ON EXPLICIT CONNECTIONS BETWEEN DYNAMICAL AND PARAMETER SPACES

Genadi LEVIN

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Février 2002

IHES/M/02/05

On explicit connections between dynamical and parameter spaces

Genadi Levin *

Inst. of Math., Hebrew University, Jerusalem 91904, Israel

Abstract

Studying holomorphic dynamical systems, we find some new relations between dynamical and parameter spaces, in particular, between velocity of motions in the parameter space and areas in dynamical ones. Applications to the Mandelbrot sets are given.

Contents

1. Main results.
2. Theorem 2 and its corollaries.
3. Proof of Theorem 1 for H-components.
4. Non-hyperbolic components.
5. Example 1.
6. On parabolic maps.
7. Generalizations to polynomial-like maps.

1 Main results

Since [5], [6], the idea of connections between dynamical and parameter spaces plays one of the central roles in modern holomorphic dynamics. For example, it is used in [5] to prove the connectivity of the Mandelbrot set.

1.1 An inequality and examples

Let M_ℓ be the Mandelbrot set of the family $f_c : z \mapsto z^\ell + c$, where $\ell \geq 2$, i.e. the set of $c \in \mathbf{C}$ such that the Julia set J_c of f_c is connected. For every ℓ , it is a connected compact in the plane [5], [6]. Let W be a component of the interior

*Supported in part by IMPAN-BC European Community Centre of Excellence

$\text{int}(M_\ell)$ of M_ℓ . By the maximal principle, W is simply-connected. A priori, W is of one of the following two types:

- H-component, i.e. hyperbolic one: for every $c \in W$, the map f_c has an attracting periodic orbit $P(c) = \{a_1(c), \dots, a_p(c)\}$ of period $p = p(W)$, so that the period p is fixed throughout W , and every $a_j(c)$ is a holomorphic function on $c \in W$. Moreover, the Julia set J_c of f_c has zero Lebesgue measure (area), and the union of the bounded components of $\mathbf{C} \setminus J_c$, which we will call the (bounded) *Fatou set* F_c of f_c , coincides with the basin of attraction of $P(c)$.
- N-component, i.e. non-hyperbolic one: it can be characterized so that, for every $c \in W$, the Julia set J_c of f_c has a positive Lebesgue measure (area), [14]. The Fatou set F_c is empty.

The Fatou, or Density of Hyperbolicity Conjecture says that N-components do not exist. Since it is an open problem, we consider N-components as well.

Denote by K_c the filled-in Julia set of f_c , i.e. $K_c = J_c \cup F_c$. In particular, if c is in N-component, then $K_c = J_c$.

Denote by $\mathbf{dist}(z, A)$ the Euclidean distance between a point z and a set A of the plane.

At last, denote by σ the Lebesgue measure on the plane, and by $\mathbf{area}(A)$ the Lebesgue measure of a subset A of the plane.

Theorem 1 *There exists an absolute constant $\mathbf{L}_*$, such that, for every $\ell \geq 2$ and every c in the interior of M_ℓ ,*

$$\mathbf{dist}(c, \partial M_\ell) \leq \mathbf{L}_* \cdot \ell^2 \int_{K_c} |z|^{\ell-2} d\sigma_z. \quad (1)$$

Moreover,

(H) *there exist absolute constants $\mathbf{L}_H > 0$ and $a_* > 0$ such that, for every $\ell \geq 2$, for every H-component W of M_ℓ , and every $c \in W$, if the multiplier ρ of the periodic orbit $P(c)$ is such that $|\rho| < \ell a_*^\ell$, then*

$$\mathbf{dist}(c, \partial M_\ell) \leq \mathbf{L}_H \cdot \ell \int_{F_c} |z|^{\ell-2} d\sigma_z. \quad (2)$$

(N) *if c is in an N-component of M_ℓ , then*

$$\mathbf{dist}(c, \partial M_\ell) \leq \frac{4\ell}{\pi} \int_{J_c} |z|^{\ell-2} d\sigma_z. \quad (3)$$

For polynomial-like maps, see last Section.

Comment 1 It is easy to check that if $c \in M_\ell$ and $z \in J_c$, then $|z| \leq 2^{\frac{1}{\ell-1}}$. Therefore,

$$\int_{K_c} |z|^{\ell-2} d\sigma_z < 2 \cdot \mathbf{area}(K_c).$$

Comment 2 Let c be in a hyperbolic component of M_ℓ . Then 0 lies in a component Ω of F_c , $r := \mathbf{dist}(0, J_c) > 0$, and the “first approximation” of $\ell \int_\Omega |z|^{\ell-2} d\sigma_z$ is

$$\ell \int_{|z| < r} |z|^{\ell-2} d\sigma_z = 2\pi r^\ell = 2\pi \mathbf{dist}(c, J_c)$$

Comment 3 If $\rho = 0$, then one can take $L_H = 4/\pi$.

Consider 3 examples.

Example 1. It is the following statement¹, which will be proved in Section 5. It shows that the inequality (1) is exact in the following sense:

For every Misiurewicz value of parameter $c_* \in M_2$ there is a constant L and a sequence c_n , which converges to c_* , such that, for every c_n , 0 is a periodic point of f_{c_n} and

$$\frac{\mathbf{dist}(c_n, \partial M_2)}{\mathbf{area}(K_{c_n})} \geq L$$

and, moreover,

$$L \leq \frac{\mathbf{dist}(c_n, \partial M_2)}{\mathbf{dist}(c_n, J_{c_n})} \leq \frac{1}{L}.$$

In **Examples 2, 3** below we use the parameter plane to get information about dynamical pictures.

Example 2. It shows that, in general, $\mathbf{dist}(c, \partial M_\ell)$ and $\mathbf{dist}(c, J_c)$ are not comparable. For the Mandelbrot set M_2 , consider the Feigenbaum parameter $c^F = -1.4\dots$ and the sequence c_p^F , $p > 0$, of real parameters which converges to the Feigenbaum parameter c^F , so that $c_p^F > c^F$ and 0 is a periodic point of the map $f_{c_p^F}$ of period 2^p , $p = 0, 1, \dots$. According to the Feigenbaum-Couillet-Tresser Universality [9],[3],[19],[17],[13], $\mathbf{dist}(c_p^F, J_{c_p^F}) \sim \text{const} \alpha^{-2p}$ while $\mathbf{dist}(c_p^F, \partial M_2) > \text{const} \delta^{-p}$ so that $\mathbf{dist}(c_p^F, \partial M_2) / \mathbf{dist}(c_p^F, J_{c_p^F}) \rightarrow \infty$ as $p \rightarrow \infty$ (where $\delta = 4.66\dots$ and $\alpha = -2.5\dots$ are two universal constants). The inequality (1) then gives that $\mathbf{area}(K_{c_p^F}) \geq (1/L_*) \delta^{-p}$.

¹Jointly with F. Przytycki

Example 3. Let $W_{p/q}$ be the hyperbolic component of M_2 , which is tangent to the boundary of the main cardioid at the point c where f_c has a fixed point with multiplier $\exp(2\pi ip/q)$. There is a conjecture saying that the “size” of $W_{p/q}$ (outside of a neighborhood of the cusp $c = 1/4$) is of precise order $1/q^2$. Then (1) would imply that $\text{area}(K_{c_{p/q}}) \geq \text{const}/q^2$, where $c_{p/q}$ is the center of $W_{p/q}$, and const does not depend on p/q .

1.2 Velocity of motion of periodic orbits

Let us fix the degree $\ell \geq 2$.

We use tools similar to those in [11], see also [15], [8]. Consider Ruelle transfer operator $T = T_c$ defined for the map $f = f_c$ as follows:

$$Tg(z) = \sum_{w:f(w)=z} \frac{g(w)}{(f'(w))^2},$$

acting on functions g , wherever the right-hand side is defined.

It is related to so-called backward invariant Beltrami forms:

Definition 1.1 (cf. [14], [16]).

Let $f = f_c$, and V be a measurable set, such that $f^{-1}(V) \subset V$. Backward invariant Beltrami form on the set V is a function $\mu \in L_\infty(V)$, such that, for a.e. $x \in f^{-1}(V)$,

$$\mu(f(x)) = \mu(x) \cdot \frac{(f'(x))^2}{|f'(x)|^2} \quad (4)$$

Then μ like this is closely related to the operator T via the following duality:

Fact 1.1 For every function g , if $\mu \cdot g$ is integrable on the set V , then

$$\int_V \mu(z) Tg(z) d\sigma_z = \int_{f^{-1}(V)} \mu(z) g(z) d\sigma_z.$$

Proof.

$$\int_{f^{-1}(V)} \mu(w) g(w) d\sigma_w = \int_V \left(\sum_{f(w)=z} \frac{\mu(w) g(w)}{|f'(w)|^2} \right) d\sigma_z = \int_V \mu(z) Tg(z) d\sigma_z.$$

□

If Ω is a component of an immediate basin of attraction of an attracting or parabolic periodic orbit of $f = f_c$, then we use the following construction of backward invariant Beltrami forms μ . Let $C \subset \Omega$ be a fundamental region of f , i.e. every grand orbit of f enters C at most once. Now define

$$V = V_C = \cup_{n \geq 0} f^{-n}(C).$$

Then choose an *arbitrary* $\mu \in L_\infty(C)$ and define extension of μ from C to a backward invariant form on V according to the equation (4).

A relation of all this to the parameter plane is described in the following theorem. Let $O = \{b_k\}_{k=1}^n$ be *any* periodic orbit $O = \{b_k\}_{k=1}^n$ of $f = f_c$ of *exact* period n with the multiplier $\rho \neq 1$. First, since $\rho \neq 1$, there is a holomorphic extension of the periodic orbit $O = O(c)$ of f_c to a periodic orbit $O(\hat{c}) = \{b_k(\hat{c})\}_{k=1}^n$ of $f_{\hat{c}}$ for $\hat{c}$ in a neighborhood of c . In particular, if $\rho(\hat{c})$ denotes the multiplier of $O(\hat{c})$, it is holomorphic in $\hat{c}$ in this neighborhood. We denote by b'_k , $k = 1, \dots, n$, and by ρ' the derivative of $b_k(\hat{c})$ and $\rho(\hat{c})$ respectively w.r.t. $\hat{c}$ at the point $\hat{c} = c$. Second, assuming also that $\rho \neq 0$, let us associate to O the following rational function:

$$A(z) = A_O(z) = \sum_{k=1}^n \frac{1}{(z - b_k)^2} + \frac{1}{\rho(1 - \rho)} \sum_{k=1}^n \frac{(f^n)''(b_k)}{z - b_k} \quad (5)$$

where $(f^n)''(z)$ is, of course, the second derivative of the n -iterate of f w.r.t. z .

