Seiberg-Witten Prepotential from Instanton Counting

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(To Arkady Vainshtein on his 60th anniversary)

Abstract

In my lecture I consider integrals over moduli spaces of supersymmetric gauge field configurations (instantons, Higgs bundles, torsion free sheaves).

The applications are twofold: physical and mathematical; they involve supersymmetric quantum mechanics of D-particles in various dimensions, direct computation of the celebrated Seiberg-Witten prepotential, sum rules for the solutions of the Bethe ansatz equations and their relation to the Laumon's nilpotent cone. As a by-product we derive some combinatoric identities involving the sums over Young tableaux.

1. Introduction

The dynamics of gauge theories is a long and fascinating subject. The dynamics of supersymmetric gauge theories is a subject which is better understood [1] yet may teach us something about the real QCD. The solution of Seiberg and Witten [2] of $\mathcal{N} = 2$ gauge theory using the constraints of special geometry of the moduli space of vacua led to numerous achievements in understanding of the strong coupling dynamics of gauge theory and also in string theory, of which the gauge theories in question arise as low energy limits. The low energy effective Wilsonian action for the massless vector multiplets $a$ is governed by the prepotential $\mathcal{F}$, which receives one-loop perturbative and instanton non-perturbative corrections:

$$\mathcal{F}(a) = \mathcal{F}^{pert}(a; \Lambda) + \mathcal{F}^{inst}(a; \Lambda). \quad (1.1)$$

In spite of the fact that these instanton corrections were calculated in many indirect ways, their gauge theory calculation is lacking beyond two instantons[3][4]. The
The problem is that the instanton measure seems to get very complicated with the growth of the instanton charge, and the integrals are hard to evaluate.

The present paper attempts the solution of this problem via the localization technique, proposed long time ago in [5][6][7]. Our method can be explained rather simply in the physical terms. We calculate the vacuum expectation value of certain gauge theory observables. These observables are annihilated by a combination of the supercharges, and their expectation value is not sensitive to various parameters. In particular, one can do the calculation in the ultraviolet, where the theory is weakly coupled and the instantons dominate. Or, one can do the calculation in the infrared, where it is rather simple to relate the answer to the prepotential of the effective low-energy theory. By equating these two calculations we obtain the desired formula.

Remark. We can also formulate our results in a more mathematical language. We study $G \times T^2$ equivariant cohomology of the (suitably partially compactified) moduli space $\tilde{\mathcal{M}}_k$ of framed $G$-instantons on $\mathbb{R}^4$, where $G$ is the gauge group, which acts by rotating the gauge orientation of the instantons at infinity, and $T^2$ is the maximal torus of $SO(4)$ – the group of rotations of $\mathbb{R}^4$ which also acts naturally on the moduli space. We consider the following quantity:

$$Z(a, \epsilon_1, \epsilon_2; q) = \sum_{k=0}^{\infty} q^k \oint_{\tilde{\mathcal{M}}_k} 1$$

where $\oint 1$ denotes the localization of 1 in $H^*_G \times T^2(\tilde{\mathcal{M}}_k)$. The latter takes values in the field of fractions of the ring $H^*_G \times T^2(pt)$ which is identified with the space of $G \times T^2$ invariant polynomial functions on the Lie algebra of $G \times T^2$. By the Chevalley theorem the latter is isomorphic to the ring of Weyl invariant functions on the Cartan subalgebra of $G$ and $T^2$. We denote the coordinates on the Cartan of $G$ by $a$ and the coordinates on the Lie algebra of $T^2$ by $\epsilon_1, \epsilon_2$. In explicit calculations we represent 1 by a cohomologically equal form which allows to replace $\oint 1$ by an ordinary integral:

$$\oint_{\tilde{\mathcal{M}}_k} 1 = \int_{\tilde{\mathcal{M}}_k} \exp \omega + \mu_G(a) + \mu_{T^2}(\epsilon_1, \epsilon_2)$$

where $\omega$ is a symplectic form on $\tilde{\mathcal{M}}_k$, invariant under the $G \times T^2$ action, and $\mu_G, \mu_{T^2}$ are the corresponding moment maps.

Our first claim is

$$Z(a, \epsilon_1, \epsilon_2; q) = \exp \left( \frac{\mathcal{F}^{\text{inst}}(a, \epsilon_1, \epsilon_2; q)}{\epsilon_1 \epsilon_2} \right)$$

where the function $\mathcal{F}^{\text{inst}}$ is analytic in $\epsilon_1, \epsilon_2$ near $\epsilon_1 = \epsilon_2 = 0$. 
We also have the following explicit expression for $Z$ in the case $\epsilon_1 = -\epsilon_2 = \hbar$ (in the general case we also have a formula, but it looks less transparent) for $G = SU(N)$ (a simple generalization to $SO$ and $Sp$ cases will be presented in [8]):

$$Z(a, h, -h; q) = \sum_k q^{|k|} \prod_{(l, i) \neq (m, j)} \frac{a_{lm} + h (k_{l,i} - k_{m,j} + j - i)}{a_{lm} + \hbar (j - i)}.$$  (1.5)

Here the sum is over all colored partitions: $\vec{k} = (k_1, \ldots, k_N)$, $k_l = \{k_{l,1} \geq k_{l,2} \geq \ldots k_{l,n} \geq k_{l,n+1} = k_{l,n+2} = \ldots = 0\}$, and

$$|\vec{k}| = \sum_{l, i} k_{l,i},$$

and the product is over $1 \leq l, m \leq N$, and $i, j \geq 1$.

Already (1.5) can be used to make rather powerful checks of the Seiberg-Witten solution. But the checks are more impressive when one considers the theory with fundamental matter.

To get there one studies the bundle $V$ over $\tilde{\mathcal{M}}_k$ of the solutions of the Dirac equation in the instanton background. Let us consider the theory with $N_f$ flavors. It can be shown that the gauge theory instanton measure calculates in this case (cf. [9]):

$$Z(a, m, \epsilon_1, \epsilon_2; q) = \sum_k q^{|k|} \int_{\tilde{\mathcal{M}}_k} \text{Eu}_{G \times T^2 \times U(N_f)}(V \otimes M)$$  (1.6)

where $M = \mathfrak{g}^{N_f}$ is the flavor space, where the flavor group $U(N_f)$ acts, $m = (m_1, \ldots, m_{N_f})$ are the masses = the coordinates on the Cartan subalgebra of the flavor group Lie algebra, and finally $\text{Eu}_{G \times T^2 \times U(N_f)}$ denotes the equivariant Euler class.

