Excited Boundary TBA
in the Tricritical Ising Model

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Abstract

By considering the continuum scaling limit of the $A_4$ RSOS lattice model of Andrews-Baxter-Forrester with integrable boundaries, we derive excited state TBA equations describing the boundary flows of the tricritical Ising model. Fixing the bulk weights to their critical values, the integrable boundary weights admit a parameter $\xi$ which plays the role of the perturbing boundary field $\varphi_{1,3}$ and induces the renormalization group flow between boundary fixed points. The boundary TBA equations determining the RG flows are derived in the $B_{(1,2)} \rightarrow B_{(2,1)}$ example. The induced map between distinct Virasoro characters of the theory are specified in terms of distribution of zeros of the double row transfer matrix.

1 Introduction

Quantum Field Theories with a nontrivial boundary have received recently a lot of attention, due to their applications in Condensed Matter, Solid State Physics and String Theory (D-branes). A problem of great interest is the Renormalization Group flow between different boundary fixed points of a CFT that remains conformal in the bulk. Many interesting results have been achieved, and flows have been studied for minimal models of Virasoro algebra and for $c = 1$ CFT (see e.g. [1] and references therein). Numerical scaling functions for the flow of states interpolating two different boundary conditions can be systematically explored by use of the Truncated Conformal Space Approach [2, 3].

A beta function can be defined for the boundary deformations, much the same as for the bulk perturbations of conformal field theories [4]. The conformal boundary conditions can thus play the role of ultraviolet (UV) and infrared (IR) points of the flow. One gets out of the UV point by perturbing with a relevant boundary operator and gets into an IR one attracted by irrelevant boundary operators.

Among the possible boundary perturbations of a CFT there are some that keep an infinite number of conservation laws. These are referred to as integrable boundary perturbations. In this particular case experience suggest that there should be some exact

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method of investigation, based on transfer matrix and Bethe ansatz techniques. One of
the most celebrated of these methods is the Thermodynamic Bethe Ansatz (TBA) giving a set of non-linear coupled integral equations governing the scaling functions along the RG flow. In the following we illustrate how one can obtain TBA equations from a lattice construction for the integrable boundary flows.

The simplest nontrivial example to consider is the second model in the Virasoro minimal series, the Tricritical Ising Model (TIM) with central charge \( c = \frac{7}{10} \). It has interesting applications in Solid State Physics and Statistical Physics. Its Kac table is given here below:

\[
\begin{array}{ccc}
2 & 1 & 3 \\
10 & 3 & 5 \\
1 & 7 & 3 \\
16 & 2 &
\end{array}
\]

We have drawn only half of the table, by taking into account the well known \( \mathbb{Z}_2 \) symmetry \((r, s) \equiv (4 - r, 5 - s)\) that reduces the independent chiral primaries to 6 in the TIM.

Conformal boundary conditions for minimal models with diagonal modular invariant partition function of type \((A_p, A_p')\) are classified to be in one to one correspondence with chiral primary fields in the Kac table. So there are 6 different conformal boundary conditions \( B_{(r,s)} \), \( r = 1, 2, 3 \) and \( s = 1, 2 \) for the TIM \((A_3, A_4)\).

Let us consider in the two-dimensional plane \((x,t)\) the TIM defined on a strip of thickness \( L \), i.e in the region \( 0 \leq x \leq L \) and \( -\infty < t < +\infty \), with non-trivial b.c. \( B_{(r,s)} \) on the left \((x = 0)\) and \( B_{(r',s')} \) on the right \((x = L)\) edge respectively. This situation will be denoted \( B_{(r,s)}|_{(r',s')} \). In the following we are interested in boundaries of type \((r, s)|(1,1)\). We shall denote for short \( B_{(r,s)}|(1,1) = B_{(r,s)} \). For this particular class of boundary conditions the partition function simply reduces to one single character

\[
Z_{B_{(r,s)}}(q) = \chi_{(r,s)}(q)
\]