Comment 4 *The numbers $(f^n)''(a_k)$ are connected directly to the uniformization of f^n at the points a_k . For instance, if the orbit is hyperbolic (i.e. $|\rho| \neq 0, 1$, then $(f^n)''(a_k)/\rho(\rho - 1) = K''(0)$ where K is the Konigs linearization function for the fixed point a_k of f^n normalizaed so that $K'(0) = 1$.*

The above mentioned precise relation between the dynamical plane (function A) and the parameter plane (the derivative ρ') is described in

Theorem 2 *Let O be a periodic orbit of $f = f_c$ with multiplier $\rho \neq 1, 0$, and $A = A_O$. Then:*

(a) *for $T = T_c$,*

$$A(z) = (TA)(z) + \frac{\rho'}{\rho} v_*(z), \quad (6)$$

where

$$v_*(z) = \frac{1}{z - c} = (Tu_*)(z),$$

with $u_*(z) = \ell z^{\ell-2}$.

(a*) *(dual to (a)) for every backward invariant Beltrami form μ on a set V , if $\mu \cdot A$ is integrable on V , then*

$$\int_{V \setminus f^{-1}(V)} \mu(z) A(z) d\sigma_z = \frac{\rho'}{\rho} \ell \int_{f^{-1}(V)} z^{\ell-2} \mu(z) d\sigma_z. \quad (7)$$

Comment 5 *In fact, Theorem 2 covers also the limit cases $\rho = 0$ and $\rho = 1$. Then one should consider (6) for $\hat{c}$ close to c , multiply both sides by $\rho(\hat{c})$ and $1 - \rho(\hat{c})$ resp., and pass to a limit. Compare Section 2.2 and Section 6.*

As an almost immediate corollary, we have

Corollary 1.1 (*Douady-Hubbard-Sullivan Theorem, see [4], [2] for $\ell = 2$*)

Let W be an H -component of M_ℓ , and $\rho(c)$ be the multiplier of the attracting periodic orbit $P(c) = \{a_1(c), \dots, a_p(c)\}$ of f_c , $c \in W$. Then $\rho : W \rightarrow \mathbf{D}$ is a $\ell - 1$ -covering map of W onto the unit disc with a single branching point $c_W \in W$ where $\rho = 0$.

See next Section for the proof.

By Corollary 1.1, any branch of the map $\rho^{\frac{1}{\ell-1}}$ extends to a conformal isomorphism of W onto $\mathbf{D}$.

Another corollary of Theorem 2 is the following

Theorem 3 *Let O be a periodic orbit of $f = f_c$ with multiplier ρ , such that O is either attracting or parabolic (with $\rho \neq 1$). Denote by Y the complex plane with the iterates $f^n(0)$, $n > 0$, and the orbit O deleted. Then, for every $z \in Y$, the function*

$$\tilde{\mathbf{T}}(z) := \sum_{n=0}^{\infty} T^n \frac{1}{z - c} = \sum_{n>0} (T^n u_*)(z)$$

is well defined, and

$$\tilde{\mathbf{T}}(z) = \frac{\rho}{\rho'} A(z).$$

Note that for the quadratic family ($\ell = 2$), $\tilde{\mathbf{T}}(z) = 2 \sum_{n>0} \sum_{f^n(w)=z} \frac{1}{((f^n)'(w))^2}$.

See next Section for the proof.

For an application of Theorem 2 to parabolic periodic orbits, see Section 6.

In fact, the inequality (1) for the H -components can be considered as a refinement of the Douady-Hubbard-Sullivan Theorem. Indeed, for any simply-connected bounded domain Ω , the value $\mathbf{dist}(x, \partial\Omega)$, for $x \in \Omega$, is closely related to the conformal radius $r(x)$ of Ω at x . The latter is defined as $r(x) = R'_x(0)$, where, in turn, R_x is the univalent map from the unit disc onto Ω normalized so that $R_x(0) = x$ and $R'_x(0) > 0$. By 1/4-Koebe theorem,

$$1/4 \leq \mathbf{dist}(x, \partial\Omega)/r(x) \leq 4.$$

That is, to prove Theorem 1, we need to estimate from above $r(x)$. On the other hand, for an H -component W , the conformal radius $r(c)$ is connected to the conformal isomorphism $\rho^{\frac{1}{\ell-1}} : W \rightarrow \mathbf{D}$ by the relation:

$$r(c) = \frac{(\ell - 1)|\rho|^{1 - \frac{1}{\ell-1}}(1 - |\rho|^2)}{|\rho'|}, \quad (8)$$

where $\rho = \rho(c)$,

The inequality (1) for H -components will be an easy consequence from Propositions 2 and 3 on the behavior of the conformal radius, see Section 3.

1.3 Non-hyperbolic components

It remains to consider N-components of M_ℓ . Then c in such a component is characterized so that, there exists a subset E of J_c of a positive Lebesgue measure and a Beltrami form μ supported on E , which is backward (and forward) invariant, and $\|\mu\|_{L_\infty} = 1$ [14], [16]. Moreover, μ is unique, in the sense, that for any other $\tilde{\mu}$ as above, there is a complex number h of the modulus 1, such that $\mu = h\tilde{\mu}$ a.e. (see Section 4)

Theorem 4 *Let W be an N-component of M_ℓ , $c \in W$, $f = f_c$, and μ as above. Let r be the conformal radius of W at c . Then*

$$r = \left| \frac{1}{\pi} \int \frac{\mu(z)}{z - c} d\sigma_z \right| = \left| \frac{\ell}{\pi} \int z^{\ell-2} \mu(z) d\sigma_z \right|. \quad (9)$$

See Section 4 for the proof. Clearly, (9) implies (1) for N-components.

1.4 Acknowledgements.

Most of this work has been done and written down during author's visits to IM-PAN, Warsaw (September-November, 2001), IHES, Orsay (December, 2001), and Warwick Univ. (January-February, 2002). The author thanks A. Avila, A. Cheritat, A. Douady, H. Epstein, V. Marcovich, F. Przytycki, D. Ruelle, and J.-C. Yoccoz for stimulating and helpful discussions.

2 Theorem 2 and its corollaries

2.1 The Ruelle transfer operator

The statement and a hint. Remind that

$$Tg(z) = \sum_{w:f(w)=z} \frac{g(w)}{(f'(w))^2}.$$

In view of Fact 1.1, we need to prove only Part (a) of Theorem 2:

$$A(z) = (TA)(z) + \frac{\rho'}{\rho} \frac{1}{z - c}, \quad (10)$$

where:

$$A(z) = \sum_{k=1}^n \frac{1}{(z - b_k)^2} + \frac{1}{\rho(1 - \rho)} \sum_{k=1}^n \frac{(f^n)''(b_k)}{z - b_k},$$

$O = \{b_1, \dots, b_n\}$ is a periodic orbit of $f(z) = z^\ell + c$ of exact period n and with the multiplier $\rho \neq 1, 0$, and ρ' is the derivative of ρ as a function of c .

There is a way to “guess” that something like the formula (10) should exist. Assume, for a moment, that the periodic orbit O is attracting, more exactly, $0 < |\rho| < 1$, and consider the series (cf. [12])

$$D_\lambda(z) = \sum_{k \geq 0} \frac{\lambda^{k-1}}{(f^{k-1})'(c)(z - f^k(0))}.$$

It converges for $|\lambda| < |\rho|$ and for every z in, say, the basin of infinity of f . On the other hand,

$$\lambda(TD_\lambda)(z) = D_\lambda(z) - \frac{S(\lambda)}{z - c},$$

where $S(\lambda) = \sum_{k \geq 0} \lambda^{k-1}/(f^{k-1})'(c)$. It can be checked with help of the Konigs Linearization Theorem and the Cauchy formula, that, for fixed z , the function $D_\lambda(z)$ (as well as $S(\lambda)$) extends to a meromorphic function on the complex plane, with simple poles at the points λ , so that $\lambda^n = \rho^{j-1}$, for $j = 1, 2, \dots$. If $R(z)$ and R_S denote the residues of $D_\lambda(z)$ and $S(\lambda)$ resp. at one of those poles λ_0 , then we come to the identity $\lambda_0 TR(z) = R(z) - \frac{R_S}{z-c}$. In particular, take $\lambda_0 = 1$. Then calculations show that

$$R(z) = \sum_{k=1}^n \frac{1}{(z - b_k)^2} - \sum_{k=1}^n \frac{E_k}{z - b_k}, \quad (11)$$

where

$$E_k = \frac{\ell - 1}{(f^{k-1})'(b_1)} \left(\frac{1}{\rho - 1} \sum_{j=0}^{n-1} \frac{(f^j)'(b_1)}{b_{1+j}} + \sum_{j=0}^{k-2} \frac{(f^j)'(b_1)}{b_{1+j}} \right).$$

Key fact is that the residue R_S of $S(\lambda)$ at the pole $\lambda_0 = 1$ turns out to be equal to ρ'/ρ . Although we prove this fact below, the reasoning behind this is unclear.

Action of T on Cauchy kernels The operator T acts on functions $1/(z - a)$, where a is a parameter, as follows:

$$T \frac{1}{z - a} = \frac{1}{f'(a)} \left[\frac{1}{z - f(a)} - \frac{1}{z - c} \right]. \quad (12)$$

It can be checked easily, for example, calculating the integral $(1/2\pi i) \int_\Gamma dw/(f'(w)(f(w) - z)(w - a))$ with the contour Γ enclosing ∞ . Taking the derivative of (12) with respect to a , we get

$$T \frac{1}{(z - a)^2} = \frac{1}{(z - f(a))^2} - \frac{\ell - 1}{af'(a)} \left[\frac{1}{z - f(a)} - \frac{1}{z - c} \right] \quad (13)$$

Let us show that

$$\frac{1}{z-c} = (Tu_*)(z), \quad (14)$$

where $u_*(z) = \ell z^{\ell-2}$. Applying (12) and the Cauchy formula, we can write:

$$(Tu_*)(z) = \frac{1}{2\pi i} \int_{\Gamma} u_*(a) \left(T \frac{1}{a-z}\right) da = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{a} \left[\frac{1}{z-c} - \frac{1}{z-f(a)}\right] da = \frac{1}{z-a}.$$

Proof of (10). First, let's check that $A(z) = R(z)$, i.e. we need to show that $\frac{(f^n)''(b_k)}{\rho(\rho-1)} = E_k$. One writes:

$$\begin{aligned} (\rho-1)E_k &= \frac{\ell-1}{(f^{k-1})'(b_1)} \left[\sum_{j=0}^{n-1} \frac{(f^j)'(b_1)}{b_{1+j}} + \rho \sum_{j=0}^{k-2} \frac{(f^j)'(b_1)}{b_{1+j}} - \sum_{j=0}^{k-2} \frac{(f^j)'(b_1)}{b_{1+j}} \right] = \\ &= \frac{\ell-1}{(f^{k-1})'(b_1)} \left[\sum_{j=k-1}^{n-1} \frac{(f^j)'(b_1)}{b_{1+j}} + \rho \sum_{j=0}^{k-2} \frac{(f^j)'(b_1)}{b_{1+j}} \right] = (\ell-1) \sum_{j=0}^{n-1} \frac{(f^j)'(b_k)}{b_{j+k}} = \frac{(f^n)''(b_k)}{(f^n)'(b_k)} = \frac{(f^n)''(b_k)}{\rho}. \end{aligned}$$

