{i*})
\]

\[
- \left( \dot{X}_n^i X_n^i + (\omega_n^i)^2 X_n^i X_n^i \right) a_n^{i*} a_n^i - \left( \dot{X}_n^{i*} X_n^{i*} + (\omega_n^{i*})^2 X_n^{i*} X_n^{i*} \right) a_n^{i*} a_n^i \right] .
\] (5.16)

In the rest of this subsection we suppress for simplicity the index \( i \).

It is convenient to write \( \chi = re^{i\phi} \) and \( X_n = r_n e^{i\phi_n} \) and then consider the equations for the amplitudes \( r, r_n \) and the phases \( \phi \) and \( \phi_n \). Using (5.7) and (5.9) we find that

\[
\dot{r} + \omega_0^2 r - \frac{1}{4r^3} = 0 , \quad \dot{\phi} = -\frac{1}{2r^2} ,
\] (5.17)
as well as
\[ r_n + \omega_n^2 r_n - \frac{n^2}{4r_n^3} = 0 \, , \quad \phi_n = \frac{n}{2r_n^2} \, . \tag{5.18} \]

Let us also note that the stringy-part of the Hamiltonian is not diagonal, but it can be diagonalized by means of the transformation
\[
\begin{pmatrix} A_n \\ \tilde{A}_n^\dagger \end{pmatrix} = \begin{pmatrix} f_n & f_{-n} \\ -f_{-n}^* & -f_n^* \end{pmatrix} \begin{pmatrix} a_n \\ a_{-n} \end{pmatrix},
\]
and its conjugation, where
\[
f_n = \frac{1}{n\sqrt{2}\omega_n} (\dot{X}_n + i\omega_n X_n) = \frac{1}{n\sqrt{2}\omega_n} e^{i\phi_n} \left[ \dot{r}_n + i \left( \omega_n r_n + \frac{n}{2r_n^2} \right) \right]. \tag{5.20} \]

Then we have the commutation rules
\[
[A_n, A_m^\dagger] = \delta_{n-m} \, , \quad [\tilde{A}_n, \tilde{A}_m^\dagger] = \delta_{n-m}
\]
and zero otherwise. Then the expansion (5.5) becomes
\[
X(\sigma, \tau) = X_0(\tau) + i \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}\omega_n} e^{in\sigma} (A_n - \tilde{A}_n^\dagger) + e^{-in\sigma} (A_n^\dagger - A_n) \, . \tag{5.22} \]

In addition, (5.16) assumes the diagonal form
\[
H_{\text{string}} = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n (A_n^\dagger A_n + \tilde{A}_n^\dagger \tilde{A}_n) \, . \tag{5.23} \]

The previous discussion was purely bosonic and in the supersymmetric case should be supplemented by the inclusion of world-sheet fermions. In addition, we have the level matching conditions arising from the expression for the light-cone variable \(v\) in terms of the transverse coordinates \(X^i\). As we are interested in some rather generic features for string propagation in our backgrounds such discussions will be omitted in this paper. We also note that in the case of constant frequency \(\omega_n^2 = n^2\) we have, according to our normalizations, that \(X_n = e^{inr}/\sqrt{2}\) and we recover familiar results for strings propagation in flat Minkowski space-time in the light-cone gauge.

### 5.1.1 The particle-like spectrum

Let's us discuss in some detail the particle spectrum corresponding to the zero-mode Hamiltonian \(H_0\). The problem at hand is to determine a complete set of solutions to the time-dependent Schrödinger equation
\[
(i\partial_\tau - H_0)\Psi_n(x, \tau) = 0 \, , \tag{5.24} \]
where $H_0$ is given in (5.13). This Hamiltonian is that for a quantum oscillator with time dependent frequency $\omega_0(\tau)$. It is possible to solve (5.24) for any $\omega_0(\tau)$.