where \( \chi_{(r,s)}(q) \) denotes the character of the irreducible Virasoro representation labeled by \((r, s)\) at central charge \( c = \frac{7}{10} \).
For each given boundary condition $B_{(r,s)}$ there is a set of boundary operators $\phi_{(u,v)}$ that live on the edge. If we want to keep the b.c. unchanged all along the edge, these operators must be restricted to the conformal families appearing in the OPE fusion of the Virasoro family $(r, s)$ with itself: $(u, v) \in (r, s) \times (r, s)$. They are distinguished in terms of their conformal dimensions $h_{r,s}$ as relevant ($h_{r,s} < 1$) and irrelevant ($h_{r,s} > 1$). Of course only relevant perturbations break scale invariance at the boundary in such a way to get out of the fixed boundary point of a specific boundary condition and flow to another one. So if we want to consider possible relevant boundary perturbations of the TIM, i.e. QFT's described by the action

$$S = S_{(r,s)} + \lambda \int_{-\infty}^{+\infty} dt \phi_{(u,v)}(x = 0, t)$$

where $S_{(r,s)}$ denotes the action of TIM with boundary condition $B_{(r,s)}$ we have to restrict to the following possibilities

<table>
<thead>
<tr>
<th>boundary condition</th>
<th>boundary perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{(1,1)} \equiv B_{(3,4)}$</td>
<td>$-$</td>
</tr>
<tr>
<td>$B_{(2,1)} \equiv B_{(3,4)}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$B_{(3,1)} \equiv B_{(1,4)}$</td>
<td>$+$</td>
</tr>
<tr>
<td>$B_{(1,2)} \equiv B_{(3,3)}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$B_{(2,2)} \equiv B_{(2,3)}$</td>
<td>$d$</td>
</tr>
<tr>
<td>$B_{(3,2)} \equiv B_{(1,3)}$</td>
<td>$0+$</td>
</tr>
</tbody>
</table>

Actually there are two physically different flows for each “pure” perturbation (i.e. containing only one operator $\phi_{(12)}$ or $\phi_{(13)}$), flowing to two possibly different IR destines. This is achieved by taking different signs in the coupling constant in the case of $\phi_{(13)}$ flows, and real or purely imaginary coupling constant in case of $\phi_{(12)}$ ones. The boundary condition $B_{(2,2)}$ can be perturbed by any linear combination of the fields $\phi_{(1,2)}$ and $\phi_{(1,3)}$. The symbols $+, -, 0+, -0, 0, d$ represent other ways to denote the TIM conformal boundary conditions present in the literature [8, 9], and we report them here only to facilitate translations to the reader.

Along the flow, going from UV to IR, the boundary entropy associated to each $B_{(r,s)}$

$$g_{(r,s)} = \left(\frac{2}{5}\right)^{1/4} \sin \frac{\pi r}{5} \sin \frac{\pi s}{5} \sqrt{\sin \frac{\pi r}{\pi} \sin \frac{\pi s}{5}}$$

decreases [4]. Therefore we expect to have flows only between boundary conditions where the starting conformal boundary entropy is higher than the final one.

The possible conformal boundary conditions have been studied extensively by Chim [8] and the flows connecting them by Affleck [9]. The picture can be summarized as in fig. [10] Integrability can be investigated in a manner similar to the bulk perturbations and it turns out that the flows generated by pure $\phi_{(1,3)}$ and $\phi_{(1,2)}$ perturbations are integrable.

3
Figure 1: The boundary flows between TIM conformal b.c. Pure $\phi_{(1,3)}$ and $\phi_{(1,2)}$ flows are integrable. More information can be found in [9].
Instead, the flow starting at $B_{(2,2)}$ as a perturbation which is a linear combination of $\phi_{(1,2)}$ and $\phi_{(1,3)}$ is strongly suspected to be non-integrable.

Notice the $\mathbb{Z}_2$ symmetry of fig. II which is strictly related to the supersymmetry of the TIM. Investigation of the supersymmetric aspects of the TIM boundary flows is, however, out of the scope of the present talk.

We deduce TBA equations for the integrable flows from the functional relations for the transfer matrix of a lattice RSOS realization of the model. We will focus, as an example, on the flow obtained by perturbing the $B_{(1,2)}$ boundary TIM by the relevant boundary operator $\varphi_{(1,3)}$

$$UV = B_{(1,2)} \mapsto B_{(2,1)} = IR, \quad \chi_{1,2}(q) \mapsto \chi_{2,1}(q)$$

(1.1)

This flow induces a map between Virasoro characters $\chi_{r,s}(q)$ of the theory, where $q$ is the modular parameter. The physical direction of the flow from the ultraviolet (UV) to the infrared (IR) is given by the relevant perturbations and is consistent with the $g$-theorem [I]. The approach outlined here, however, is quite general and should apply, for example, to all integrable boundary flows of minimal models.