The formula (1.5) generalizes in this case to:

$$Z(a, m, \epsilon_1, \epsilon_2; q) = \sum_k (q h^{N_f})^{|k|} \prod_{(l, i) \neq (m, j)} \frac{\Gamma(\frac{a_{l,m} + m_j}{h} + 1 + k_{l,i} - i)}{\Gamma(\frac{a_{l,m} + m_j}{h} + 1 - i)} \times \prod_{(l, i) \neq (m, j)} \frac{a_{lm} + \hbar (k_{l,i} - k_{m,j} + j - i)}{a_{lm} + \hbar (j - i)}.$$  (1.7)

Again, we claim that

$$\mathcal{F}^{\text{inst}}(a, m, \epsilon_1, \epsilon_2; q) = \epsilon_1 \epsilon_2 \log Z(a, m, \epsilon_1, \epsilon_2; q)$$  (1.8)

is analytic in $\epsilon_{1,2}$.

The formulae (1.5)(1.7) were checked against the Seiberg-Witten solution [10]. Namely, we claim that $\mathcal{F}^{\text{inst}}(a, m, \epsilon_1, \epsilon_2)|_{\epsilon_1 = \epsilon_2 = 0} =$ the instanton part of the prepotential of the low-energy effective theory of the $\mathcal{N} = 2$ gauge theory with the
gauge group $G$ and $N_f$ fundamental matter hypermultiplets. We have checked this claim by an explicit calculation for up to five instantons, against the formulae in [11]. There is also a generalization of (1.5) to the case of adjoint matter. We shall present it in the main body of the paper.

2. Field theory expectations

In this section we explain our approach in the field theory language. We exploit the fact that the supersymmetric gauge theory on flat space has a large collection of observables whose correlation functions are saturated by instanton contribution in the limit of weak coupling. In addition, in the presence of the adjoint scalar vev these instantons tend to shrink to zero size. Moreover, the observables we choose have the property that the instantons which contribute to their expectation values are localized in space. This solves the problem of the runaway of point-like instantons, pointed out in [5].

2.1. Supersymmetries and twisted supersymmetries

The $\mathcal{N} = 2$ theory has eight conserved supercharges, $Q^i_{\alpha}, Q^i_{\dot{\alpha}}$, which transform under the global symmetry group $SU(2)_L \times SU(2)_R \times SU(2)_I$ of which the first two factors belong to the group of spatial rotations and the last one is the $R$-symmetry group. The indices $\alpha, \dot{\alpha}, i$ are the doublets of these respective $SU(2)$ factors. The basic multiplet of the gauge theory is the vector multiplet. It is useful to work in the notations which make only $SU(2)_L \times SU(2)_d$ part of the global symmetry group manifest. Here $SU(2)_d$ is the diagonal subgroup of $SU(2)_R \times SU(2)_I$. If we call this subgroup a “Lorentz group”, then the supercharges, superspace, and the fermionic fields of the theory will split as follows:

**Fermions:** $\psi_\mu, \chi^+_{\mu\nu}, \eta$;

**Superspace:** $\theta^\mu, \theta^+_{\mu\nu}, \bar{\theta}$;

**Superfield:** $\Phi = \phi + \theta^\mu \psi_\mu + \frac{1}{2} \theta^\mu \theta^\nu F_{\mu\nu} + \ldots$;

**Supercharges:** $Q, Q^+_{\mu\nu}, G_\mu$.

The supercharge $Q$ is a scalar with respect to the “Lorentz group” and is usually considered as a BRST charge in the topological quantum field theory version of the susy gauge theory. It is conserved on any four-manifold.

In [12] E. Witten has employed a self-dual two-form supercharge $Q^+_{\mu\nu}$ which is conserved on Kähler manifolds.

Our idea is to use other supercharges $G_\mu$ as well. Their conservation is tied up with the isometries of the four-manifold on which one studies the gauge theory. Of course, the idea to regularize the supersymmetric theory by subjecting it to the twisted boundary conditions is very common both in physics [13], and in mathematics [14][15][16].
2.2. Good observables: UV

With respect to the standard topological supercharge $Q$ the observables one is usually interested in are the gauge invariant polynomials $O^{(k)}_{P,x} = P(\phi(x))$ in the adjoint scalar $\phi$, evaluated at space-time point $x$, and its descendants: $O^{(k)}_{P,\Sigma} = \int_{\Sigma} P(\phi + \psi + F)$, where $\Sigma$ is a $k$-cycle. Unfortunately for $k > 0$ all such cycles are homologically trivial on $\mathbb{R}^4$ and no non-trivial observables are constructed in such a way. One construct an equivalent set of observables by integration over $\mathbb{R}^4$ of a product of a closed $4-k$-form $\omega = \frac{1}{(4-k)!} \omega_{\mu_1...\mu_{4-k}} \theta^{\mu_1}...\theta^{\mu_{4-k}}$ and the $k$-form part of $P(\phi + \psi + F)$:

$$O^\omega_P = \int d^4x d^4\theta \, \omega(x,\theta) \, P(\Phi).$$  \hspace{1cm} (2.1)

Again, most of these observables are $Q$-exact, as any closed $k$-form on $\mathbb{R}^4$ is exact for $k > 0$.

However, if we employ the rotational symmetries of $\mathbb{R}^4$ and work equivariantly, we find new observables.

Namely, consider the fermionic charge

$$\tilde{Q} = Q + E a \Omega^a_{\mu\nu} x^\mu \partial_\nu.$$  \hspace{1cm} (2.2)

Here $\Omega^a = \Omega^a_{\mu\nu} x^\mu \partial_\nu$ for $a = 1...6$ are the vector fields generating $SO(4)$ rotations, and $E \in Lie(SO(4))$ is a formal parameter.

With respect to the charge $\tilde{Q}$ the observables $O^{(k)}_{P,\Sigma}$ are no longer invariant (except for $O^{(0)}_{P,0}$ where $0 \in \mathbb{R}^4$ is the origin, left fixed by the rotations.