Using the oscillator operators $a$ and $a^\dagger$ we can define, at a given $\tau$ (and $i$), a Fock space of states as usual

$$a|0,\tau\rangle = 0, \quad |n,\tau\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0,\tau\rangle. \quad (5.25)$$

In configuration space the eigenstates $\Psi_n(x, \tau) = \langle x|n,\tau\rangle$ can be constructed in terms of Hermite polynomials. Using the representation of $a$ and $a^\dagger$ as first order differential operators we find that the normalized to one zero-mode eigenstate satisfies

$$a|0,\tau\rangle = 0 \Rightarrow \left(\chi^* \partial_x - i\dot{\chi}^* x\right)\Psi_0 = 0 \Rightarrow \Psi_0(x, \tau) = (\sqrt{2\pi}|\chi|)^{-1/2} e^{i\chi^* x^2}. \quad (5.26)$$

Then using (5.25) and the defining differential relation for Hermite Polynomials, we find that in configuration space

$$\Psi_n(x, \tau) = (\sqrt{2\pi}2^n2^n|\chi|)^{-1/2} \left(\frac{\chi}{\chi^*}\right)^{n/2} e^{i\chi^* x^2} H_n\left(\frac{x}{\sqrt{2}|\chi|}\right)$$

$$= (\sqrt{2\pi}2^n n!r)^{-1/2} e^{i\phi} e^{i\left(\frac{1}{4}r^2 - \frac{1}{4}r^2\right)} x^2 H_n\left(\frac{x}{\sqrt{2r}}\right), \quad (5.27)$$

where in the last step we have used (5.17). The advantage of this space is that we can write down immediately states $|\bar{n},\tau\rangle$ that obey the operator equation (5.24), simply as

$$\bar{\Psi}_n(x, \tau) = e^{i\theta(\tau)}\Psi_n(x, \tau), \quad (5.28)$$

where the phase $\theta(\tau)$ is independent of the quantum number $n$ and is determined by integrating the equation

$$\dot{\theta} = -\dot{r}^2 - \omega_0^2 r^2. \quad (5.29)$$

We note that in the case of constant frequency we have: $\chi = e^{-i\omega_0 \tau}/\sqrt{2\omega_0}$ and $\theta = -\frac{1}{2}\omega_0 \tau$. Then the total time dependence in (5.28) is in a phase factor which reads $-\omega_0(n + \frac{1}{2})\tau$, as it should be.

### 5.2 Application for our plane wave backgrounds

We now apply the general formalism for our backgrounds (3.13) and (3.18). For the modes corresponding to the spatial brane directions $y_a$ we get the expected eigenfrequencies $\omega_{an} = n^2$. For the eigenfrequencies corresponding to the transverse coordinates $z_i$, $i = 1, 2$ the general formalism we have developed is not adequate since we assumed zero

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4 For the description of the method see [45] and references therein. We also note that the method has been used recently in the present context in [41, 46].

5 We correct below an apparent typo in eq. (A8) of [45].
antisymmetric tensor field. However, it is easy to see that we get a system of two coupled differential equations

\[ \frac{d^2 z_{in}}{d\tau^2} + (n^2 + P^2 J^2) z_{in} = 2iPJn\epsilon_{ij} z_{jn}, \]  

(5.30)

where the term on the right hand side is due to the non-vanishing antisymmetric tensor. We easily see that the combinations \( z_{\pm n} = z_{1n} \pm iz_{2n} \) diagonalize the system. The corresponding eigenfrequencies are

\[ \omega_{\pm n}^2 = (n \pm PJ)^2 \]  

(5.31)

and coincide with those found in [47] for the light-cone treatment of strings in the plane wave background of [10] (for a recent discussion that takes into account the world-sheet fermions see [48]).

In the rest of this subsection we set the parameter \( J = 0 \) and concentrate on the three-dimensional plane wave, arising from the non-trivial parts of (3.13) and (3.18).

For the modes corresponding to the coordinate \( x \) we get a Schrödinger equation as in (5.8) with potential that depends on the specific plane wave background. In particular, in the case of the background (3.13) we have

\[ V = -2 \frac{P^2}{\cosh^2 (P\tau)} . \]  

(5.32)

This belongs to a class of reflectionless potentials which can support a finite number of bound states. The reason for this is that this potential and the corresponding eigenvalue problem is related to the problem with constant potential within the context of supersymmetric quantum mechanics [49]. In our case there is exactly one bound state. However, it is not admissible to the perturbative string spectrum since it corresponds to setting \( n \) equal to the imaginary unit and that destroys the periodicity in the spatial string world-sheet variable \( \sigma \). The explicit solution for the states is (we follow the normalization of (5.9))

\[ x_n(\tau) = \frac{1}{\sqrt{2}} \frac{i n - P \tanh a \tau}{i n + P} e^{in\tau}, \quad n \neq 0, \quad -\infty < \tau < +\infty , \]  