2 Scaling of Critical $A_4$ Model with Boundary Fields

It is well known that the TIM is obtained as the continuum scaling limit of the generalized hard square model of Baxter [10] on the Regime III/IV critical line, i.e. the $A_4$ RSOS integrable lattice model of Andrews-Baxter-Forrester [11], with critical Boltzmann weights

$$W \left( \begin{array}{ccc} d & c \\ a & b & u \end{array} \right) = \frac{\sin \left( \frac{\pi}{5} - u \right)}{\sin \frac{\pi}{5}} \delta_{a,c} + \frac{\sin u}{\sin \frac{\pi}{5}} \left( \frac{\sin \frac{\pi a}{5} \sin \frac{\pi c}{5}}{\sin \frac{\pi b}{5} \sin \frac{\pi d}{5}} \right)^{1/2} \delta_{b,d}$$

satisfying the Yang-Baxter equation. The indices $a, b, c, d = 1, \ldots 4$ take values on the $A_4$ Dynkin diagrams, whose adjacency dictates which are the nonzero elements of the Boltzmann weights.

In the presence of integrable boundaries, one must also introduce the so called boundary Boltzmann weights, satisfying the Boundary Yang-Baxter equations. Their general form has been given in [12], for our purposes here we select
\[
K \left( \frac{a}{a} a + 1 \mid u \right) = \left( \sin \frac{\pi(a \pm 1)}{5} \frac{\sin \frac{\pi a}{5}}{\sin \frac{\pi}{5}} \right)^{1/2} \sin(u \pm \xi_L) \sin(u \mp \frac{\pi a}{5} \mp \xi_L) \sin^2 \frac{\pi}{5}
\]

All these Boltzmann weights are periodic under \( u \rightarrow u + \pi \). The scaling energies of the TIM are obtained from the scaling limit of the eigenvalues of the commuting double-row transfer matrices \( D(u) \).

\[
\dim D^{(N)}_{(a_L, a_R)} = (\mathcal{A}^N)_{a_L, a_R}
\]

where the matrix \( \mathcal{A} \) is the incidence matrix of the \( A_4 \) Dynkin diagram. Here the notation \( D_{(a_L, a_R)} \) means the sub-matrix of \( D \) for fixed \( a_L, a_R \) boundary spins. It is convenient to introduce the normalized double row transfer matrix

\[
t(u) = \mathcal{P}(u, \xi_L) D(u)
\]
that satisfies the following functional equation:

\[ t(u)t \left( u + \frac{\pi}{5} \right) = 1 + t \left( u + \frac{3\pi}{5} \right). \]  

(2.1)

For the interested reader, the expression for the normalization factor \( \mathcal{P}(u, \xi_L) \) is given in [13]. Even if the equation for \( t \) looks boundary independent, the solutions are fixed by the behavior of the zeros, that are boundary dependent. The transfer matrix \( \mathbf{D}(u) \) is entire and has only zeros; \( t(u) \) has instead two complex conjugate poles (coming from \( \mathcal{P}(u, \xi_L) \)) in the strip \(-\frac{\pi}{10} < \text{Im}u < \frac{3\pi}{10}\). We shall see that they will affect the boundary driving term in the TBA equations we are going to deduce.

In a lattice of \( N \) horizontal and \( M \) vertical sites, with periodic boundary conditions on the vertical direction and \( a_L, a_R \) boundary conditions on the horizontal one, the partition function of the tricritical hard square model can be written as

\[ Z_{MN}(u) = \text{Tr}_\mathcal{H} \left( \mathbf{D}(a_L, a_R)^M \right) \sim e^{-NMf_{\text{bulk}}(u) - Mf_{\text{surf}}(u)} Z(q) \]

where \( f_{\text{bulk}} \) and \( f_{\text{surf}} \) are the bulk and surface contributions to the free energy respectively. \( Z(q) \) is the universal conformal partition function, with modular parameter

\[ q = e^{-2\pi M N \sin 5u} \]

Therefore the behavior of each transfer matrix eigenvalue is

\[ \log d(u) = -NF_{\text{bulk}}(u) - F_{\text{surf}}(u) - \frac{2\pi}{N} \left( \Delta - \frac{c}{24} + n \right) \sin 5u + o \left( \frac{1}{N} \right) \]

with \( n \) a nonnegative integer. It is then logical to consider for the eigenvalues \( t(u) \) of \( t(u) \), the decomposition for \( N \to \infty \)

\[ t(u) = f(u)g(u)l(u) \]

where \( f(u) \sim O(N), \ g(u) \sim O(1) \) and \( l(u) \sim O(1/N) \).