We use here notation $b_{n+j} = b_j$ and the identities $\rho/(f^{k-1})'(b_1) = (f^{n-k+1})'(b_k)$ and $(f^{n-k+1})'(b_k)(f^j)'(b_1) = (f^{n-k+1+j})'(b_k)$. Now, using (12) and (13), we see that

$$TR(z) = R(z) + \frac{F}{z-c}$$

where

$$F = \sum_{k=1}^n \left[\frac{E_k}{f'(b_k)} + \frac{\ell-1}{b_k f'(b_k)} \right].$$

It remains to show that $F = -\rho'/\rho$. Make the following transformations:

$$\begin{aligned} F &= \sum_{k=1}^n \left[\frac{E_k}{f'(b_k)} + \frac{\ell-1}{b_k f'(b_k)} \right] = \sum_{k=1}^n \left[\frac{\ell-1}{(f^k)'(b_1)} \left(\frac{1}{\rho-1} \sum_{j=0}^{n-1} \frac{(f^j)'(b_1)}{b_{1+j}} + \sum_{j=0}^{k-2} \frac{(f^j)'(b_1)}{b_{1+j}} \right) + \frac{\ell-1}{b_k f'(b_k)} \right] \\ &= \sum_{k=1}^n \frac{\ell-1}{(f^k)'(b_1)} \left(\frac{1}{\rho-1} \sum_{j=0}^{n-1} \frac{(f^j)'(b_1)}{b_{1+j}} + \sum_{j=0}^{k-1} \frac{(f^j)'(b_1)}{b_{1+j}} \right) = \\ &= (\ell-1) \sum_{j=0}^{n-1} \frac{(f^j)'(b_1)}{b_{1+j}} \left(\frac{1}{\rho-1} \sum_{k=1}^n \frac{1}{(f^k)'(b_1)} + \sum_{k=j+1}^n \frac{1}{(f^k)'(b_1)} \right) = \\ &= (\ell-1) \sum_{j=0}^{n-1} \frac{(f^j)'(b_1)}{b_{1+j}} \left(\frac{\rho}{\rho-1} \sum_{k=1}^n \frac{1}{(f^k)'(b_1)} - \sum_{k=1}^j \frac{1}{(f^k)'(b_1)} \right). \end{aligned}$$

Now we use identities

$$\frac{\partial f^m}{\partial c}(z) = (f^m)'(z) \sum_{j=1}^m \frac{1}{(f^j)'(z)}$$

so that the derivative of b_1 w.r.t. c ,

$$b'_1 = \frac{\frac{\partial f^n}{\partial c}(b_1)}{1 - \rho} = \frac{\rho}{1 - \rho} \sum_{k=1}^n \frac{1}{(f^k)'(b_1)},$$

and continue:

$$\begin{aligned} F &= -(\ell-1) \sum_{j=0}^{n-1} \frac{(f^j)'(b_1)}{b_{1+j}} (b'_1 + \sum_{k=1}^j \frac{1}{(f^k)'(b_1)}) = -(\ell-1) \sum_{j=0}^{n-1} \frac{1}{b_{1+j}} ((f^j)'(b_1)b'_1 + \frac{\partial f^j}{\partial c}(b_1)) \\ &= -(\ell-1) \sum_{j=0}^{n-1} \frac{b'_{j+1}}{b_{j+1}} = -\frac{\rho'}{\rho}. \end{aligned}$$

2.2 Douady-Hubbard-Sullivan Theorem

Let W be an H-component of M_ℓ , and $\rho(c)$ be the multiplier of the attracting periodic orbit $P(c) = \{a_1(c), \dots, a_p(c)\}$ of f_c , $c \in W$. We prove that $\rho : W \rightarrow \mathbf{D}$ is a $\ell - 1$ -covering map of W onto the unit disc with a single branching point $c_W \in W$ where $\rho(c_W) = 0$.

A priori, the map $\rho : W \rightarrow \mathbf{D}$ is a branching covering, because its limit values are of the modulus 1. Hence, it is enough to prove two facts:

- (1) $\rho' \neq 0$ for any $c \in W$ where $\rho \neq 0$,
- (2) if $\rho(\tau) = 0$ for some $c = \tau \in W$, then necessarily $\rho'(\tau) = \rho''(\tau) = \dots = \rho^{(\ell-2)}(\tau) = 0$ while $\rho^{(\ell-1)}(\tau) \neq 0$.

To prove (1), assume $\rho' = 0$ for some $c \in W$. Then we get, for $f = f_c$ and $P = P(c)$, according to Theorem 2(a^*), that $\int_C \mu A d\sigma = 0$, where $A = A_P$, and C is a fundamental annulus in the basin of attraction of the periodic orbit P of f around, say, the point a_1 of P , and for any $\mu \in L_\infty(C)$. Observe that A is integrable on V because V is disjoint with a neighborhood of the set P . Now, choosing μ so that $\mu A = |A|$ on C , we get that $A = 0$ on C , a contradiction with the fact that A is a non-zero rational function.

Let's prove (2). Let $\rho(\tau) = 0$, for some $c = \tau \in W$. Denote by a_p the point of $P(\tau)$ such that $a_p = 0$, and $c_p = f_\tau^p(0)$. Denote also by a'_p and c'_p the derivatives of a_p, c_p w.r.t. c at τ . Differentiating the equality $\rho(c) = \ell^p(a_1(c) \dots a_p(c))^{\ell-1}$ $\ell - 1$ times at the point $c = \tau$, we have:

$$\rho' = \dots = \rho^{(\ell-2)} = 0$$

while $\rho^{(\ell-1)} = \ell! (f^{p-1})'(\tau) (a'_p)^{\ell-1}$. (Here, as always, $(f^k)'(x) = f'(x) \dots f'(f^{k-1}(x))$ is the derivative w.r.t. z .) Thus it is enough to prove

Proposition 1 *At $c = \tau$,*

$$c'_p = a'_p \neq 0.$$

Proof. (cf. [11], [20]). We have at $c = \tau$:

$$\begin{aligned} a'_p &= \frac{(\partial f^p / \partial c)(a_p)}{1 - \rho} = \frac{\partial f^p}{\partial c}(a_p) = \frac{\partial f^p}{\partial c}(0) = c'_p \\ &= (f^{p-1})'(f(0))S_p, \end{aligned}$$

where

$$S_p = \sum_{k=0}^{p-1} \frac{1}{(f^k)'(f(0))}.$$

Introduce

$$\psi(z) = \sum_{k=0}^{p-1} \frac{1}{(f^k)'(f(0))} \frac{1}{z - f^{k+1}(0)}.$$

Applying (12), we see that

$$T\psi(z) = \psi(z) - \frac{S_p}{z - \tau}.$$

Assuming $S_p = 0$ and then repeating the argument in the proof that $\rho' \neq 0$ if $\rho \neq 0$, we get a contradiction. Thus $S_p \neq 0$, and, therefore,

$$a'_p = c'_p = (f^{p-1})'(f(0))S_p \neq 0.$$

□

2.3 The function $\sum_{n=0}^{\infty} T^n \frac{1}{z-c}$.

Denote

$$\tilde{\mathbf{T}}(z) = \sum_{n=0}^{\infty} T^n \frac{1}{z-c} = \sum_{n=0}^{\infty} \sum_{f^n(w)=z} \frac{1}{((f^n)'(w))^2(w-c)}.$$

According to (14),

$$\tilde{\mathbf{T}}(z) = \sum_{n>0} (T^n u_*)(z) = \ell \sum_{n>0} \sum_{f^n(w)=z} \frac{w^{\ell-2}}{((f^n)'(w))^2}.$$

If $z \in Y$, i.e. z is not in the closure of the iterates of the critical point, then it is easy to see that $\tilde{\mathbf{T}}(z)$ is well defined and holomorphic in a neighborhood of z ; this follows from the Koebe Distortion Theorem applied to a neighborhood C of z combined

with the estimate $\int_C \sum_{n>0} |(T^n u_*)(z)| d\sigma_z \leq L \sum_{n>0} \text{area}(f^{-n}(C)) < \infty$, where the constant $L = \ell[\text{diam}(K_c)]^{\ell-2}$. Now we have from Theorem 2,

$$\sum_{k=0}^{n-1} T^k \frac{1}{z-c} = \frac{\rho}{\rho'} (A(z) - (T^n A)(z)) \quad (15)$$

Since Y is open and connected and $\tilde{T}$ is holomorphic in Y , by analytic continuation, it is enough to prove the statement for z in the basin of infinity of f . For each such z , find a neighborhood C such that $f^{-n}(C)$ are pairwise disjoint. If O is attracting, then the set $V := \cup_{n \geq 0} f^{-n}(C)$ stays away from a neighborhood of O , hence, A is integrable on V in this case. If O is parabolic, this is true as well. Indeed, then functions $1/|z-a|^2$, for $a \in O$, are integrable in the intersection of a neighborhood of the point a with the basin of infinity, as it follows from the local dynamics at parabolic points. Thus, in any case, one can write $\sum_{n>0} \int_C |T^n A| d\sigma \leq \int_{\cup_{n>0} f^{-n}(C)} |A| d\sigma < \infty$, in particular, $\int_C |(T^n A)(z)| d\sigma_z \rightarrow 0$ as $n \rightarrow \infty$. Since $(T^n A)(z)$ are holomorphic in C , $(T^n A)(z) \rightarrow 0$ uniformly on compacts in C , and Theorem 3 follows from (15).

Comment 6 *Considering limits of corresponding functions when the parameter tends to the boundary of H-component from inside, one can generalize the above statement for Siegel as well as Cremer periodic orbit.*

3 Proof of Theorem 1 for H-components

For the hyperbolic components of M_ℓ , Theorem 1 will be a consequence of Propositions 2 and 3 below, where we will apply Proposition 2 for $|\rho|$ close to 1 and Proposition 3 for $|\rho|$ close to zero. Let W be a H-component of M_ℓ , $c \in W$, $f = f_c$, $P = \{a_1, \dots, a_p\}$ be the attracting periodic orbit of f with multiplier ρ , and $A(z)$ be the function associated with the orbit P .

3.1 A special Beltrami form: Proposition 2

We assume here that $\rho \neq 0$. Pick up the point a_1 . Then let Φ be the Konigs linearization function, which is defined and univalent in a neighborhood U of a_1 , normalized so that $\Phi(a_1) = 0$ and $\Phi'(a_1) = 1$, and

$$\Phi \circ f^p(z) = \rho \Phi(z)$$

for $z \in U$. One can assume that $\Phi(U)$ is a round disc centered at 0. Let $K = \Phi^{-1}$. Fix real $m > 0$ small enough, and define a fundamental annulus

$$C := K(\{|\rho|m < |w| < m\}).$$

Now set, for every $z \in C$,

$$\mu(z) := \left\{ \frac{\Phi(z)}{|\Phi(z)|} \frac{|\Phi'(z)|}{\Phi'(z)} \right\}^2.$$

Then spread μ_C out to the set $V = \cup_{n \geq 0} f^{-n}(C)$ according to the rule (backward invariance): if $f^n(z) \in C$, for some $n > 0$, then $\mu(z) = \mu(f^n(z)) \frac{|(f^n)'(z)|^2}{((f^n)'(z))^2}$.