(5.33)

which clearly exhibits the reflectionless behaviour of the potential since the wavefunctions at \( \tau \to -\infty \) and at \( \tau \to +\infty \) differ only by a phase factor. Hence in our plane-wave background we do not have string-mode creation. For the string part of the Hamiltonian we get

\[ H_{\text{string}} = \frac{1}{2} \sum_{n=1}^{\infty} \Gamma_n (a_n a_{-n} + \tilde{a}_n \tilde{a}_{-n}) + \Delta_n a_n \tilde{a}_n + \Delta_n^* a_{-n} \tilde{a}_{-n} , \]  

(5.34)
\[ \Gamma_n = 1 + \frac{P^2}{n^2 \cosh^2 P \tau} - \frac{P^4}{2n^2(P^2 + n^2) \cosh^4 P \tau}, \]
\[ \Delta_n = \frac{1}{2} \frac{P^3}{n^2 \cosh P \tau} \frac{e^{2in\tau}}{P \cosh 2P \tau - i(n + \frac{1}{2})}. \] (5.35)

For \( \tau \to \pm \infty \) this part of the Hamiltonian becomes diagonal and takes the same form as in the case of strings in flat space, i.e. \( H_{\text{string}}(\pm \infty) = \frac{1}{2} \sum_{n=1}^{\infty} (a_n a_{-n} + \tilde{a}_n \tilde{a}_{-n}). \)

For the zero mode we have the, properly normalized, solution
\[ \chi(\tau) = \left( \frac{2}{\sqrt{2}} \right)^{-1/2} \exp \left( \frac{1}{2} \left( P\tau \tanh P\tau - 1 \right)^2 + \tanh^2 P\tau \right), \] (5.36)
from which we deduce the expressions for the amplitude \( r(\tau) \) and the phase \( \phi(\tau) \)
\[ r^2 = \frac{1}{\sqrt{2} P} \left( \frac{1}{2} (P\tau \tanh P\tau - 1)^2 + \tanh^2 P\tau \right), \]
\[ \tan \phi = \frac{\sqrt{2} \tanh P\tau}{P\tau \tanh P\tau - 1}. \] (5.37)

This Hamiltonian has of course the form (5.13) where the coefficients of the quadratic operators are quite complicated to written down explicitly. We note that the constants of integration in (5.36) have been chosen so that the Hamiltonian at \( \tau = 0 \) is diagonal and has the form \( H_0(\tau = 0) = \sqrt{2} P(a^\dagger a + \frac{1}{2}). \) With this choice the solution at \( \tau \to \pm \infty \) represents outgoing freely moving particle states.

Of particular interest is the amplitude to find the particle at a state \( \Psi_n(x, \tau) \) having started at \( \tau \to -\infty \), where the potential vanishes, as a plane wave, i.e. \( \Psi_p^{\text{in}}(x, -\infty) = \frac{1}{\sqrt{2\pi}} e^{ipx}, \) where \( p \) is the momentum. The matrix element describing this is given by
\[ \langle \Psi_n(\tau) | \Psi_p^{\text{in}}(-\infty) \rangle = (2\pi \sqrt{2n} n!)^{-1/2} e^{-i(n\phi + \theta)} \int_{-\infty}^{\infty} dx e^{ipx - (i \frac{x^2}{2r} + \frac{1}{4} \frac{r^2}{x^2})} H_n \left( \frac{x}{\sqrt{2r}} \right). \] (5.38)

The integral is computed using the generating function for Hermite polynomials. We find
\[ \langle \Psi_n(\tau) | \Psi_p^{\text{in}}(-\infty) \rangle = \left( \frac{2}{\pi} \right)^{1/4} \sqrt{\frac{r}{2n n!}} e^{-i(n\phi + \theta)} e^{2 \xi_p^2 x^2 + i} \left( \frac{2r \dot{r} + i}{2r \dot{r} - i} \right)^{n/2} e^{-\frac{1}{2} \xi_p^2} H_n(\xi_p), \] (5.39)
where \( \xi_p \) is a time-dependent function
\[ \xi_p(\tau) = \frac{\sqrt{2rp}}{\sqrt{4r^2 \dot{r}^2 + 1}}, \] (5.40)
Therefore the corresponding amplitude is
\[ |\langle \Psi_n(\tau) | \Psi_p^{\text{in}}(-\infty) \rangle|^2 = \left( \frac{2}{\pi} \right)^{1/2} \frac{r}{2n n!} e^{-\xi_p^2} H_n^2(\xi_p). \] (5.41)
This expression is completely general and holds for all time-dependent frequencies that go to zero at $\tau \to -\infty$. In our case we may find the explicit $\tau$-dependence of the solution by computing $\xi_p$ using (5.37). The result is not particularly simple and will not be written down here.