However, the observables (2.1) can be generalized to the new setup, producing a priori nontrivial $Q$-cohomology classes. Namely, let us take any $SO(4)$-equivariant form on $\mathbb{R}^4$. That is, take an inhomogeneous differential form $\Omega(E)$ on $\mathbb{R}^4$ which depends also on an auxiliary variable $E \in Lie(SO(4))$ which has the property that for any $g \in SO(4)$:

$$g^* \Omega(E) = \Omega(g^{-1}Eg)$$  \hspace{1cm} (2.3)

where we take pullback defined with the help of the action of $SO(4)$ on $\mathbb{R}^4$ by rotations. Such $E$-dependent forms are called equivariant forms. On the space of equivariant forms acts the so-called equivariant differential,

$$D = d + \iota_{V(E)}$$  \hspace{1cm} (2.4)

where $V(E)$ is the vector field on $\mathbb{R}^4$ representing the infinitesimal rotation generated by $E$. For equivariantly closed (i.e. $D$-closed) form $\Omega(E)$ the observable:

$$O^{\Omega(E)}_P = \int_{\mathbb{R}^4} \Omega(E) \wedge P(\Phi)$$  \hspace{1cm} (2.5)

is $\tilde{Q}$-closed.
Any $SO(4)$ invariant polynomial in $E$ is of course an example of the $D$-closed equivariant form. Such a polynomial is characterized by its restriction onto the Cartan subalgebra of $SO(4)$, where it must be Weyl-invariant. The Cartan subalgebra of $SO(4)$ is two-dimensional. Let us denote the basis in this subalgebra corresponding to the decomposition $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ into a orthogonal direct sum of two dimensional planes, by $(\epsilon_1, \epsilon_2)$. Under the identification $\text{Lie}(SO(4)) \approx \text{Lie}(SU(2)) \oplus \text{Lie}(SU(2))$ these map to $(\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2)$.

Let us fix in addition a translationally invariant symplectic form $\omega$ on $\mathbb{R}^4$. It's choice breaks $SO(4)$ down to $U(2)$ – the holonomy group of a Kähler manifold. Let us fix this $U(2)$ subgroup. Then we have a moment map:

$$\mu : \mathbb{R}^4 \to \text{Lie}(U(2))^*, \quad d\mu(E) = \iota_V(E)\omega, \quad E \in \text{Lie}(U(2)).$$ (2.6)

And therefore, the form $\omega - \mu(E)$ is $D$-closed. One can find such euclidean coordinates $x^\nu, \nu = 1, 2, 3, 4$ that the form $\omega$ reads as follows:

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4.$$ (2.7)

The Lie algebra of $U(2)$ splits as a direct sum of one-dimensional abelian Lie algebra of $U(1)$ and the Lie algebra of $SU(2)$. Accordingly, the moment map $\mu$ splits as $(h, \mu^1, \mu^2, \mu^3)$. In the $x^\mu$ coordinates

$$h = \sum_\mu (x^\mu)^2, \quad \mu^a = \frac{1}{2} \eta^a_{\mu\nu} x^\mu x^\nu,$$ (2.8)

where $\eta^a_{\mu\nu}$ is the anti-self-dual 't Hooft symbol.

Finally, the choice of $\omega$ also defines a complex structure on $\mathbb{R}^4$, thus identifying it with $\mathbb{C}^2$ with complex coordinates $z_1, z_2$ given by: $z_1 = x^1 + ix^2, z_2 = x^3 + ix^4$. For $E$ in the Cartan subalgebra $H = \mu(E)$ is given by the simple formula:

$$H = \epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2.$$ (2.9)

After all these preparations we can formulate the correlation function of our interest:

$$Z(a, \epsilon) = \langle \exp \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} (\omega \wedge \text{Tr} (\phi F + \frac{1}{2} \psi \psi) - H \text{ Tr} (F \wedge F)) \rangle_a$$ (2.10)

where we have indicated that the vacuum expectation value is calculated in the vacuum with the expectation value of the scalar $\phi$ in the vector multiplet given by $a \in \mathfrak{t}$. More precisely, $a$ will be the central charge of $\mathcal{N} = 2$ algebra corresponding to the $W$-boson states.

**Remarks.** 1) Note that the observable in (2.10) makes the widely separated instantons suppressed. More precisely, if the instantons form clusters around points $\vec{r}_1, \ldots, \vec{r}_l$ then they contribute $\sim \exp - \sum_m H(\vec{r}_m)$ to the correlation function.
2) One can expand (2.10) as a sum over different instanton sectors:

\[ Z(a, \epsilon) = \sum_{k=0}^{\infty} q^k Z_k(a, \epsilon) \]

where \( q \sim \Lambda^{2N} \) is the dynamically generated scale – for us – simply the generating parameter.

3) The supersymmetry guarantees that (2.10) is saturated by instantons. Moreover, the superspace of instanton zero modes is acted on by a finite dimensional version of the supercharge \( \tilde{Q} \) which becomes an equivariant differential on the moduli space of framed instantons. Localization with respect to this supercharge reduces the computation to the counting of the isolated fixed points and the weights of the action of the symmetry groups (a copy of gauge group and \( U(2) \) of rotations) on the tangent spaces. This localization can be understood as a particular case of the Duistermaat-Heckman formula [17], as (2.10) calculates essentially the integral of the exponent of the Hamiltonian of a torus action (Cartan of \( G \) times \( T^2 \)) against the symplectic measure.

The counting of fixed points can be nicely summarized by a contour integral (see below). This contour integral also can be obtained by transforming the integral over the ADHM moduli space of the observable (2.10) evaluated on the instanton configuration, by adding \( \tilde{Q} \)-exact terms, as in [7][6]. It also can be derived from Bott’s formula [18].

### 2.3. Good observables: IR

The nice feature of the correlator (2.10) is its simple relation to the prepotential of the low-energy effective theory. In order to derive it let us think of the observable (2.10) as of a slow varying changing of the microscopic coupling constant. If we could completely neglect the fact that \( H \) is not constant, then its adding would simply renormalize the effective low-energy scale \( \Lambda \rightarrow \Lambda e^{-H} \).

However, we should remember that \( H \) is not constant, and regard this renormalization as valid up to terms in the effective action containing derivatives of \( H \). Moreover, \( H \) is really a bosonic part of the function \( \mathcal{H}(x, \theta) \) on the (chiral part of) superspace (in [5] we considered such superspace-dependent deformations of the theory on curved four-manifolds):

\[ \mathcal{H}(x, \theta) = H(x) + \frac{i}{2} \omega_{\mu\nu} \theta^\mu \theta^\nu. \]

Together these terms add up to the making the standard Seiberg-Witten effective action determined by the prepotential \( \mathcal{F}(a; \Lambda) \) to the one with the superspace-dependent prepotential

\[ \mathcal{F}(a; \Lambda e^{-\mathcal{H}(x, \theta)}) = \mathcal{F}(a; \Lambda e^{-H}) + \omega \Lambda \partial_\Lambda \mathcal{F}(a; \Lambda e^{-H}) + \frac{1}{4} \omega^2 (\Lambda \partial_\Lambda)^2 \mathcal{F}(a; \Lambda e^{-H}), \]

(2.11)
This prepotential is then integrated over the superspace (together with the conju-
gate terms) to produce the effective action.