Comment 7 *It is easy to check that μ defined near a_1 as above is automatically a forward invariant form as well. It allows us to consider μ as a function on the whole basin of attraction F of f .*

Proposition 2

$$2\rho \log \frac{1}{|\rho|} = \ell \frac{\rho'}{\pi} \int_{f^{-1}(V)} z^{\ell-2} \mu(z) d\sigma_z = \ell \frac{\rho'}{\pi} \int_F z^{\ell-2} \mu(z) d\sigma_z, \quad (16)$$

where $f^{-1}(V) = \cup_{n > 0} f^{-n}(C)$.

Proof. In polar coordinates, $w = \tau \exp(i\theta)$ and then

$$\begin{aligned} \int_C \mu_C(z) A(z) d\sigma_z &= \int_{\{|\rho|m < |w| < m\}} \mu_C(K(w)) A(K(w)) |K'(w)|^2 d\sigma_w = \\ \int_{\{|\rho|m < |w| < m\}} \frac{w^2}{|w|^2} (K'(w))^2 A(K(w)) d\sigma_w &= 2\pi \int_{|\rho|m}^m \frac{d\tau}{\tau} \frac{1}{2\pi i} \int_{|w|=\tau} w (K'(w))^2 A(K(w)) dw \\ &= 2\pi \log \frac{1}{|\rho|}, \end{aligned}$$

where we use that $w(K'(w))^2 A(K(w)) \sim \frac{1}{w}$ as $w \rightarrow 0$. It remains to apply Theorem 2 (a^*).

□

3.2 Proposition 3

Blaschke products. Pick up any $c \in W$, $f = f_c$, $P = P(c)$, $\rho = \rho(c)$. Let Ω be a component of the immediate basin of attraction of the attracting periodic orbit P of f , which, for definitivity, contains 0 and the point a of P . Let $\phi : \mathbf{D} \rightarrow \Omega$ be a Riemann map, normalized so that $\phi(0) = a$. Then the conjugate map $B(w) := \phi^{-1} \circ f^p \circ \phi(w)$ is a Blaschke product, such that $B(0) = 0$, $\rho = B'(0)$, and B has a single critical point of order ℓ . Then it is easy to see that ϕ can be chosen so that

$$B(w) = B_{a,\ell}(w) = K_1([K_0(w)]^\ell),$$

where

$$K_0(w) = \frac{w+a}{1+\bar{a}w}, K_1(w) = \frac{w-b}{1-\bar{b}w}$$

and $b = a^\ell$, for some $0 < |a| < 1$. Hence,

$$B'(w) = K_1'([K_0(w)]^\ell) \ell [K_0(w)]^{\ell-1} K_0'(w),$$

so that

$$\rho = \frac{\ell a^{\ell-1}(1-|a|^2)}{1-|a|^{2\ell}}. \quad (17)$$

Therefore, given ρ , $0 \leq |\rho| < 1$, the number $|a|$ is determined as the unique solution of the equation

$$|\rho| = \frac{\ell |a|^{\ell-1}(1-|a|^2)}{1-|a|^{2\ell}},$$

such that $0 \leq |a| < 1$, and then a is unique determined as well from (17), up to a rotation by $1/(\ell-1)$ root of unity.

By continuity, there exists $a_* > 0$, $0 < a_* < 0.5$, be so that:

for every a with $|a| \leq a_*$ and every w with $|w| \leq \frac{1}{2}$, we have:

$$\left| \frac{w+a}{1+\bar{a}w} \right|^2 \leq \frac{1}{3} - \frac{1}{24},$$

for every b with $|b| \leq a_*^2$ and every w with $|w| \leq \frac{1}{3} - \frac{1}{24}$, we have:

$$\left| \frac{w-b}{1-\bar{b}w} \right| \leq \frac{1}{3}.$$

Then we have

Fact 3.1 *For every a , $|a| \leq a_*$, and every $\ell \geq 2$,*

$$\left\{ \frac{1}{3} < |w| < \frac{1}{2} \right\} \subset B_{a,\ell}^{-1}(\mathbf{D}_{1/3}) \setminus \mathbf{D}_{1/3},$$

where $\mathbf{D}_r = \{|w| < r\}$.

Uniformization of the function A We carry out some calculations, which are collected in

Lemma 3.1 *Let $0 < |\rho| < 1$ and $B(w) = B_{a,\ell}(w) = \phi^{-1} \circ f^p \circ \phi(w)$. Then:*

•

$$(\phi'(w))^2 A(\phi(w)) = \frac{1}{w^2} - \frac{B''(0)}{\rho(\rho-1)} \frac{1}{w} + l(w), \quad (18)$$

where l is a function, which is holomorphic in the unit disc $\mathbf{D}$.

•

$$B''(0) = \frac{\ell a^{\ell-2}(1-|a|^2)}{(1-|a|^{2\ell})^2} \cdot \{(\ell-1)(1-|a|^{2\ell+2}) - (\ell+1)|a|^2(1-|a|^{2\ell-2})\}. \quad (19)$$

Proof. From the definition of A ,

$$(\phi'(w))^2 A(\phi(w)) = \frac{(\phi'(w))^2}{(\phi(w) - \phi(0))^2} - \frac{1}{\rho(\rho-1)} \frac{(f^p)''(\phi(w))(\phi'(w))^2}{\phi(w) - \phi(0)} + l_1(w),$$

where l_1 is a function holomorphic in $\mathbf{D}$. After differentiating twice the identity $f^p \circ \phi(w) = \phi \circ B(w)$ at $w = 0$, we get:

$$(f^p)''(\phi(0))(\phi'(0))^2 = \phi''(0)\rho(\rho-1) + \phi'(0)B''(0). \quad (20)$$

Then

$$\begin{aligned} & \frac{(\phi'(w))^2}{(\phi(w) - \phi(0))^2} - \frac{1}{\rho(\rho-1)} \frac{(f^p)''(\phi(w))(\phi'(w))^2}{\phi(w) - \phi(0)} \\ &= \frac{1}{w^2} \left(\frac{1 + w\phi''(0)/\phi'(0) + O(|w|^2)}{1 + w \cdot \phi''(0)/2\phi'(0) + O(|w|^2)} \right)^2 - \frac{1}{w\phi'(0)} (\phi''(0) + \phi'(0) \cdot \frac{B''(0)}{\rho(\rho-1)}) (1 + O(|w|)) \\ &= \frac{1}{w^2} (1 + w \frac{\phi''(0)}{\phi'(0)} + O(|w|^2)) - \frac{1}{w} (\frac{\phi''(0)}{\phi'(0)} + \frac{B''(0)}{\rho(\rho-1)} + O(|w|)) + O(1) \\ &= \frac{1}{w^2} - \frac{1}{w} \frac{B''(0)}{\rho(\rho-1)} + O(1). \end{aligned}$$

Let us prove the second part. We have:

$$B'(w) = K_1'([K_0(w)]^\ell) \ell [K_0(w)]^{\ell-1} K_0'(w),$$

$$B''(0) = K_1''(b) [\ell a^{\ell-1}]^2 [K_0'(0)]^2 + K_1'(b) \ell(\ell-1) a^{\ell-2} [K_0'(0)]^2 + K_1'(b) \ell a^{\ell-1} K_0''(0).$$

$$\rho = K_1'(b) \ell a^{\ell-1} K_0'(0) = \frac{\ell a^{\ell-1}(1-|a|^2)}{1-|a|^{2\ell}},$$

and finally,

$$\begin{aligned} B''(0) &= \frac{\ell a^{\ell-2}(1-|a|^2)}{(1-|a|^{2\ell})^2} \cdot \{2\ell|a|^{2\ell}(1-|a|^2) + (\ell-1)(1-|a|^2)(1-|a|^{2\ell}) - 2|a|^2(1-|a|^{2\ell})\} \\ &= \frac{\ell a^{\ell-2}(1-|a|^2)}{(1-|a|^{2\ell})^2} \cdot \{(\ell-1)(1-|a|^{2\ell+2}) - (\ell+1)|a|^2(1-|a|^{2\ell-2})\}. \end{aligned}$$

□

The statement.

Proposition 3 *If ρ is such that*

$$|\rho| \leq \frac{\ell a_*^{\ell-1}(1-a_*^2)}{1-a_*^{2\ell}},$$

then

$$2\pi\left(\frac{1}{2} - \frac{1}{3}\right)\left|\frac{B''(0)}{1-\rho}\right| \leq |\rho'| \cdot \ell \cdot \int_{F_c} |z|^{\ell-2} d\sigma_z. \quad (21)$$

Proof. For every ρ as above, we find $B = B_{a,\ell}$, where $|a| \leq a_*$. According to Fact 3.1, we can define a fundamental annulus $C := \phi^{-1}(C_B)$, where $C_B := B^{-1}(\mathbf{D}_{1/3}) \setminus \mathbf{D}_{1/3}$ contains the round annulus $C_{1/3,1/2} = \{\frac{1}{3} < |w| < \frac{1}{2}\}$. Define $\mu(z)$ for $z = \phi(w)$, where $w \in C_B$, to be equal:

- $\mu(\phi(w)) = 0$ if $w \in C_B \setminus C_{1/3,1/2}$,
-

$$\mu(\phi(w)) = \frac{(\phi'(w))^2}{|\phi'(w)|^2} \frac{w}{|w|},$$

if $w \in C_{1/3,1/2}$.

Then we have, in polar coordinate $w = \tau \exp(i\theta)$:

$$\begin{aligned} \int_C \mu(z) A(z) d\sigma_z &= \int_{\{1/3 < |w| < 1/2\}} \mu(\phi(w)) A(\phi(w)) |\phi'(w)|^2 d\sigma_w = \\ 2\pi \int_{1/3}^{1/2} d\tau \frac{1}{2\pi i} \int_{|w|=\tau} (\phi'(w))^2 A(\phi(w)) dw &= 2\pi \left(\frac{1}{2} - \frac{1}{3}\right) \frac{B''(0)}{\rho(1-\rho)}, \end{aligned}$$

where Lemma 3.1 is used. It remains to apply Theorem 2 (a^*).