In general the conformal partition function expands over characters of the \( c = \frac{7}{10} \) Virasoro algebra

\[ Z(q) = \sum_{r,s} N_{r,s} \chi_{r,s}(q) \]

where \( N_{r,s} \) are nonnegative integers describing the multiplicities of Virasoro representations in the Hilbert space.

Bauer and Saleur [13] used Coulomb gas techniques to show that if one chooses the boundary conditions leading to the configuration depicted in fig.2, the single character \( \chi_{r,s} \), i.e. the partition function of TIM with \( \mathcal{B}_{r,s} \) boundary conditions, can be realized in the scaling limit \( N \to \infty \). The configuration can be built by taking the boundary Boltzmann weights described above, with \( a_L = r, \ a_R = s \) and \( \text{Re}\xi_L = \frac{\pi}{10} \). This latter choice kills the weight alternating \( r \) and \( r-1 \) on the left at the isotropic point \( u = \frac{\pi}{10} \).
At criticality there exist integrable boundary conditions associated with each conformal boundary condition \((r, s)\) carrying 0, 1 or 2 arbitrary boundary parameters \(\xi_L, \xi_R\) \[\text{[12]}\]. To consider boundary flows we must fix the bulk at its critical point and vary the boundary parameters \(\xi_L, \xi_R\). However, these “fields” are irrelevant in the sense that they do not change the scaling energies if they are real and restricted to appropriate intervals \[\text{[13]}\]. Explicitly, the cylinder partition functions obtained \[\text{[13]} \text{[12]}\] from \((r, s)\) integrable boundaries with \(\xi_L, \xi_R\) real are independent of \(\xi_L, \xi_R\) and given by single Virasoro characters

\[
B_{(r,s)} : \quad Z(q) = \chi_{r,s}(q). \tag{2.2}
\]

The reason for this is that in the lattice model the fields \(\xi_L\) and \(\xi_R\) control the location of zeros of eigenvalues of \(D(u)\) on the real axis in the complex plane of the spectral parameter \(u\) whereas only the zeros in the scaling regime, a distance \(i \log N\) out from the real axis, contribute in the scaling limit \(N \to \infty\). The solution is to scale the imaginary part of \(\xi_L, \xi_R\) as \(\log N\). This is allowed because \(\xi_L, \xi_R\) are arbitrary complex fields.

We find that only one parameter at the time (we call it \(\xi_L\)) is needed to induce the \(\varphi_{(1,3)}\) RG flows and is identified as the thermal boundary field:

\[
\xi_L \sim \varphi_{(1,3)}. \tag{2.3}
\]

For the flow \[\text{[11]}\], following \[\text{[13]}\], we consider the \((r, s) = (2, 1)\) boundary with the \((1, 1)\) boundary on the right (with no boundary field) and the \((2, 1)\) boundary on the left with boundary field \(\xi_L\). We scale \(\xi_L\) as

\[
\text{Re}(\xi_L) = \frac{3}{10} \pi, \quad \text{Im}(\xi_L) = -\frac{\xi + \log N}{5} \tag{2.4}
\]

where \(\xi\) is real. In terms of boundary weights, the \((1, 2)\) boundary is reproduced in the limit \(\text{Im}(\xi_L) \to \pm \infty\) whereas the \((2, 1)\) boundary is reproduced in the limit \(\text{Im}(\xi_L) \to 0\).
The scaling behavior of the $A_4$ lattice model at the boundary fixed points is described by the TBA equations of O’Brien, Pearce and Warnaar [13] (OPW). We generalize their analysis to include the scaling boundary field [2,4].