Now, let us go to the extreme infrared, that is let us scale the metric on \( \mathbb{R}^4 \) by a very large factor \( t \) (keeping \( \omega \) intact). On flat \( \mathbb{R}^4 \) the only term which may contribute to the correlation function in question in the limit \( t \to \infty \) is the last term in (2.10) as the rest will (after integration over superspace) necessarily contain couplings to the gauge fields which will require some loop diagrams to get non-trivial contractions, which all will be suppressed by inverse powers of \( t \). The last term, on the other hand, gives:

\[
Z(a; \epsilon) = e^{\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \omega \wedge \omega \frac{\partial^2 F(a; \Lambda e^{-H})}{(\partial \log \Lambda)^2} + O(\epsilon)}
\]  

(2.12)

where we used the fact that the derivatives of \( H \) are proportional to \( \epsilon_1, \epsilon_2 \). Recalling (2.7)(2.8) the integral in (2.12) reduces to:

\[
Z(a; \epsilon_1, \epsilon_2) = e^{\frac{\mathcal{F}_{\text{inst}}(a; \Lambda) + O(\epsilon)}{\epsilon_1 \epsilon_2}}
\]  

(2.13)

where

\[
\mathcal{F}_{\text{inst}}(a; \Lambda) = \int_{0}^{\infty} \partial_{H}^{2} F(a; \Lambda e^{-H}) \, H \, dH,
\]

thereby explaining our claim about the analytic properties of \( Z \) and \( \mathcal{F} \).

3. Instanton measure and its localization

3.1. ADHM data

The moduli space \( \mathcal{M}_{k,N} \) of instantons with fixed framing at infinity has dimension \( 4kN \). It has the following convenient description. Take two complex vector spaces \( V \) and \( W \) of the complex dimensions \( k \) and \( N \) respectively. These spaces should be viewed as Chan-Paton spaces for \( D(p - 4) \) and \( Dp \) branes in the brane realization of the gauge theory with instantons.

Let us also denote by \( L \) the two dimensional complex vector space, which we shall identify with the Euclidean space \( \mathbb{R}^4 \approx \mathbb{C}^2 \) where our gauge theory lives.

Then the ADHM [19] data consists of the following maps between the vector spaces:

\[
V \xrightarrow{\tau} V \otimes L \oplus W \xrightarrow{\sigma} V \otimes \Lambda^2 L
\]  

(3.1)

where

\[
\tau = \begin{pmatrix} B_2 \\ -B_1 \\ J \end{pmatrix}, \quad \sigma = \begin{pmatrix} B_1 & B_2 & I \end{pmatrix},
\]

(3.2)

\( B_{1,2} \in \text{End}(V), \ I \in \text{Hom}(W, V), \ J \in \text{Hom}(V, W) \).
The ADHM construction represents the moduli space of $U(N)$ instantons on $\mathbb{R}^4$ of charge $k$ as a hyperkähler quotient [20] of the space of operators $(B_1, B_2, I, J)$ by the action of the group $U(k)$ for which $V$ is a fundamental representation, $B_1, B_2$ transform in the adjoint, $I$ in the fundamental, and $J$ in the anti-fundamental representations.

More precisely, the moduli space of proper instantons is obtained by taking the quadruples $(B_1, B_2, I, J)$ obeying the so-called ADHM equations:

$$\mu_c = 0, \quad \mu_r = 0,$$

(3.3)

where:

$$\mu_c = [B_1, B_2] + IJ,$$

$$\mu_r = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J$$

(3.4)

and with the additional requirement that the stabilizer of the quadruple in $U(k)$ is trivial. This produces a non-compact hyperkähler manifold $M_{k,N}$ of instantons with fixed framing at infinity.

The framing is really the choice of the basis in $W$. The group $U(W) = U(N)$ acts on these choices, and acts on $M_{k,N}$, by transforming $I$ and $J$ in the anti-fundamental and the fundamental representations respectively.

This action also preserves the hyperkähler structure of $M_{k,N}$ and is generated by the hyperkähler moment maps:

$$m_r = I^\dagger I - JJ^\dagger, \quad m_c = JI.$$

(3.5)

Actually, $\text{Tr}_W m_{r,c} = \text{Tr}_V \mu_{r,c}$, thus the central $U(1)$ subgroup of $U(N)$ acts trivially on $M_{k,N}$. Therefore it is the group $G = SU(N)/\mathbb{Z}_N$ which acts non-trivially on the moduli space of instantons.

3.2. Instanton measure

The supersymmetric gauge theory measure can be regarded as an infinite-dimensional version of the equivariant Matthei-Quillen representative of the Thom class of the bundle $\Gamma(\Omega^{2,+} \otimes g_P)$ over the infinite-dimensional space of all gauge fields $A_P$ in the principal $G$-bundle $P$ (summed over the topological types of $P$). In physical terms, in the weak coupling limit we are calculating the supersymmetric delta-function supported on the instanton gauge field configurations. In the background of the adjoint Higgs vev, this supersymmetric delta-function is actually an equivariant differential form on the moduli space $M_{k,N}$ of instantons. It can be also represented using the finite-dimensional hyperkähler quotient ADHM construction of $M_{k,N}$ (as opposed to the infinite-dimensional quotient of the space of all gauge fields by the action of the group of gauge transformations, trivial at infinity) [7]:

$$\int D\phi D\bar{\phi} D\bar{H} D\chi D\eta D\psi DB D\bar{D} D\bar{I} D\bar{J} e^{\tilde{Q}(\bar{\chi} \bar{\bar{\mu}}(B, I, J) + \psi V(\bar{\phi}) + \eta [\phi, \bar{\phi}])}$$