□

Proof of Theorem 1 for H-components. It is enough to estimate the conformal radius r at the point c of a hyperbolic component W of M_ℓ . As we know,

$$r = \frac{(\ell-1)|\rho|^{1-\frac{1}{\ell-1}}(1-|\rho|^2)}{|\rho'|}.$$

Given $\ell \geq 2$, consider two cases:

(1)

$$|\rho| > \frac{\ell a_*^{\ell-1}(1-a_*^2)}{1-a_*^{2\ell}}.$$

Denote $t = |\rho|^{1/(\ell-1)}$, then

$$t \geq a_* \left[\frac{\ell}{1 + a^2 + \dots + a_*^{2\ell-2}} \right]^{1/(\ell-1)} > a_*,$$

and applying Proposition 2,

$$r = \frac{(\ell-1)|\rho|^{1-\frac{1}{\ell-1}}(1-|\rho|^2)}{|\rho| \log \frac{1}{|\rho|}} \cdot \frac{|\rho| \log \frac{1}{|\rho|}}{|\rho'|} \leq$$

$$\frac{(\ell-1)|\rho|^{1-\frac{1}{\ell-1}}(1-|\rho|^2)}{|\rho| \log \frac{1}{|\rho|}} \cdot \frac{\ell}{2\pi} \int_{F_c} |z^{\ell-2}| d\sigma_z \leq \frac{1-t^{\ell-1}}{t \log \frac{1}{t}} \cdot \frac{\ell}{\pi} \int_{F_c} |z^{\ell-2}| d\sigma_z \leq (\ell-1) \mathbf{L}_1 \ell \int_{F_c} |z^{\ell-2}| d\sigma_z,$$

where $\mathbf{L}_1 = \sup\{(1-t)/t \log(1/t) : a_* \leq t < 1\}$ is an absolute constant.

(2)

$$|\rho| \leq \frac{\ell a_*^{\ell-1} (1-a_*^2)}{1-a_*^{2\ell}}.$$

Then $\rho = \frac{\ell a^{\ell-1} (1-|a|^2)}{1-|a|^{2\ell}}$, where $|a| \leq a_*$. To estimate r , we now apply Proposition 3 and Lemma 3.1:

$$\begin{aligned} r &= \frac{(\ell-1)|\rho|^{1-\frac{1}{\ell-1}}(1-|\rho|^2)}{|\rho'|} \leq \frac{(\ell-1)|\rho|^{1-\frac{1}{\ell-1}}(1-|\rho|^2)}{|B''(0)|} \cdot \frac{3|1-\rho|}{\pi} \cdot \ell \int_{F_c} |z^{\ell-2}| d\sigma_z \leq \\ &\frac{(\ell-1) \left(\frac{\ell |a|^{\ell-1} (1-|a|^2)}{1-|a|^{2\ell}} \right)^{1-\frac{1}{\ell-1}}}{\left| \frac{\ell a^{\ell-2} (1-|a|^2)}{(1-|a|^{2\ell})^2} \cdot \{(\ell-1)(1-|a|^{2\ell+2}) - (\ell+1)|a^2|(1-|a|^{2\ell-2})\} \right|} \cdot \frac{6}{\pi} \cdot \ell \int_{F_c} |z^{\ell-2}| d\sigma_z \leq \\ &\left(\frac{1+|a|^2+\dots+|a|^{2\ell-2}}{\ell} \right)^{\frac{1}{\ell-1}} \cdot \frac{1}{(1-\frac{\ell+1}{\ell-1}|a|^2) + |a|^{2\ell}(\frac{\ell+1}{\ell-1}-|a|^2)} \cdot \frac{6}{\pi} \cdot \ell \int_{F_c} |z^{\ell-2}| d\sigma_z \leq \\ &\frac{12}{\pi} \cdot \ell \int_{F_c} |z^{\ell-2}| d\sigma_z. \end{aligned}$$

End of the proof of Theorem 1 for the H-components.

4 Non-hyperbolic components

Here we prove Theorem 5 thus completing the proof of Theorem 1. Let W be an N-component of M_ℓ , and $c = \tau \in W$. Let R_τ be the univalent map from the unit disc $\mathbf{D}$ onto W normalized so that $R_\tau(0) = \tau$. In the first step, we construct the Riemann map $R_\tau : t \in \mathbf{D} \mapsto c(t) \in W$. In the second step we calculate the conformal radius $r = |R'_\tau(0)|$ of W at τ .

4.1 The Riemann map

The construction we describe in this subsection is known, at least, implicitly, see e.g. [16]. As it follows from the λ -lemma [14], for any $c_0 \in W$, $c_0 \neq \tau$, the maps f_{c_0} and f_τ are conjugate by a quasi-conformal homeomorphism h of the plane:

$$h \circ f_\tau = f_{c_0} \circ h,$$

h is conformal off J_τ , and $h(z) \sim z$ at infinity. Let $\mu = \mu_h$ be the Beltrami coefficient of h : $\mu = (\partial h / \partial \bar{z}) / (\partial h / \partial z)$. Then:

- (1) $\mu = 0$ outside of J_τ , $0 < \|\mu\|_{L_\infty} < 1$,
- and, from the conjugacy equation,
- (2)

$$\mu(f_\tau(z)) = \mu(z) \left(\frac{f'_\tau(z)}{|f'_\tau(z)|} \right)^2$$

a.e.

First, we prove the uniqueness:

Proposition 4 *If ν satisfies (1)-(2), then there exists a number $\psi \neq 0$ such that $\nu = \psi \mu$ a.e.*

Proof. For any ν which satisfies (1)-(2), define $\hat{\nu}(z) = \int \nu(w) d\sigma_w / (z - w)$. Then (see [11]) $\hat{\nu}$ is a continuous function on the plane, $\hat{\nu}(\tau) \neq 0$, and $Q(z) := \hat{\nu}(z) / \hat{\nu}(\tau)$ satisfies the functional equation $f'_\tau(z) Q(z) - Q(f_\tau(z)) = -1$. Iterating this equation for z in the basin of infinity of f_τ , we obtain that $Q(z) = -\sum_{n \geq 0} 1 / (f_\tau^n)'(\tau)$ i.e. is defined uniquely. Therefore, $\hat{\nu}(z) = \psi \hat{\mu}(z)$ with $\psi := \hat{\nu}(\tau) / \hat{\mu}(\tau)$. Now, define $k(z) := \nu(z) - \psi \hat{\mu}(z)$. Then, for every z , $\int k(w) d\sigma_w / (z - w) = \hat{\nu}(z) - \psi \hat{\mu}(z) = 0$, which implies that $k = 0$ a.e. □

From now on, let's fix any μ which satisfies (1)-(2) and normalized so that $\|\mu\|_{L_\infty} = 1$. For every t , $|t| < 1$, denote by μ_t the Beltrami coefficient, which is equal to $t\mu(z)$ for $z \in E$ and 0 otherwise. Since $\|\mu_t\|_{L_\infty} = |t| < 1$ and μ_t is zero off J_τ , there exists a unique so-called normal solution of the Beltrami equation (see [1], Ch. V, Th. 1), i.e. a quasi-conformal homeomorphism h^t of the plane with the Beltrami coefficient μ_t and such that $h^t(0) = 0, h^t(z)/z \rightarrow 1$ as $z \rightarrow \infty$. Obviously, h^0 is the identity. Since μ_t satisfies the invariance condition (2), and by the normalizations,

$$h^t \circ f_\tau \circ (h^t)^{-1}(z) = z^\ell + c(t).$$

The correspondence $t \mapsto c(t)$ obeys the following properties:

- $c(0) = \tau$,

- $t \mapsto c(t)$ is injective.

Proof. Assume that $c(t_1) = c(t_2) = c$ while $t_1 \neq t_2$. Then $H \circ f_\tau = f_\tau \circ H$ where $H = (h^{t_2})^{-1} \circ h^{t_1}$. We know that H is quasi-conformal, H is conformal off J_τ , and $H(z)/z \rightarrow 1$ as $z \rightarrow \infty$. passing to the Bottcher coordinate of f_τ at infinity, we see that actually H is the identity map off J_τ , hence, everywhere. Then $h^{t_1} = h^{t_2}$ and $t_1 = t_2$, a contradiction.

□

- $t \in \mathbf{D} \mapsto c(t) \in W$ is surjective. This follows from Proposition 4.
- $c(t)$ is holomorphic in t . Indeed, $c(t) = f_{c(t)}(0) = h^t \circ f_\tau \circ (h^t)^{-1}(0) = h^t(\tau)$ is holomorphic in t , for fixed τ , by the Ahlfors-Bers theorem on the dependence on parameters.

Thus we can put $R_\tau(t) = c(t)$.

4.2 The conformal radius

Now we calculate the conformal radius of W at the point τ . In other words, we need to find $|c'(0)|$.

Theorem 5

$$c'(0) = -\frac{1}{\pi} \int \frac{\mu(z)}{z - \tau} d\sigma_z = -\frac{\ell}{\pi} \int z^{\ell-2} \mu(z) d\sigma_z. \quad (22)$$

Proof. Since $c(t) = h^t(\tau)$, we find

$$c'(0) = \frac{\partial h^t}{\partial t} \Big|_{t=0}(\tau).$$

We apply general theory of quasi-conformal mappings [1] and particularly the proof of Th. 1 of [1], Ch. V. One can write:

$$h^t(z) = P(\mu_t + \mu_t H_t)(z)$$

where:

$P : L_p \rightarrow C(\mathbf{C})$, $p > 2$, is the integral operator

$$Pg(z) = -\frac{1}{\pi} \int g(w) \left[\frac{1}{w - z} - \frac{1}{w} \right] d\sigma_w,$$

$H_t = R\mu_t + R\mu_t R\mu_t + R\mu_t R\mu_t R\mu_t + \dots$, where, in turn, R is a linear operator in L_p (an extension of an integral singular operator, see [1]). Given t , $|t| < 1$, the number $p > 2$ can be chosen so that the L_p -norm of the operator $g \mapsto \mu_t Rg$ is less

than 1. Therefore, the series in the definition of H_t converges in L_p absolutely. Now, since $\mu_t = t\mu$ and R is linear, $\mu_t + \mu_t H_t = t\mu + t^2 B(t, z)$, where

$$B(t, z) = R\mu + tR\mu R\mu + t^2 R\mu R\mu R\mu + \dots$$

is bounded in L_p uniformly on t from a neighborhood of $t = 0$. Thus,

$$h^t(z) = z + t(P\mu)(z) + t^2(PB)(t, z).$$

Since $P\mu$ and PB are continuous in z and t , we finally get:

$$\frac{\partial h^t}{\partial t}|_{t=0}(\tau) = (P\mu)(z) = -\frac{1}{\pi} \int \mu(w) \left[\frac{1}{w-\tau} - \frac{1}{w} \right] d\sigma_w.$$

Now, since μ is backward invariant Beltrami form on J_τ , one can write

$$\int \frac{\mu(w)}{w} d\sigma_w = \int \mu(w) T \frac{1}{w} d\sigma_w = 0$$

because

$$T \frac{1}{z} = \lim_{a \rightarrow 0} \frac{1}{f'(a)} \left[\frac{1}{z-f(a)} - \frac{1}{z-\tau} \right] = \lim_{a \rightarrow 0} \frac{a}{\ell(z-f(a))(z-\tau)} = 0.$$

On the other hand, as we know after (14), $\frac{1}{z-\tau} = (Tu_*)(z)$, where $u_*(z) = \ell z^{\ell-2}$. Using again the backward invariance of μ , we are left with the formula

$$\begin{aligned} c'(0) &= \frac{\partial h^t}{\partial t}|_{t=0}(\tau) = -\frac{1}{\pi} \int \mu(w) \frac{1}{w-\tau} d\sigma_w = -\frac{1}{\pi} \int \mu(w) T u_*(w) d\sigma_w \\ &= -\ell \frac{1}{\pi} \int \mu(w) w^{\ell-2} d\sigma_w. \end{aligned}$$

□

5 Example 1

In this section, we consider the Mandelbrot set M_2 (denoted by M). For $f = f_c$, denote by $(f^k)'(z)$ the derivative of $f^k(z)$ w.r.t. z , $c_k(c) = f^k(0)$, and $c'_k(c) = \frac{d}{dc} c_k(c)$.