3 Classification of zeros

The zeros of $D_{(a_L,a_R)}(u)$ can be classified in two strips, the first given by $-\frac{\pi}{10} < \text{Re}(u) < \frac{3\pi}{10}$ and the second by $\frac{2\pi}{5} < \text{Re}(u) < \frac{4\pi}{5}$. We refer to them as strip I and II respectively. Each zero has to be accompanied by its complex conjugate, and there are only two zeros on the real axis depending on $\text{Re}\xi_L$ that do not play any role in the following. So we are led to consider only zeros in the upper half $u$ complex plane excluding the real axis. For $N$ sufficiently large general considerations supported by numerical inspection allow to say that there exists

- $m_1$ 1-strings at $\frac{\pi}{10} + iy_k^{(1)}$ and $n_1$ 2-strings at $\frac{(2j+1)\pi}{10} + iz_k^{(1)}$ with $j = 0, 1$ in strip I
- $m_2$ 1-strings at $\frac{3\pi}{5} + iy_k^{(2)}$ and $n_2$ 2-strings at $\frac{(2j+1)\pi}{5} + iz_k^{(2)}$ with $j = 0, 1$ in strip II

The precise classification can be done in terms of the relative ordering (with respect to their imaginary parts) of the 1 and 2-strings. This can be achieved by giving the sequence of numbers $I_k^{(i)}$ of 2-strings with $z_k^{(i)} > y_k^{(i)}$. In the case of $B_{(1,2)}$ this must be supplemented [13] by a parity of strings $\sigma = \pm 1$. Because we order the zeros as $y_{k+1}^{(i)} > y_k^{(i)}$, the quantum numbers must satisfy $I_1^{(i)} \geq I_2^{(i)} \geq \ldots \geq I_{m_1}^{(i)} \geq 0$. Moreover, in the second strip the largest quantum number must be less or equal the number of 2-strings of type 2, $n_2 \geq I_1^{(i)}$. The 2-strings in strip I are infinite in number at the scaling limit (see below), and their quantum numbers are unbounded. We write this sequence as

$$I = (I_1^{(1)}, I_2^{(1)}, \ldots, I_{m_1}^{(1)}, I_1^{(2)}, I_2^{(2)}, \ldots, I_{m_2}^{(2)})$$

with $\sigma = \begin{cases} \pm 1 & \text{for } B_{(1,2)} \\ 0 & \text{for } B_{(2,1)} \end{cases}$

There are constraints on the numbers of strings, analyzed in detail in [13], depending on the specific $B_{(r,s)}$ condition

- for $B_{(1,2)}$ ($m_1$ odd, $m_2$ even)
  $$n_1 = \frac{N + m_2 + \sigma}{2} - m_1 \quad , \quad n_2 = \frac{m_1 - \sigma}{2} - m_2$$

- for $B_{(2,1)}$ ($m_1$ odd, $m_2$ odd)
  $$n_1 = \frac{N + m_2}{2} - m_1 \quad , \quad n_2 = \frac{m_1 + 1}{2} - m_2$$

9
In the scaling limit $N \to \infty$ the number of 2-strings in strip I becomes infinite: $n_1 \sim O(N)$, while the number of the other three types of roots (1-strings in both strips and 2-strings in strip II) remains of order $O(1)$. In [13] it was shown that the pattern of zeros fixes the energies (here $m = (m_1, m_2)$ and $C$ is the $A_4$ Cartan matrix)

$$E(B_{(1,2)}) = \frac{1}{10} + \frac{mCm}{4} + \sigma \frac{m_1 + m_2}{2} + \sum_{i=1}^{m_1} \sum_{k=1}^{m_2} I_k^{(i)}$$

$$E(B_{(2,1)}) = \frac{7}{16} - \frac{1}{2} + \frac{mCm}{4} + \sum_{i=1}^{m_1} \sum_{k=1}^{m_2} I_k^{(i)}$$

Out of these energy expressions it is possible to reconstruct the characters $\chi_{1,2}$ and $\chi_{2,1}$. First consider the so called finitized characters [16]

$$\chi_{r,s}^N(q) = q^{-\frac{r^2}{2}} \sum_{E} q^E = q^{-\frac{r^2}{2}} \text{Tr} \left( \frac{D^{(N)}(u)}{D_0^{(N)}(u)} \right)^{\frac{M}{2}}$$

where $D_0^{(N)}$ is the largest eigenvalue of the double row transfer matrix with $(r, s)$ boundary conditions and $N$ sites. The limit $N \to \infty$ of such objects can be shown to give the well known formulae for the TIM characters (for details see [17]).