(3.6)
where, say:

\[
\begin{align*}
\tilde{Q} B_{1,2} &= \Psi B_{1,2}, \quad \tilde{Q} \Psi B_{1,2} = [\phi, B_{1,2}] + \epsilon_{1,2} B_{1,2}, \\
\tilde{Q} I &= \Psi I, \quad \tilde{Q} \Psi I = \phi I - I a, \\
\tilde{Q} J &= \Psi J, \quad \tilde{Q} \Psi J = -J \phi + J a - (\epsilon_1 + \epsilon_2) J, \\
\tilde{Q} \chi_r &= H_r, \quad \tilde{Q} H_r = [\phi, \chi_r], \quad \tilde{Q} \chi_c = H_c, \quad \tilde{Q} H_c = [\phi, \chi_c] + (\epsilon_1 + \epsilon_2) \chi_c, \\
\Psi \cdot V (\tilde{\phi}) &= \text{Tr} \left( \Psi B_{1} [\tilde{\phi}, B_{1}^\dagger] + \Psi B_{2} [\tilde{\phi}, B_{2}^\dagger] + \Psi I [\tilde{\phi}, I^\dagger] - \Psi J [\tilde{\phi}, J^\dagger] + \text{c.c.} \right)
\end{align*}
\]

(3.7)

(we refer to [7] for more detailed explanations). If the moduli space \( M_{k,N} \) was compact and smooth one could interpret (3.6) as a certain topological quantity and apply the powerful equivariant localization techniques [21] to calculate it.

The non-compactness of the moduli space of instantons is of both ultraviolet and of infrared nature. The UV non-compactness has to do with the instanton size, which can be made arbitrarily small. The IR non-compactness has to do with the non-compactness of \( \mathbb{R}^4 \) which permits the instantons to run away to infinity.

### 3.3. Curing non-compactness

The UV problem can be solved by relaxing the condition on the stabilizer, thus adding the so-called point-like instantons. A point of the hyperkähler space \( \tilde{M}_{k,N} \) with orbifold singularities which one obtains in this way (Uhlenbeck compactification) is an instanton of charge \( p \leq k \) and a set of \( k - p \) points on \( \mathbb{R}^4 \):

\[
\tilde{M}_{k,N} = M_{k,N} \cup M_{k-1,N} \times \mathbb{R}^4 \cup M_{k-2,N} \times \text{Sym}^2(\mathbb{R}^4) \cup \ldots \cup \text{Sym}^k(\mathbb{R}^4).
\]

(3.8)

The resulting space \( \tilde{M}_{k,N} \) is a geodesically complete hyperkähler orbifold. Its drawback is the non-existence of the universal bundle with the universal instanton connection over \( \tilde{M}_{k,N} \times \mathbb{R}^4 \). We actually think that in principle one can still work with this space. Fortunately, in the case of SU(N) gauge group there is a nice space \( \tilde{M}_{k,N} \) which is obtained from \( \tilde{M}_{k,N} \) by a sequence of blowups (resolution of singularities) which is smooth, and after some modification of the gauge theory (noncommutative[22][23][24][25] deformation) becomes a moduli space with the universal instanton. Technically this space is obtained [26] by the same ADHM construction except that now one performs the hyperkähler quotient at the non-zero level of the moment map:

\[
\mu_r = \zeta_r 1_V, \quad \mu_c = 0
\]

(one can also make \( \mu_c \neq 0 \) but this does not give anything new). The cohomology theory of \( \tilde{M}_{k,N} \) is richer then that of \( \tilde{M}_{k,N} \) because of the exceptional divisors. However, our goal is to study the original gauge theory. Therefore we are going to consider the (equivariant) cohomology classes of \( \tilde{M}_{k,N} \) lifted from \( \tilde{M}_{k,N} \).
As we stated in the introduction, we are going to utilize the equivariant symplectic volumes of $\tilde{M}_{k,N}$. This is not quite precise. We are going to consider the symplectic volumes, calculated using the closed two-form lifted from $\tilde{M}_{k,N}$. This form vanishes when restricted onto the exceptional variety. This property ensures that we don’t pick up anything not borne in the original gauge theory (don’t pick up freckle contribution in the terminology of [27]).

The ADHM construction from the previous section gives rise to the instantons with fixed gauge orientation at infinity (fixed framing). The group $G = SU(N)/\mathbb{Z}_N$ acts on their moduli space $M_{N,k}$ by rotating the gauge orientation. Also, the group of Euclidean rotations of $\mathbb{R}^4$ acts on $M_{N,k}$. We are going to apply localization techniques with respect to both of these groups.

In fact, it is easier to localize first with respect to the groups $U(k) \times G \times T^2$ acting on the vector space of ADHM matrices, and then integrate out the $U(k)$ part of the localization multiplet, to incorporate the quotient.

The action of $T^2$ is free at “infinities” of $\tilde{M}_k$, thus allowing to apply localization techniques without worrying about the IR non-compactness. Physically, the integral (2.10) is Gaussian-like and convergent in the IR region.

3.4. Reduction to contour integrals

After all the manipulations as in [7][6] we end up with the following integral[27]:

$$Z_k(a, \epsilon_1, \epsilon_2) = \frac{1}{k! (2\pi i \epsilon_1 \epsilon_2)^k} \int \prod_{i=1}^k \frac{d\phi_i}{P(\phi_i)P(\phi_i + \epsilon)} \prod_{1 \leq i < j \leq k} \frac{\phi_{ij}^2 (\phi_{ij}^2 - \epsilon^2)}{(\phi_{ij}^2 - \epsilon_1^2)(\phi_{ij}^2 - \epsilon_2^2)}$$

(3.9)

where:

$$Q(x) = \prod_{j=1}^{N_f} (x + m_j),$$

$$P(x) = \prod_{l=1}^N (x - a_l),$$

(3.10)

$\phi_{ij}$ denotes $\phi_i - \phi_j$ and $\epsilon = \epsilon_1 + \epsilon_2$.

We went slightly ahead of time and presented the formula which covers the case of the gauge theory with $N_f$ fundamental multiplets. In fact, the derivation is rather simple if one keeps in mind the relation to the Euler class of the Dirac zeromodes bundle over the moduli space of instantons, stated in the introduction.