Let $\tau \in \partial M$ be a Misiurewicz value. Assume, for simplicity, that, for some $p > 1$, $f_\tau^p(0) = a$, where a is a repelling fixed point of f_τ with multiplier λ_τ , $|\lambda_\tau| > 1$. First, we construct the sequence of parameters $c_n \rightarrow c_\tau$ as follows. Fix a small neighborhood Y of the point a , so that $\psi(Y) \subset Y$, for an inverse branch ψ of f_τ . Let $c_{-1}, \dots, c_{-r}$ be a segment of a backward orbit of 0 by f_τ , such that $c_{-r} \in Y, c_{-j} \notin Y, j \neq r$. Fix a small neighborhood U_τ of the parameter τ . Then we

have in U_τ analytic continuations $a(c), \lambda(c), \psi_c, c_{-j}(c)$, and sequence of analytic functions $g_n(c) := \psi_c^n(c_{-r}(c)) - f_c^p(0)$ in U_τ , which converges, as $n \rightarrow \infty$, to a non-zero function $g(c) := a(c) - f_c^p(0)$. Since $g(\tau) = 0$, there is a sequence $c_n \rightarrow \tau$ ($n > n_0$), such that $g_n(c_n) = 0$, i.e.

$$f_{c_n}^{p_n}(0) = 0$$

where $p_n = p + n + r$.

Denote by Ω_n the component of the (bounded) Fatou set F_{c_n} of f_{c_n} , which contains 0. Let δ_n be the conformal radius of Ω_n at 0, i.e. $\delta_n = |\phi_n'(0)|$, where $\phi_n : \mathbf{D} \rightarrow \Omega_n$, $\phi_n(0) = 0$, is the Bottcher coordinate of $f_{c_n}^{p_n}(0) = 0$.

Remind $\mathbf{D}_e = \{z : |z| < e\}$.

Proposition 5 (*jointly with F. Przytycki*)

(1) *There is a sequence $e_n \rightarrow 0$, such that*

$$\mathbf{D}_{(1-e_n)\delta_n} \subset \Omega_n \subset \mathbf{D}_{(1+e_n)\delta_n}.$$

(2) *There is a constant L , such that, for every n ,*

(2a)

$$\frac{\text{area}(F_{c_n})}{\text{area}(\Omega_n)} \leq L.$$

(2b)

$$\frac{1}{L} \leq \frac{\text{area}(\Omega_n)}{\text{dist}(c_n, \partial M)} \leq L.$$

Proof. Let us consider a Yoccoz puzzle structure [10] for f_τ which starts with a periodic orbit other than the fixed point a . Now fix a level of the structure, such that:

each point $f_\tau^k(0)$, $0 \leq k \leq p$, is contained in a different (open) puzzle piece, $c_{-r} \in Y_p$ and $c_{-j} \notin Y_p$, $0 \leq j < p$.

Let $Y_0, \dots, Y_p$ be those pieces, so that $f_\tau^k(0) \in Y_k$. One can assume that Y_p is the chosen above neighborhood Y of a .

The same picture persists in the neighborhood U_τ of τ . Denote by $Y_k(c)$ corresponding piece of f_c . Clearly, $Y_k(c)$ tends to Y_k as $c \rightarrow \tau$.

Fix $f = f_{c_n}$. Let's pull back $Y_0(c_n)$ by f along the orbit $f^j(0)$, $0 \leq j \leq p_n$. Let $Y^n(c_n)$ be the pullback: $0 \in Y^n(c_n)$ and $f^{p_n}(Y^n(c_n)) = Y_0(c_n)$. Then $f^{p_n} : Y^n(c_n) \rightarrow Y_0(c_n)$ is a quadratic-like map [7]: it is conjugate by a quasi-conformal map h_n to the map $z \mapsto z^2$, and $h_n : \mathbf{D} \rightarrow \Omega_n$ is conformal, i.e. $h_n|_{\mathbf{D}}$ can be chosen to be equal to the Bottcher coordinate ϕ_n . Since $\psi_{c_n} : Y_p(c_n) \rightarrow Y_p(c_n)$ is a contraction, $Y^n(c_n)$ shrink to the point 0 as $n \rightarrow \infty$. Therefore, the modulus of the fundamental annulus $Y_p(c_n) \setminus Y_p(c_n)$ tends to infinity, and the maximal dilatation of h_n tends to zero as $n \rightarrow \infty$. It implies (1).

Let's prove (2a). Let V be a component of F_{c_n} other than Ω_n . Then V contains a unique point x such that $f_{c_n}^k(x) = 0$, for some $k > 0$. The branch $f_{c_n}^{-k} : \Omega_n \rightarrow V$ extends univalently to the neighborhood $Y_0(c_n)$. Hence, the distortion of all these inverse branches are uniformly bounded, in fact, tend to zero as $n \rightarrow \infty$ (because Ω_n shrink to 0 while $Y_0(c_n)$ is roughly fixed). Therefore,

$$\mathbf{area}(F_{c_n}) \sim \mathbf{area}(\Omega_n) \cdot L_n$$

where

$$L_n = \sum_{n \geq 0} \sum_{f_{c_n}^n(x)=0} \frac{1}{|(f_{c_n}^n)'(x)|^2}.$$

Thus to prove (2a) we show that the sequence of L_n is bounded from above. Since a definite neighborhood K of Y_0 (and, by continuity, all $Y_0(c_n)$) contains no iterates of the critical point 0, any branch $f_{c_n}^{-k} : \Omega_n \rightarrow V$ as above extends univalently to the neighborhood K . It follows that L_n is comparable to the number $\frac{L'_n}{\mathbf{area}(Y_0(c_n))}$, where

$$L'_n = \sum_{k \geq 0} \sum_{f_{c_n}^{-k}} \mathbf{area}(f^{-k}(Y_0(c_n))),$$

and $f_{c_n}^{-k}$ runs over the all univalent branches defined on the neighborhood K . Since $\mathbf{area}(Y_0(c_n))$ is away from zero, L_n is comparable to L'_n . Thus we need to prove that the sum of the areas of all univalent preimages $\{Y\}$ of $Y_0(c_n)$ by the iterates of f_{c_n} is bounded uniformly in n . All such preimages are ordered by inclusion. If Y is such a preimage, its level is defined to be the number of other univalent preimages Y' which contain Y and so that the next one contains the previous one. Sum of the areas of the preimages of the zero level is uniformly (in n) bounded because they are disjoint. Consider the preimages of the first level. By the bounded distortion, it is enough to consider them in $Y_0(c_n)$. Thus we get a map R from the union of the preimages $V_1 \subset Y_0(c_n)$ of the first level onto $Y_0(c_n)$. Since a definite part of $Y_0(c_n)$ (close to an equipotential in the boundary of $Y_0(c_n)$) never returns to it by the iterates of f_{c_n} , there is a constant $\alpha > 0$ independent on n , such that the sum of the areas of the preimages $V_1 \subset Y_0(c_n)$ of the first level is less than $\mathbf{area}(Y_0(c_n))(1 - \alpha)$. Now, the preimages $V_2 \subset Y_0(c_n)$ of the second level are preimages of V_1 by R . Because of the bounded distortion, the sum of the areas of $V_2 \subset Y_0(c_n)$ is less than $\mathbf{area}(Y_0(c_n))(1 - \alpha')^2$, where $\alpha' < \alpha$ is independent on n . Continue this process, we see that the sum of the areas of the preimages $V_k \subset Y_0(c_n)$ of the level k is less than $\mathbf{area}(Y_0(c_n))(1 - \alpha')^k$. Thus L'_n are uniformly bounded, and the proof of (2a) is finished.

To prove (2b), let's calculate the conformal radius δ_n . Differentiating twice the equation $f_{c_n}^{p_n}(\phi_n(w)) = \phi_n(w^2)$ at $w = 0$, we get

$$\delta_n = \frac{2}{|(f_{c_n}^{p_n})''(0)|} = \frac{1}{|(f_{c_n}^{p_n-1})'(c_n)|}.$$

On the other hand, c_n is a center of a component W_n of the interior of M , and the conformal radius r_n of W_n at c_n is equal to (cf. Subsection 2.2)

$$r_n = |(f_{c_n}^{p_n-1})'(c_n)| \cdot \frac{d}{dc} |_{c=c_n} f_{c_n}^{p_n}(0)|^{-1} = |[(f_{c_n}^{p_n-1})'(c_n)]^2 \cdot S_{p_n}|^{-1} = \delta_n^2 \cdot |S_{p_n}|^{-1} \sim \frac{1}{\pi} \text{area}(\Omega_n) |S_{p_n}|^{-1},$$

where

$$S_{p_n} = \sum_{k=0}^{p_n-1} \frac{1}{(f_{c_n}^k)'(c_n)}.$$

We complete the proof if we show that S_{p_n} tend to a non-zero number. Let's prove that S_{p_n} tend to S_τ , where

$$S_\tau = \sum_{k=0}^{\infty} \frac{1}{(f_\tau^k)'(\tau)}.$$

By [18], [11], $S_\tau \neq 0$.