Once the characters have been related in such a strict way to the distribution of zeros in strips I and II, it is also possible to see how they are mapped one another along the flow $B_{(1,2)} \to B_{(2,1)}$. For this purpose it is more convenient to consider the reverse flow $B_{(2,1)} \to B_{(1,2)}$ from IR to UV. The counting of the total number of zeros in the upper $u$ plane for a $B_{(2,1)}$ configuration is $N$, while for $B_{(1,2)}$ is $N - 1$. Therefore one zero must disappear to infinity along this reverse flow (or appear from infinity in the physical flow). We have found 3 mechanisms that seem to exhaust the possible ways for such a phenomenon to be realized [17]

- **A.** The top 1-string in strip 2 flows to $+\infty$, decoupling from the system while $m_1$, $n_1$ and $n_2$ remain unchanged. This mechanism applies in the IR when $I_{m_1}^{(1)} = I_{m_2}^{(1)} = 0$ and produces frozen states in the UV with $\sigma = 1$ and $I_{m_1}^{(1)} = 0$.

- **B.** The top 2-string in strip 1 and the top 1-string in strip 2 flow to $+\infty$ and a 2-string comes in from $+\infty$ in strip 2 becoming the top 2-string. Consequently, each $I_j^{(1)}$ decreases by 1 and each $I_k^{(2)}$ increases by 1. This mechanism applies in the IR when $I_{m_1}^{(1)} > 0$ and $I_{m_2}^{(2)} = 0$ and produces states in the UV with $\sigma = -1$ and either $m_2 = 0$ or $I_{m_2}^{(2)} > 0$.

- **C.** The top 2-string in strip 2 flows to $+\infty$ and a 1-string in strip 2 comes in from $+\infty$. Consequently, each $I_k^{(2)}$ decreases by 1. This mechanism applies in the IR when $I_{m_2}^{(2)} > 0$ and produces states in the UV with $\sigma = -1$ and $I_{m_2}^{(2)} = 0$. 
Observe that

- these mappings are in fact one-to-one
- the 1-strings in strip 1 are never involved
- the change in parity of $m_2$ is consistent with (3.1) and (3.2).
- for A,B: $m_2^{UV} = m_2^{IR} - 1$ and for C: $m_2^{UV} = m_2^{IR} + 1$.

By doing such substitutions $m_2^{UV} \to m_2^{IR} \pm 1$ in the finitized characters, it is possible to show explicitly that $\chi_{1,2}(q) \to \chi_{2,1}(q)$ under the physical flow $\mathcal{B}_{(1,2)} \to \mathcal{B}_{(2,1)}$. See [17] for details.

4 Derivation of Boundary TBA Equations

4.1 On the lattice

The recurrence relation (2.41) for the $O(N)$ piece yields the following functional equation for the function $f(u)$

$$f(u)f(u + \frac{\pi}{5}) = 1$$

Plugging its solution (for details see [13]) back into the functional equation (2.41) it yields an equation for $g(u)$ that in turn can be solved in the first analyticity strip $-\frac{\pi}{10} < \text{Re}u < \frac{\pi}{10}$. In the following, for any $h(u) = f(u), g(u), l(u)$ we introduce the functions

$$h_1(x) = h\left(\frac{\pi}{10} + i\frac{x}{5}\right), \quad h_2(x) = h\left(\frac{3\pi}{5} + i\frac{x}{5}\right)$$

each one being analytic in the corresponding strip I or II.

The recurrence relation for $g(u)$ yields

$$g_1(x, \xi_L) = \tanh^2 \frac{x}{2} \tanh \frac{x + 5i\xi_L}{2} \tanh \frac{x - 5i\xi_L}{2}$$

and an analogous formula for $g_2(x)$ that we do not need in the following. The lattice TBA equations contain the boundary contribution encoded in the function $g_1(x)$. It becomes exactly 1 at the conformal points and takes $\xi_L$ dependent values during the flow. Notice that $g_1(x)$ depends on $\xi_L$ only through its imaginary part.