3.5. Classification of the residues

The integrals (3.9) should be viewed as contour integrals. As explained in [6] the poles at $\phi_{ij} = \epsilon_1, \epsilon_2$ should be avoided by shifting $\epsilon_1, \epsilon_2 \to \epsilon_1, \epsilon_2 + i0$, those at $\phi_i = a_l$ similarly by $a_l \to a_l + i0$ (this case was not considered in [6] but actually was considered (implicitly) in [7]). The interested reader should consult [28] for more mathematically sound explanations of the contour deformations arising in the similar context in the study of symplectic quotients.
The poles which with non-vanishing contributions to the integral must have 
$\phi_{ij} \neq 0$, for $i \neq j$, otherwise the numerator vanishes. This observation simplifies the 
classification of the poles. They are labelled as follows. Let $k = k_1 + k_2 + \ldots + k_N$ be
a partition of the instanton charge in $N$ summands which have to be non-negative 
(but may vanish), $k_l \geq 0$. In turn, for all $l$ such that $k_l > 0$ let $Y_l$ denote a partition 
of $k_l$:

$$k_l = k_{l,1} + \ldots + k_{l,\nu_l,1}, \quad k_{l,1} \geq k_{l,2} \geq \ldots \geq k_{l,\nu_l,1} > 0.$$

Let $\nu^{l,1} \geq \nu^{l,2} \geq \ldots \nu^{l,k_l,1} > 0$ denote the dual partition. Pictorially one represents 
these partitions by the Young diagram with $k_{l,1}$ rows of the lengths $\nu^{l,1}, \ldots, \nu^{l,k_l,1}$. 
This diagram has $\nu_l^{l,1}$ columns of the lengths $k_{l,1}, \ldots, k_{l,\nu_l,1}$.

In total we have $k$ boxes distributed among $N$ Young tableaux (some of which 
could be empty, i.e. contain zero boxes). Let us label these boxes somehow (the ordering is not important as it is cancelled in the end by the factor $k!$ in (3.9)). Let us denote the collection of $N$ Young diagrams by $\vec{Y} = (Y_1, \ldots, Y_N)$. We denote by $|Y_l| = k_l$ the number of boxes in the $l$'th diagram, and by $|\vec{Y}| = \sum_l |Y_l| = k$.

Then the pole of the integral (3.9) corresponding to $\vec{Y}$ is at $\phi_s$ with $s$ labelling 
the box $(\alpha, \beta)$ in the $l$'th Young tableau (so that $0 \leq \alpha \leq \nu^{l,\beta}$, $0 \leq \beta \leq k_{l,\alpha}$) equal to:

$$\vec{Y} \rightarrow \phi_s = a_l + \epsilon_1(\alpha - 1) + \epsilon_2(\beta - 1). \quad (3.11)$$

### 3.6. Residues and fixed points

The poles in the integral (3.9) correspond to the fixed points of the action of
the groups $G \times T^2$ on the resolved moduli space $\tilde{M}_{k,N}$. Physically they correspond 
to the $U(N)$ (noncommutative) instantons which split as a sum of $U(1)$ noncommutative instantons corresponding to $N$ commuting $U(1)$ subgroups of $U(N)$. The 
instanton charge $k_l$ is the charge of the $U(1)$ instanton in the $l$'th subgroup. 
Moreover, these abelian instantons are of special nature – they are fixed by the group 
of space rotations. If they were commutative (and therefore point-like) they had 
to sit on top of each other, and the space of such point-like configurations would have been rather singular. Fortunately, upon the noncommutative deformation the singularities are resolved. The instantons cannot sit quite on top of each other. Instead, they try to get as close to each other as the uncertainty principle lets them. 
The resulting abelian configurations were classified (in the language of torsion free sheaves) by H. Nakajima [29].

Now let us fix a configuration $\vec{Y}$ and consider the corresponding contribution 
to the integral over instanton moduli. It is given by the residue of the integral (3.9) 
corresponding to (3.11):
where we have used the following notations: $a_l = a_{l,m}$,
\[
S_l(x) = \frac{Q(a_l + x)}{\prod_{m \neq l}(x + a_{l,m})(x + \epsilon + a_{l,m})}, \quad S_l(x) = \frac{Q(a_l + x)}{\prod_{m \neq l}(x + a_{l,m})^2},
\]
and
\[
\ell(s) = k_{l,m} - \beta, \quad h(s) = k_{l,m} + \nu^{l,m} - \alpha - \beta + 1.
\]

Now, if we set $\epsilon_1 = h = -\epsilon_2$ the formula (3.12) can be further simplified. After some reshuffling of the factors it becomes exactly the summand in (1.7).

3.7. The first three nonabelian instantons

We shall now give the formulae for the first three instanton contributions to the prepotential for the general $SU(N)$ case, with $N_f < 2N$.

We shall work with $\epsilon_1 = h = -\epsilon_2$. It will be sufficient to derive the gauge theory prepotential.

Directly applying the rules (3.9)(3.12) we arrive at the following expressions for the moduli integrals:

\[
Z_1 = \frac{1}{\epsilon_1 \epsilon_2} \sum_l S_l,
\]
\[
Z_2 = \frac{1}{(\epsilon_1 \epsilon_2)^2} \left( \frac{1}{4} \sum_l S_l (S_l(+h) + S_l(-h)) + \frac{1}{2} \sum_{l \neq m} \frac{S_l S_m}{(1 - \frac{h^2}{a_{l,m}})^2} \right),
\]
\[
Z_3 = \frac{1}{(\epsilon_1 \epsilon_2)^3} \left( \sum_l \frac{S_l (S_l(+h) S_l(+2h) + S_l(-h) S_l(-2h) + 4S_l(+h) S_l(-h))}{36} \right.
\]
\[
+ \sum_{l \neq m} \frac{S_l S_m}{4} \left( \frac{S_l(+h)}{1 - \frac{2h^2}{(a_{l,m} + h)^2}} \right)^2 + \frac{S_l(-h)}{1 - \frac{2h^2}{(a_{l,m} - h)^2}} \right)
\]
\[
+ \sum_{l \neq m \neq n} \frac{S_l S_m S_n}{6} \left( \frac{1 - \frac{h^2}{a_{l,m}}}{1 - \frac{h^2}{a_{l,n}}} \left( \frac{1 - \frac{h^2}{a_{m,n}}} \right) \right)^2, \quad (3.15)
\]
from which we immediately derive:

\[ F_1 = \sum_l S_l, \]

\[ F_2 = \sum_l \frac{1}{4} S_l S_l^{(2)} + \sum_{l \neq m} \frac{S_l S_m}{a_{1m}^2} + O(h^2), \]

\[ F_3 = \sum_l \frac{S_l}{36} \left( S_l S_l^{(4)} + 2 S_l^{(1)} S_l^{(3)} + 3 S_l^{(2)} S_l^{(2)} \right) \]

\[ + \sum_{l \neq m} \frac{S_l S_m}{a_{1m}^2} \left( 5 S_l - 2 a_{1m} S_l^{(1)} + a_{1m}^2 S_l^{(2)} \right) \]

\[ + \sum_{l \neq m \neq n} \frac{2 S_l S_m S_n}{3(a_{1m}a_{1n}a_{mn})^2} \left( a_{ln}^2 + a_{im}^2 + a_{mn}^2 \right) + O(h^2). \] (3.16)