Let us divide the sum S_{p_n} into 3 parts: $\sum_n^1 = \sum_{k=0}^{p-1}$, $\sum_n^2 = \sum_{k=p}^{p+n-1}$, and $\sum_n^3 = \sum_{k=p+n}^{p+n+r-1}$. Obviously, as $n \rightarrow \infty$, $\sum_n^1 \rightarrow \sum_{k=0}^{p-1} 1/(f_\tau^k)'(\tau)$. The sum $\sum_n^3$ contains a fixed number of summands, and each of them tends (as $n \rightarrow \infty$) to zero (because $|(f_{c_n}^{p_n-1})'(c_n)| = 1/\delta_n \rightarrow 0$), hence, $\sum_n^3 \rightarrow 0$. Let us prove that $\sum_n^2 \rightarrow \sum_{k=p}^{\infty} 1/(f_\tau^k)'(\tau)$. Denote $z_n = f_{c_n}^{p+n-1}(c_n)$. as $n \rightarrow \infty$. One writes:

$$\sum_{k=p}^{p+n-1} \frac{1}{(f_{c_n}^k)'(c_n)} = \frac{1}{(f_{c_n}^p)'(c_n)} \sum_{j=0}^{n-1} (\psi_{c_n}^j)'(\psi_{c_n}^{n-j-1}(z_n)).$$

Now, for $c \in U_\tau$, by the Konigs Linearization Theorem, $\psi_c \circ K_c(w) = K_c(w/\lambda_c)$, where K_c is holomorphic at zero, $K(0) = a(c)$, $K'(0) = 1$. Moreover, $K_c(z)$ is holomorphic in (w, c) for $c \in U_\tau$, $|w| < \epsilon$, and therefore $1/K_c(w) = 1 + b_1(c)w + b_2(c)w^2 + \dots$, where all $b_k(c)$ are holomorphic in U_τ . Since $z_n = c_{-r}(c_n) \rightarrow c_{-r}$, one can assume that $z_n = K_{c_n}(w_n)$, where $w_n \rightarrow w_* \neq 0$. Then

$$(\psi_{c_n}^j)'(\psi_{c_n}^{n-j-1}(z_n)) = K_{c_n}'((\lambda(c_n))^{-(n-1)}) \frac{\lambda(c_n)^{-j}}{K_{c_n}'(w_n \lambda(c_n)^{-(n-j-1)})},$$

and

$$\begin{aligned} & \sum_{j=0}^{n-1} (\psi_{c_n}^j)'(\psi_{c_n}^{n-j-1}(z_n)) = \\ & (1+O(|\lambda|^{-n})) \cdot \sum_{j=0}^{n-1} \lambda(c_n)^{-j} (1+a_1(c_n)w_n \lambda(c_n)^{-(n-j-1)} + a_2(c_n)(w_n \lambda(c_n)^{-(n-j-1)})^2 + \dots) \sim \end{aligned}$$

$$\sum_{j=0}^{n-1} \lambda(c_n)^{-j} \sim \sum_{j=0}^{n-1} \lambda^{-j}.$$

□

6 On parabolic maps.

Here we consider a quadratic map $f : z \mapsto z^2 + c$ with a neutral parabolic periodic orbit $O = \{a_1, \dots, a_p\}$ of exact period p . As usual, ρ is the multiplier of O .

A formula for $1/\rho'$. We consider here the case $\rho \neq 1$ (for $\rho = 1$, see below). Then the derivative ρ' of ρ w.r.t. c exists. By Theorem 3, $\rho/\rho' = 2 \sum_{n>0} \sum_{f^n(w)=z} [(f^n)'(w)]^{-2}/A(z)$, and one can try to pass to the limit as $z \rightarrow O$ using local dynamics. We make use of Theorem 2 instead. So, let $q > 1$ be the minimal integer such that $\rho^q = 1$. Denote $a = a_0 = a_p \in O$. To state the result, we need the following notations.

- Complex number K_a is defined by the expansion

$$f^q(z) = z + K_a \cdot (z - a)^{q+1} + O|z - a|^{q+2}.$$

$K_a \neq 0$ because f has a unique critical point.

- Function $R(z)$ defined in a small enough neighborhood of a as follows. Given $j = 0, 1, \dots, p-1$, denote by $f_{a_j}^{-1}$ the branch of f^{-1} sending a_j to $-a_{j-1}$ ($a_{-1} := a_{p-1}$). Then let

$$R_j(z) = \sum_{f^{-n}} ((f^{-n} \circ f_{a_j}^{-1})'(z))^2, \quad (23)$$

where f^{-n} , $n = 0, 1, 2, \dots$, run over all branches and all n from a neighborhood of $-a_{j-1}$; R_j is defined and holomorphic in a neighborhood of a_j . Denote by f_0^{-j} the branch of f^j sending a_p to a_j , $j = 0, 1, \dots, p-1, p$. In particular, f_0^{-p} maps a_p to itself. At last, define

$$R(z) = \sum_{j=0}^{p-1} \sum_{k=0}^{q-1} R_j(f_0^{-j} \circ f_0^{-pk}(z)) \cdot [(f_0^{-j} \circ f_0^{-pk})'(z)]^2. \quad (24)$$

In other words, $R(z) = \sum [(f_O^{-n})'(z)]^2$ where f_O^{-n} run over all those branches of f^{-n} , $n \geq 0$, which spend near the cycle O at most pq backward iterates, and then go away of O . Then R is holomorphic at a . In fact, R has at the point a zero of order at least $q-2$. To see this, find a local holomorphic change of variable h , $h(0) = a$, such that $f_0^{-p}(h(u)) = h(g(u))$ where $g(u) = \rho u + O|u|^{q+1}$, and use the following fact which is easy to check: If Ψ is holomorphic at 0, then

$$\sum_{k=0}^{q-1} \Psi(g^k(u)) [(g^k)'(u)]^2 = q \Psi_{q-2} u^{q-2} + O|u|^q,$$

where $\Psi(u) = \sum_{n \geq 0} \Psi_n u^n$ is the power series. Thus there exists a complex number $P(a)$, such that, as $z \rightarrow a$,

$$R(z) \sim P(a) \cdot (z - a)^{q-2}. \quad (25)$$

Theorem 6

$$\frac{1}{\rho'} = \frac{1}{\rho} \frac{P(a)}{qK_a}. \quad (26)$$

Proof. Let $A(z)$ be the function associated with the periodic orbit O . To apply Theorem 2, Part (a^*) , we need some preparations. Let L be an attracting petal of f^p at the point a , and let $\Phi : \{w : \operatorname{Re}(w) > M\} \rightarrow L$ be the Fatou coordinate, i.e. a holomorphic isomorphism, such that $f^{pq}(\Phi(w)) = \Phi(w+1)$. It is known, that, as $w \rightarrow \infty$, $\Phi(w) \sim a + \frac{B}{w^{1/q}}$, where the number B is such that $B^q = -1/(qK_a)$. Given a real number $m > M + 1$, choose the fundamental region $C(m)$ as the image of vertical strip $\Pi(m)$: $C(m) = \Phi(\Pi(m))$, where $\Pi(m) = \{w : m-1 < \operatorname{Re}(w) < m\}$. Now let $\mu(z)$ be any function from $L_\infty(C(m))$, $|\mu(z)| \leq 1$ a.e. $z \in C(m)$. It is easy to see that one can spread it out backward as well as forward and get an extension (denoted again by μ) of μ from $C(m)$ to the whole (bounded) Fatou set F of f (i.e. F is the basin of attraction of O), so that $\mu(f(z)) = \mu(z)[f'(z)/|f'(z)|]^2$ a.e. in F .

Fact 6.1

$$\int_F \mu(z) d\sigma_z = 0. \quad (27)$$

This follows, for example, from Theorem 2, Part (a^*) . Indeed, it gives us that

$$\int_{C(m)} \mu(z) A(z) d\sigma_z = \frac{2\rho'}{\rho} \int_{V(m)} \mu(z) d\sigma(z), \quad (28)$$

where $V(m) = \cup_{n>0} f^{-n}(C(m))$. As $m \rightarrow +\infty$, $V(m)$ tends to F , so that it is enough to show that $\int_{C(m)} \mu(z) A(z) d\sigma_z \rightarrow 0$ as $m \rightarrow \infty$. As $w \rightarrow \infty$ ($\operatorname{Re}(w) > M$),

$$A(\Phi(w))(\Phi'(w))^2 \sim \frac{(\Phi'(w))^2}{(\Phi(w) - a)^2} \sim \frac{B^2}{q^2 w^{2+2/q}} \frac{1}{(B/w^{1/q})^2} = \frac{1}{q^2 w^2}.$$

It follows, as $m \rightarrow \infty$,

$$\int_{\Pi(m)} |A(\Phi(w))| \cdot |(\Phi'(w))^2| d\sigma_w \sim \frac{1}{q^2} \int_{\Pi(m)} \frac{d\sigma_w}{|w|^2} \rightarrow 0.$$

Since $|\int_{C(m)} \mu(z) A(z) d\sigma_z| \leq \int_{C(m)} |A(z)| d\sigma_z = \int_{\Pi(m)} |A(\Phi(w))| \cdot |(\Phi'(w))^2| d\sigma_w$, it proves the fact.

This fact and (28) give us:

$$\int_{C(m)} \mu(z) A(z) d\sigma_z = -\frac{2\rho'}{\rho} \int_{F \setminus V(m)} \mu(z) d\sigma(z). \quad (29)$$

This is the main equation to analyse. Make the following choice of $\mu = \mu_m$ on $C(m)$. Namely, let

$$Q(m) = \{w : m-1 < \operatorname{Re}(w) < m, -1 < \operatorname{Im}(w) < 1\}.$$

Now choose μ_m so that:

- $\mu_m(\Phi(w)) = 0$ for all $w \in P(m) \setminus Q(m)$,
-

$$\mu_m(\Phi(w)) = \frac{(\Phi'(w))^2}{|\Phi'(w)|^2}$$

for all $w \in Q(m)$.

First, we calculate the asymptotics (as $m \rightarrow +\infty$) of the left-hand side of (29). By the choice of μ_m ,

$$\int_{C(m)} \mu_m(z) A(z) d\sigma_z = \int_{Q(m)} A(\Phi(w)) \cdot (\Phi'(w))^2 d\sigma_w \sim \frac{1}{q^2} \int_{Q(m)} \frac{d\sigma_w}{w^2} \sim \frac{2}{q^2} \frac{1}{m^2}. \quad (30)$$

Let us look at the right-hand side of (29). Denote $B(m) = \cup_{k=1}^{\infty} f^{pqk}(C(m)) = \Phi(\{w : \operatorname{Re}(w) > m\})$. Now it is easy to understand that $F \setminus V(m) = C(m) \cup \cup_{n \geq 0} f_O^{-n}(B(m))$ where f_O^{-n} are exactly the branches from the definition of the function R . It follows from the invariance equation for μ_m ,

$$\int_{F \setminus V(m)} \mu_m(z) d\sigma_z = \int_{C(m)} \mu_m d\sigma + \int_{C(m)} \sum_{k=1}^{\infty} R(f^{pqk}(z)) [(f^{pqk})'(z)]^2 \mu_m(z) d\sigma_z.$$

By the choice of μ_m , the latter integral is equal to

$$\int_{Q(m)} \sum_{k=1}^{\infty} R(\Phi(w+k)) [\Phi'(w+k)]^2 d\sigma_w = \int_{G(m)} R(\Phi(w)) [\Phi'(w)]^2 d\sigma_w,$$

where $G(m) = \{w : \operatorname{Re}(w) > m, -1 < \operatorname{Im}(w) < 1\}$ is horizontal strip. As $w \rightarrow \infty$,

$$R(\Phi(w)) [\Phi'(w)]^2 \sim P(a) \left(\frac{B}{w^{1/q}} \right)^{q-2} \frac{B^2}{q^2 w^{2+2/q}} = \frac{1}{q^2} \frac{P(a)}{(-qK_a)} \frac{1}{w^3}.$$

But

$$\int_{G(m)} \frac{d\sigma_w}{w^3} \sim \frac{1}{m^2}.$$

Since $\int_{C(m)} \mu_m d\sigma = o(1/m^2)$, we finally get

$$-\frac{2\rho'}{\rho} \int_{F \setminus V(m)} \mu(z) d\sigma(z) \sim -\frac{2\rho'}{\rho} \frac{1}{q^2} \frac{P(a)}{(-qK_a)} \frac{1}{m^2} = \frac{2\rho'}{\rho} \frac{1}{q^2} \frac{P(a)}{qK_a} \frac{1}{m^2}. \quad (31)$$

Compare (30) with (31), we get the result.