We end this section by briefly discussing the case of strings propagating in the background (3.18), in the light-cone gauge, which has some distinct features. In this case the modes corresponding to the coordinate $x$ obey a Schrödinger equation with potential

$$V = 2\frac{P^2}{\sin^2(P\tau)}, \quad 0 \leq P\tau \leq \pi,$$

where we have shifted $P\tau$ by $\pi/2$. Similarly to (5.32), the Schrödinger problem with this potential is related, via supersymmetric quantum mechanics, to the same problem with constant potential in the same finite interval for $\tau$. It turns out that the wavefunctions that vanish at $\tau = 0$ and $\tau = \pi/P$ are

$$x_{n,m} = (m+2)P \cos((m+2)P\tau) - P \cot P\tau \sin((m+2)P\tau), \quad m = 0, 1, \ldots,$$

provided that $n^2 = (m+2)^2 P^2$. Hence the light-cone momentum $P$ is quantized in order for a non-trivial solution to exist and in particular it should be a rational number. Given such a number, for each value $m$ there is only one string mode $n$, given by the above relation, that can be excited. We also note the absence of the zero mode corresponding to $n = 0$. The phenomenon of the quantization of the light-cone parameter $P$ in the present context was found before in plane waves constructed from continuously distributed D3-branes [50]. Clearly, restricting to the finite interval $\tau \in (0, \pi/P)$ is related to the fact the singularities of the metric in (3.18), where the light-cone gauge breaks down. We expect that a covariant quantization of the string will shed light on this issue.

6 Concluding remarks

We constructed plane wave backgrounds corresponding to Penrose limits of NS5-branes. The latter have a transverse space symmetry group $SO(2) \times Z_N \in SO(4)$ and give rise to time-dependent profiles for the plane wave solutions. We identify the corresponding exact theory as the five-dimensional Logarithmic CFT arising from the contraction of the $SU(2)_N/U(1) \times SL(2,R)_{-N}$ exact CFT, times $\mathbb{R}^5$. We constructed a free field representation for this theory and studied explicitly string propagation and spectra in the light-cone gauge. In view of the delicate issues concerning the validity of a uniform choice for the light cone gauge and states with light-cone momentum $P = 0$, it will be desirable to also develop the covariant approach and construct string scattering amplitudes. Work
towards this direction has been undertaken for the plane wave of [10], corresponding to $|J| = 1$ and a current algebra in our class of models, in [40] and more recently in [51]. It will be desirable to develop a similar approach at least for the case with $J = 0$, corresponding to a purely Logarithmic CFT.

One of our motivations in considering the Penrose limits of brane distributions was to extend the considerations of [22] for the sector of $\mathcal{N} = 4$ SYM with high spin and dimension operators, to cases away from conformality as the starting point. In particular, one way to break the conformal symmetry is by means of the six scalars in the theory acquiring vacuum expectation values (vev’s). We expect that in such cases the BMN operators are the same, but they have to act on a different vacuum. On the supergravity side the centers of the D3-branes are distributed in the six-dimensional transverse to the branes space, according to the the distribution of vev’s on the gauge theory side, resulting into a deformation of the $AdS_5 \times S^5$ space. A particular such case was considered in [50], where plane wave solutions were obtained via Penrose limits on the solution representing D3-branes uniformly distributed on a disc. In this case it was possible to obtain, in the limit of very large vev’s, the perturbative string spectra in the light-cone gauge. However, there were problems related to the singularity arising due to the breakdown of the continuous approximation, similar to the case of the background (3.18) we have discussed in the present paper. However, if we start with D3-branes distributed on a circle of radius $r_0$, instead of a disc, it is possible to obtain a plane wave which is completely non-singular by choosing, as in the case of (2.9), a geodesic that goes through the center of the ring. The result is quite simple and here we just mention the case with zero angular parameter. This is purely geometrical since, as it turns out, the metric is given by

$$ds^2_{10} = 2du dv + \sum_{i=1}^{8} dx_i^2 + (3(x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2) + x_6^2 + x_7^2 + x_8^2) \frac{du^2}{(1 + u^2)^2},$$

and the self-dual five-form is zero! This solution may well serve as a starting point for investigating the Coulomb branch of $\mathcal{N} = 4$ SYM in a particular sector but beyond the supergravity limit.
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