### 3.8. Four and five instantons

To collect more “experimental data-points” we have considered the case of the gauge groups SU(2) and SU(3) with fundamental matter. We have computed explicitly the prepotential for four and five instantons and found a perfect agreement (yet a few typos) with the results of [11]. We should stress that this is a non-trivial check. Just as an example, we quote here the expression for \( F_5 \):

\[ F_5(a, m) = \frac{\mu_3}{8a^{18}}(35a^{12} - 210a^{10} \mu_2 + a^8 (207 \mu_2^2 + 846 \mu_4) \]

\[ -1210a^6 \mu_2 \mu_4 + a^4 \left( 1131 \mu_2^4 + 3698 \mu_2^2 \mu_4 \right) - 5250a^2 \mu_2^2 \mu_4 + 4471 \mu_4^2), \]

where \( 2a = a_1 - a_2, \mu_2 = m_1^2 + m_2^2 + m_3^2, \mu_3 = m_1 m_2 m_3, \mu_4 = (m_1 m_2)^2 + (m_2 m_3)^2 + (m_1 m_3)^2. \)

### 3.9. Adjoint matter and other matters

So far we were discussing \( \mathcal{N} = 2 \) gauge theories with matter in the fundamental representations. Now we shall pass to other representations. It is simpler to start with the adjoint representation. The \( \epsilon \)-integrals (3.9) reflect both the topology of the moduli space of instantons and also of the matter bundle.

The latter is the bundle of the Dirac zero modes in the representation of interest. For the adjoint representation, and on \( \mathbb{R}^4 \) this bundle can be identified with the tangent bundle to the moduli space of instantons. It has a \( U(1) \) symmetry. The equivariant Euler class of the tangent bundle (= the Chern polynomial) is the instanton measure in the case of massive adjoint matter. This reasoning leads to the following \( \epsilon \)-integral:

\[ Z_k = \frac{1}{k!} \left( \frac{\epsilon_1 + \epsilon_2}{2\pi i \epsilon_1 \epsilon_2} (\epsilon_2 + m)(\epsilon_1 + m) \right)^k \int \prod_{i=1}^k d\phi_i \frac{P(\phi_i + m)P(\phi_i + \epsilon - m)}{P(\phi_i)P(\phi_i + \epsilon)} \]

\[ \times \prod_{i < j} \frac{\phi_{ij}^2 (\phi_{ij}^2 - \epsilon_1^2)(\phi_{ij}^2 - (\epsilon_1 - m)^2)(\phi_{ij}^2 - (\epsilon_2 - m)^2)}{(\phi_{ij}^2 - \epsilon_1^2)(\phi_{ij}^2 - \epsilon_2^2)(\phi_{ij}^2 - m^2)(\phi_{ij}^2 - (\epsilon - m)^2).} \] (3.17)
Note the similarity of this expression to the contour integrals appearing\[6\] in the calculations of the bulk contribution to the index of the supersymmetric quantum mechanics with 16 supercharges (similarly, (3.9) is related to the one with 8 supercharges). This is not an accident, of course.

Proceeding analogously to the pure gauge theory case we arrive at the following expressions for the first two instanton contributions to the prepotential (which agrees with \[11\]):

\[
\mathcal{F}_1 = m^2 \sum_l T_l, \\
\mathcal{F}_2 = \sum_l \left( -\frac{3m^2}{2} T_l^2 + \frac{m^4}{4} T_l T_l^{(2)} \right) \tag{3.18}
\]

where \( T_l(x) = \prod_{n \neq l} \left( 1 - \frac{m^2}{(x + a_n)^2} \right) \), \( T_l = T_l(0) \), \( T_l^{(n)} = \partial_x^n T_l(x) \big|_{x=0} \) (cf. \[30\]).

For generic representation \( R \) of the gauge group we should use the equivariant Euler class of the corresponding (virtual) vector bundle \( \mathcal{E}_R \) over the moduli space of instantons \[8\].

### 3.10. Perturbative part

So far we were calculating the nonperturbative part of the prepotential. It would be nice to see the perturbative part somewhere in our setup, so as to combine the whole expression into something nice.

One way is to calculate carefully the equivariant Chern character of the tangent bundle to \( \tilde{\mathcal{M}}_k \) along the lines sketched in the end of the previous section\[8\]. The faster way in the \( \epsilon_1 + \epsilon_2 = 0 \) case is to note that the expression (1.5) is a sum over partition with the universal denominator, which is not well-defined without the non-universal numerator. Nevertheless, let us try to pull it out of the sum.

We get the infinite product (up to an irrelevant constant):

\[
\prod_{i,j=1}^{\infty} \prod_{l \neq m} \frac{1}{a_{im} + h(i-j)} \sim \\
\exp - \sum_{l \neq m} \int_0^{\infty} ds \frac{e^{-sa_{im}}}{s \left( e^{hs} - 1 \right) \left( e^{-hs} - 1 \right)} \tag{3.19}
\]

If we regularize this by cutting the integral at \( s \to 0 \), we get a finite expression, which actually has the form

\[
\exp \frac{\mathcal{F}_{\text{pert}}(a, \epsilon_1, \epsilon_2)}{\epsilon_1 \epsilon_2},
\]
with \( \mathcal{F}^{pert} \) being analytic in \( \epsilon_1, \epsilon_2 \) at zero. In fact
\[
\mathcal{F}^{pert}(a, 0, 0) = \sum_{l \neq m} \frac{1}{2} a_{lm}^2 \log a_{lm} + \text{ambiguous quadratic polynomial in } a_{lm}.
\]
The formula (3.19) is a familiar expression for the Schwinger amplitude of a mass \( a_{lm} \) particle in the electromagnetic field
\[
F \propto \epsilon_1 \, dx^1 \wedge dx^2 + \epsilon_2 \, dx^3 \wedge dx^4.
\] (3.20)
Its appearance be explained in the next section.