□

Cauchy transform of μ . Let again $\mu \in L_\infty(F)$ be any invariant (backward and forward) Beltrami form of f on F . Fact 6.1 asserts that $\int_F \mu d\sigma = 0$. Consider the Cauchy transform of μ : $\hat{\mu}(z) = \int_F d\sigma_u/(u - z)$, which is continuous in the plane and holomorphic off $K = F \cup J$. But $\hat{\mu}(c) = 2 \int_F \mu d\sigma = 0$. From the functional equation for μ , see [11], we then get that $\hat{\mu}(z) = 0$ for every $z \notin K$, i.e. μ is “invisible” outside the filled-in Julia set.

Cusps. In fact, very similar analysis as in Theorem 6 can be done when $\rho = 1$ (cf. Comment 5). Then, instead of ρ' (which equals to ∞ in this case), one should consider $E := \lim(1 - \rho(\hat{c}))\rho'(\hat{c})$ when $\hat{c} \rightarrow c$ from, say, inside of corresponding hyperbolic component. Then $E \neq 0, \infty$ [6]. The result is that

$$\frac{1}{E} = \frac{\sum_{\{x,n\}} [(f^n)'(x)]^{-2}}{3[(f^p)''(a)]^2},$$

where $\{x, n\}$ are all *different* points x and corresponding *minimal* n , such that $f^n(x) = a$.

Doublings. In conclusion, consider the case $\rho = -1$. Then $K_a = (Sf^p)(a)/3$, where S denotes the Schwarzian derivative, and $P(a) = 2 \sum_{\{x,n\}} [(f^n)'(x)]^{-2}$, where $\{x, n\}$ are all different points x and corresponding minimal n , such that $f^n(x) = a$. Thus

$$\frac{1}{\rho'} = -3 \frac{\sum_{\{x,n\}} [(f^n)'(x)]^{-2}}{\sum_{k=0}^{p-1} (Sf)(f^k(a))[(f^k)'(a)]^2}.$$

7 Generalizations to polynomial-like maps.

Main results are generalized to polynbomial-like maps with single critical point. Here we do this for quadratic critical point.

Let $f : U \rightarrow W$ be a polynomial-like map [7], i.e. U, W are topological discs, U is compactly contained in W , and f is a holomorphic branched covering map of U on W . We assume that $f'(x) = 0$ iff $x = 0$ and $f''(0) \neq 0$. One can assume additionally that $f'''(0) = 0$ (for example, $f(z) = Q(z^2)$, for a holomorphic Q , $Q'(0) \neq 0$). Denote by K_f the filled-in Julia set of f , and $d(K_f)$ its Euclidian diameter.

Let us imbed f in the following family of mappings:

$$f_t(z) = f(z) + t \cdot \frac{f'(z)}{z} \tag{32}$$

There exists a neighborhood of $t = 0$ such that there $f_t : U_t \rightarrow W_t$ is still a polynomial-like map with the single quadratic critical point at 0 and with U_t and W_t close to U and W resp.

Let $P = \{a_1, \dots, a_p\}$ be a periodic orbit of f with exact period p and multiplier $\rho \neq 1$. Then it has an analytic continuation to a periodic orbit $\{a_1(t), \dots, a_p(t)\}$ of f_t , so that $\rho(t) = (f_t^p)'(a_j(t))$ is analytic around $t = 0$. Denote $\rho' = \rho'(0)$. Let us introduce a "total variation" along the orbit P :

$$V(P) = p \cdot |\rho| + \frac{d(K_f)}{|1 - \rho|} \sum_{k=1}^p |(f^p)''(a_k)|.$$

For example, let the orbit P be superattracting, i.e. $\rho = 0$. In this case

$$V(P) = d(K_f) \sum_{k=1}^p |(f^p)''(a_k)| = d(K_f) \sum_{k=1}^p \frac{2}{r_k},$$

where r_k is the conformal radius of the component of the basin of P which contains a_k , at the point a_k .

Theorem 1 is generalized to polynomial-like maps as follows:

Theorem 7 *There exist an absolute constant $\hat{L}$ and a decreasing function $M : (0, \infty) \rightarrow (0, \infty)$ so that $M(m) = O(\exp(-m))$ as $m \rightarrow \infty$, such that, for any polynomial-like map f with the modulus m of the fundamental annulus $W \setminus U$, and for any attracting periodic orbit P of f with multiplier ρ , the following inequality holds:*

$$\hat{L} \cdot (1 - |\rho|) \leq |\rho'| \cdot \text{area}(K_f) + M(m) \cdot V(P) \cdot \frac{\text{area}(K_f)}{d(K_f)^2}. \quad (33)$$

Comment 8 *The family of maps f_t appears naturally in the proof of the theorem. It also has the following nice property: for every periodic orbit with multiplier $\rho \neq 0$, the quantity $|\rho'| \cdot \text{area}(K_f)$ is invariant under linear changes of variable.*

Comment 9 *The function $M(m)$ is expressed through the following data: let γ be the core curve of the fundamental annulus, $R = \text{dist}(f^{-1}(\gamma), K_f)$, and l is the Euclidian length of $f^{-1}(\gamma)$. Then $M(m)$ is equal essentially to $(d(K_f)/l)^2 \cdot (l/2\pi R) \cdot (1 + d(K_f)/l)$.*

Comment 10 *Theorem 7 gives as well a generalization of the Douady-Hubbard-Sullivan Theorem to polynomial-like maps. Namely, if attracting periodic orbit P of the polynomial-like map f is such that*

$$\hat{L} \cdot (1 - |\rho|) > M(m) \cdot V(P) \cdot \frac{\text{area}(K_f)}{d(K_f)^2}$$

then $\rho' \neq 0$.

The proof of Theorem 7 is similar to the one of Theorem 1. It is contained in the following statements. Let $P = \{a_1, \dots, a_p\}$ be a periodic orbit of f with exact period p and multiplier $\rho \neq 1, 0$. As usual, denote

$$A(z) = A_P(z) = \sum_{k=1}^p \frac{1}{(z - a_k)^2} + \frac{1}{\rho(1 - \rho)} \sum_{k=1}^p \frac{(f^p)''(a_k)}{z - a_k}.$$

Firstly, we have the following generalization of Theorem 2:

Theorem 8 *For any periodic orbit P of f as above, the function*

$$h(z) := A(z) - (TA)(z) - \rho'(T1)(z) \quad (34)$$

is holomorphic throughout the range W .

The proof begins as follows. If H and ϕ are any two functions holomorphic in $W \setminus K_f$, and γ is any contour enclosing K_f such that $f^{-1}(\gamma)$ is "inside" γ , then a change of variable gives us:

$$\frac{1}{2\pi i} \int_{\gamma} H(z)[\phi(z) - (T\phi)(z)]dz = \frac{1}{2\pi i} \int_{f^{-1}(\gamma)} \phi(z)[H(z) - \frac{H(f(z))}{f'(z)}]dz \quad (35)$$

Now we put H to be any function holomorphic in W and $\phi = A$. Then we use the Residue Theorem and the following two identities

$$\frac{f''(a_k)}{(f'(a_k))^2} - \frac{1}{\rho(1 - \rho)} \frac{(f^p)''(a_k)}{f'(a_k)} = - \frac{(f^p)''(a_{k+1})}{\rho(1 - \rho)}$$

and

$$A(0) = -\frac{\rho'}{\rho}$$

which can be proved very similar to the proofs of analogous identities in Section 2. As a result we get that $\frac{1}{2\pi i} \int_{f^{-1}(\gamma)} H(z)h(z)dz = 0$ for every holomorphic function H in W . The statement that h is holomorphic in W follows. Now we can use the Maximum Principal to estimate h on the set K_f . At last, we repeat the proofs of Propositions 2 and 3, and get the result.

References

- [1] Ahlfors, L.: *Lectures on quasiconformal mappings*. Van Nostrand, 1966
- [2] Carleson, L. & Gamelin, T.: *Complex dynamics*. Springer-Verlag, 1993
- [3] Coullet, P. & Tresser, C.: *Iteration d'endomorphismes et groupe de renormalisation*, CRAS **287 A** (1978)

- [4] Douady, A.: *Systemes dynamiques holomorphes*. Seminare Bourbaki, v. 1982-83, n. 599, Asterisque, 105-106, 1983, 39-63
- [5] Douady, A. & Hubbard, J.H.: *Iteration des polynomes quadratiques complexes*. C.R.A.S., t. 294 (1982), 123-126
- [6] Douady, A. & Hubbard, J.H.: *Etude dynamique des polynomes complexes*. Pub. Math. d'Orsay, 84-02 (1984), 84-04 (1985)
- [7] Douady, A. & Hubbard, J.H.: *On the dynamics of polynomial-like maps*. Ann. Sc. Ec. Norm. Sup., v. 18 (1985), 287-343
- [8] Epstein, A.: *Infinitesimal Thurston rigidity and the Fatou-Shishikura inequality*, preprint Stony Brook, 1999.
- [9] Feigenbaum, M.: *Qualitative universality for a class of non-linear transformations*, J. Stat. Phys. **19** (1978), 25-52
- [10] Hubbard, J. H.: *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*. In: "Topological Methods in Modern Mathematics." Publish or perish, 1993
- [11] Levin, G.: *On an analytic approach to the Fatou conjecture*. To appear in Fundamenta Mathematicae
- [12] Levin, G., Sodin, M. & Yuditski, P.: *A Ruelle operator for a real Julia set*. Comm. Math. Phys. 141 (1991), 119-132
- [13] Lyubich, M.: *Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture*. Annals of Math. v. 149 (1999), 319-420
- [14] Mane, R., Sad, P. & Sullivan, D.: *On the dynamics of rational maps*. Ann. Sci. Ec. Norm. Sup. 16 (1983), 193-217
- [15] Makienko, P.: *Remarks on Ruelle operator and invariant line field problem*. Part I: preprint FIM, 25, Zurich, July 2000. Part II: preprint, 2001.
- [16] McMullen, C.: *Complex dynamics and renormalization*, Ann. of Math. Studies **135**, Princeton University Press (1994)
- [17] McMullen, C.: *Renormalization and 3-manifolds which fiber over the circle*, Ann. of Math. Studies **142**, Princeton University Press (1996)
- [18] Rivera-Letelier, J.: *On the continuity of Hausdorff dimension of Julia sets and similarity between the Mandelbrot set and Julia sets*. Fundamenta Mathematicae, v. 170 (2001), n. 3, 287-317
- [19] Sullivan, D.: *Bounds, quadratic differentials and renormalization conjecture*. In: Mathematica into twenty-first century, AMS Centennial publications (1991)
- [20] Tsujii, M.: *A simple proof for monotonicity of entropy in the quadratic family*. Erg. Th. Dyn. Syst. 20 (2000), n. 3, 925-933