Inserting this value for $g_1(x)$ into the recurrence equation again, one can determine a constraint to be satisfied by the last pieces $l_i(x)$, in the form of an integral equation. Putting all these things together we arrive finally to two coupled integral equations to be satisfied by $t_1(x)$ and $t_2(x)$

$$-\log t_1(x) = -\log g_1(x) + \sum_{j=1}^{m_1} \log \tanh \frac{1}{2}(x - v_j^{(1)}) - K * \log(1 + t_2) \quad (4.1)$$

$$-\log t_2(x) = -N \log(\tanh^2 \frac{x}{2}) + \sum_{k=1}^{m_2} \log \tanh \frac{1}{2}(x - v_k^{(2)}) - K * \log(1 + t_1) \quad (4.2)$$
where the kernel of the convolution is:

\[ K(x) = \frac{1}{2\pi \cosh(x)}. \]

Observe that in \( t_i \) there is an implicit \( N \) dependence that will take a role performing the scaling limit. Only the 1-strings appear explicitly in the TBA equations, \( v_k^{(i)} \) being their imaginary parts. Their location is dictated by the quantization conditions

\[
-\log \left( -t_1(v_k^{(2)}) \right) = n_k^{(2)} \pi i \tag{4.3}
\]

\[
-\log \left( -t_2(v_k^{(1)}) \right) = n_k^{(1)} \pi i \tag{4.4}
\]

where \( n_k^{(1)} = 2(m_1 - k + I_k^{(1)}) + 1 - m_2 \) and \( n_k^{(2)} = 2(m_2 - k + I_k^{(2)}) + 1 - m_1 \).

### 4.2 The scaling limit

In the scaling limit \( N \to \infty \) where \( \text{Im} \xi_L \) rescales as in \([24]\) we define \( \hat{h}_i(x) = \lim_{N \to \infty} h_i(x + \log N) \). We have

\[
\hat{g}_1(x, \xi) = \tanh \frac{x + \xi}{2}
\]

with \( \xi \to -\infty \) and \( \xi \to +\infty \) corresponding to \( B_{(1,2)} \) and \( B_{(2,1)} \) respectively.

Eqs. \([1.3, 1.4]\) fix the following large \( N \) behavior for \( v^{(i)} \):

\[
y = \lim_{N \to \infty} (5v - \log N) = \text{finite val.}
\]

Performing this scaling limit on \([1.2, 1.3, 1.4]\), and introducing the so-called pseudoenergies

\[
\varepsilon_i(x) = -\log \left( -\hat{t}_i(x) \right)
\]

and the functions

\[
L_i(x) = \log(1 - e^{-\varepsilon_i(x)})
\]

we obtain the following final TBA equations:

\[
\varepsilon_1(x) = -\log \tanh \frac{x + \xi}{2} - \sum_{j=1}^{m_1} \log \tanh \left( \frac{y_j^{(1)} - x}{2} \right) - K * L_2(x) \tag{4.7}
\]

\[
\varepsilon_2(x) = 4e^{-x} - \sum_{k=1}^{m_2} \log \tanh \left( \frac{y_k^{(2)} - x}{2} \right) - K * L_1(x). \tag{4.8}
\]

The definition of the scaling theory must be completed by the energy formula

\[
E(\xi) - \frac{c}{24} = \frac{1}{\pi} \lim_{R \to 0} RE(R) =
\]

\[
\frac{1}{\pi} \left[ \sum_{j=1}^{m_1} 2e^{-y_j^{(1)}} - \int_\infty^{\infty} \frac{dx}{\pi} e^{-x} \log(1 + e^{-\varepsilon_2(x)}) \right] \tag{4.9}
\]
where the limit $R \to 0$ means that all this computations are performed at the critical temperature of the TIM. The boundary parameter $\xi$ appears only implicitly in (4.9). The last piece required to complete the scaling picture is the location of the zeros. From (4.3 4.4) we obtain:

$$
\varepsilon_2(y_j^{(1)} - \frac{\pi}{2}) = n_k^{(1)} \pi i \quad \text{first strip zeros} \tag{4.10}
$$

$$
\varepsilon_1(y_k^{(2)} - \frac{\pi}{2}) = n_k^{(2)} \pi i \quad \text{second strip zeros} \tag{4.11}
$$

The two-string zeros satisfy similar equations, using the substitution $y_k^{(i)} \to z_k^{(i)}$.

An immediate consequence is that $L_i(x)$ vanish on the zeros of the same strip:

$$
L_i(y_k^{(i)}) = 0. \tag{4.12}
$$

The inverse statement is not true, in the sense that there can be points where $L_i$ vanishes without corresponding to some zero.

We do not deal here with the numerical analysis of the TBA equations found above. Details can be found in [17] and will be an important part of our paper in preparation [18] where we explore the other integrable flows of the TIM. Generalizations to higher minimal models and to other rational CFT’s should follow the same lines, although they can become more technically involved.

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References