Let us now combine \( \mathcal{F}^{inst} \) and \( \mathcal{F}^{pert} \) into a single \( \epsilon \)-deformed prepotential
\[
\mathcal{F}(a, \epsilon_1, \epsilon_2) = \mathcal{F}^{pert}(a, \epsilon_1, \epsilon_2) + \mathcal{F}^{inst}(a, \epsilon_1, \epsilon_2)
\]
where in general we define:
\[
\mathcal{F}^{pert}(a, \epsilon_1, \epsilon_2) = \sum_{l \neq m} \int \frac{ds}{\epsilon} \frac{e^{-sa_{lm}}}{s \sinh \left( \frac{s \epsilon_1}{2} \right) \sinh \left( \frac{s \epsilon_2}{2} \right)}
\] (3.21)
with the singular in \( \epsilon \) part dropped. We define:
\[
Z(a, \epsilon_1, \epsilon_2; q) = \exp \frac{\mathcal{F}(a, \epsilon_1, \epsilon_2; q)}{\epsilon_1 \epsilon_2}.
\] (3.22)

4. \( \tau \)-function conjecture

This conjecture relates the expansion (1.5) to the dynamics of the Seiberg- Witten curve.

Consider the theory of a free complex chiral fermion \( \psi, \psi^* \),
\[
\mathcal{S} = \int_{\Sigma} \psi^* \bar{\partial} \psi
\] (4.1)
living on the curve \( \Sigma \):
\[
w + \frac{\Lambda^{2N}}{w} = \mathbf{P}(\lambda), \quad \mathbf{P}(\lambda) = \prod_{i=1}^{N} (\lambda - \alpha_i)
\] (4.2)
embedded into the space \( \mathbb{C} \times \mathbb{C}^* \) with the coordinates \( (\lambda, w) \). This curve has two distinguished points \( w = 0 \) and \( w = \infty \) which play a prominent role in the Toda integrable hierarchy [31]. Let us cut out small disks \( D_0 \) and \( D_\infty \) around these two points.
The path integral on the surface $\Sigma$ with two discs deleted will give a state in the tensor product $\mathcal{H}_0 \otimes \mathcal{H}_\infty^*$ of the Hilbert spaces $\mathcal{H}_0$, $\mathcal{H}_\infty$ associated to $\partial D_0$ and $\partial D_\infty$ respectively. It can also be viewed as an operator $G_\Sigma : \mathcal{H}_0 \to \mathcal{H}_\infty$.

Choose a vacuum state $|0\rangle \in \mathcal{H}_0$ and its dual $\langle 0| \in \mathcal{H}_\infty^*$ (we use the global coordinate $w$ to identify $\mathcal{H}_0$ and $\mathcal{H}_\infty$). Consider

$$\tau_\Sigma = \left\langle 0 \left| \exp \left( \frac{1}{\hbar} \oint_{\partial D_\infty} S J \right) G_\Sigma \exp \left( - \frac{1}{\hbar} \oint_{\partial D_0} S J \right) \right| 0 \right\rangle \quad (4.3)$$

where:

$$J = : \psi^* \psi :$$

$$dS = \frac{1}{2\pi i} \frac{dw}{w} \quad (4.4)$$

and we choose the branch of $S$ near $w = 0, \infty$ such that (cf. [32]):

$$S = \frac{N}{2\pi i} w \frac{\bar{w}}{\bar{w}} + O(\lambda^{-1}).$$

Let us represent $\Sigma$ as a two-fold covering of the $\lambda$-plane. It has branch points at $\lambda = \alpha_i^\pm$ where

$$\mathbf{P}(\alpha_i^\pm) = \pm 2\Lambda N.$$

Let us choose the cycles $A_l$ to encircle the cuts between $\alpha_i^-$ and $\alpha_i^+$. Of course, in $H_1(\Sigma, \mathbb{Z})$, $\sum_l A_l = 0$. Then, we define:

$$a_l = \oint_{A_l} dS.$$

Our final conjecture states:

$$\mathcal{Z}(a, \hbar, -\hbar) = \tau_\Sigma. \quad (4.5)$$

Note that from this conjecture the fact that $\mathcal{F}_0(a, 0, 0)$ coincides with the Seiberg-Witten expression follows as a consequence of the Krichever universal formula [33].

The remaining paragraph is devoted to the explanation of the motivation behind (4.5).

Let us assume that we are in the domain where $\alpha_l - \alpha_m \gg \Lambda$. Then the surface $\Sigma$ can be decomposed into two halves $\Sigma_{\pm}$ by $N$ smooth circles $C_l$ which are the lifts to $\Sigma$ of the cuts connecting $\alpha_i^-$ and $\alpha_i^+$. The path integral calculating the matrix element (4.3) can be evaluated by the cutting and sewing along the $C_l$'s. The path integral on $\Sigma_{\pm}$ gives a state in $\otimes_{l=1}^N \mathcal{H}_{C_l}$.
(its dual). If we first pull the $f SJ$ as close to $C_l$ as possible, we shall get the Hilbert space obtained by quantization of the fermions which have $a_l + \frac{1}{2}_{modZ}$ moding:

$$\psi(w) \sim \sum_{i \in Z} \psi_{l,i} w^{a_l+i} \left( \frac{dw}{w} \right)^{\frac{1}{2}}$$  \hfill (4.6)

near $C_l \subset \Sigma$. In addition, the states in $H_{C_l}$ of fixed total $U(1)$ charge are labelled by the partitions $k_{l,i}$. We conjecture, that

$$\langle 0 | e^{f_{SJ}} \prod_{l,i} \psi_{l,k_{l,i}-i} \psi_{l,i}^* | 0 \rangle_l \sim \prod_{l,i<k,m,j} (a_{lm} + \hbar (k_{l,i} - k_{m,j} + j - i)).$$  \hfill (4.7)

It is clear that (4.7) implies (4.5). For $N = 1$ (4.7) is of course a well-known fact (with the coefficient given by $\prod_{i<j} \frac{1}{j-i}$). It gives rise to the formula (which can also be derived using the Schur identities [34]):

$$Z_{N=1}(h, -\hbar) = e^{-\frac{1}{\hbar^2}}$$

which confirms that in spite of the fact that we worked with the resolved moduli space $\bigcup_k \bar{M}_{k,1} = \bigcup_k (\mathbb{T}^2)[k]$ the “symplectic” volume we calculated is that of $\bigcup_k \bar{M}_{k,1} = \bigcup_k Sym^k (\mathbb{R}^4)$. 
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