Supergravity in \((2 + 1)\) dimensions from \((3 + 1)\)-dimensional Supergravity

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In the context of the formalism proposed by Stelle-West and Grignani-Nardelli, it is shown that Chern-Simons supergravity can be consistently obtained as a dimensional reduction of \((3 + 1)\)-dimensional supergravity, when written as a gauge theory of the Poincaré group. The dimensional reductions are consistent with the gauge symmetries, mapping \((3 + 1)\)-dimensional Poincaré supergroup gauge transformations onto \((2 + 1)\)-dimensional Poincaré supergroup ones.

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I. INTRODUCTION

Supergravity in \((2 + 1)\) \(\mathbb{R} \mathbb{R}\) and in \((3 + 1)\) \(\mathbb{R} \mathbb{R}\) dimensions can be formulated as a gauge theory of the Poincaré superalgebra. The first-order formalism permits one to write the three dimensional supergravity as a Chern-Simons theory \(\mathbb{R}\), for which \((2 + 1)\)-dimensional supergravity is a good theoretical laboratory for the construction of a quantum theory \(\mathbb{R}\). Then it is interesting to find a link between supergravities in \((2 + 1)\) and in \((3 + 1)\) dimensions.

The action for supergravity in \((2 + 1)\)-dimensions \(S = \int \left(\varepsilon_{abc} R^{ab} \varepsilon + 4\psi D\psi\right)\), with \(\psi\) a two component Majorana spinor, is invariant under Lorentz rotations, Poincaré translations and supersymmetry transformations. The dreibein \(e^{a}_\mu\), the spin connection \(\omega^{ab}_\mu\) and the gravitino \(\psi^{a}_\mu\) transform as components of a connection for the super Poincaré group. This means that the supersymmetry algebra implied by the corresponding supersymmetry transformations is the super Poincaré algebra.

\((3 + 1)\)-dimensional supergravity invariant under the Poincaré supergroup is based on the supersymmetric extension of the Stelle-West formalism and of supergravity as a gauge theory of the Poincaré supergroup. This procedure can be used because both supergravities in \((2 + 1)\) \(\mathbb{R}\) and supergravity in \((3 + 1)\)-dimensions \(\mathbb{R} \mathbb{R}\) can be formulated as theories genuinely invariant under the Poincaré supergroup.

The purpose of the present work is to find the supersymmetric extension of the successful formalism of refs. \([10, 11]\). This means that, in the context of the procedure of refs. \([10, 11]\), \((3 + 1)\)-dimensional supergravity can be dimensionally reduced to Chern-Simons supergravity. Section \(IV\) concludes the work with brief comment.

II. SUPERGRAVITY INVARIANT UNDER THE POINCARE GROUP

In this section we shall review some aspects of the Supersymmetric extension of the Stelle-West formalism and of supergravity as a gauge theory of the Poincaré group. The dimensional reduction is carried out in sec. \(III\) where the principal features of the dimensional reduction process are presented. Section \(IV\) concludes the work with brief comment.

A. Non-Linear realizations

The non-linear realizations can be studied by the general method developed in ref. \([12, 13]\). Following these references, we consider a Lie (super)group \(G\) and a subgroup \(H\).

Let us call \(\{V_{i}\}_{i=1}^{n-d}\) the generators of \(H\). We assume that the remaining generators \(\{A_{i}\}_{i=1}^{d}\) are chosen so that they form a representation of \(H\). In other words, the commutator \([V_{i}, A_{j}]\) should be a linear combination of \(A_{i}\) alone. A group element \(g \in G\) can be represented

\[D\zeta^{A} = d\zeta^{A} + \omega^{AB}\zeta_{B}, \quad D\chi = d\chi - \frac{1}{2}\omega^{AB}\gamma_{AB}\chi\]

where \(\omega^{AB}\) is the spin connection.
(uniquely) in the form
\[ g = e^{\xi^i A_i} h \] (2)
where \( h \) is an element of \( H \). The \( \xi^i \) parametrize the coset space \( G/H \). We do not specify here the parametrization of \( h \). One can define the effect of a group element \( g_0 \) on the coset space by
\[ g_0 g = g_0 (e^{\xi^i A_i} h) = e^{\xi''^i A_i} h' \] (3)
or
\[ g_0 e^{\xi^i A_i} = e^{\xi''^i A_i} h_1 \] (4)
where
\[ \xi' = \xi'(g_0, \xi) \] (5)
\[ h_1 = h' h^{-1} \] (6)
\[ h_1 = h_1(g_0, \xi) \]. (7)

If \( g_0 - 1 \) is infinitesimal, (4) implies
\[ e^{-\xi^i A_i} (g_0 - 1) e^{\xi^i A_i} - e^{-\xi^i A_i} \delta e^{\xi^i A_i} = h_1 - 1. \] (8)
The right-hand side of (8) is a generator of \( H \).
Let us first consider the case in which \( g_0 = h_0 \in H \).
Then (1) gives
\[ e^{\xi^i A_i} = h_0 e^{\xi^i A_i} h_0^{-1} \] (9)
Since the \( A_i \) form a representation of \( H \), this implies
\[ h_1 = h_0; \quad h' = h_0 h. \] (10)
The transformation from \( \xi \) to \( \xi' \) given by (10) is linear. On the other hand, consider now
\[ g_0 = e^{\xi^0 A_i}. \] (11)
In this case (1) becomes
\[ e^{\xi^0 A_i} e^{\xi^i A_i} = e^{\xi''^i A_i} h. \] (12)
This is a non-linear inhomogeneous transformation on \( \xi \). The infinitesimal form (8) becomes
\[ e^{-\xi^i A_i} \delta \xi^0 A_i e^{\xi^i A_i} - e^{-\xi^i A_i} \delta e^{\xi^i A_i} = h_1 - 1. \] (13)
The left-hand side of this equation can be evaluated, using the algebra of the group. Since the results must be a generator of \( H \), one must set equal to zero the coefficient of \( A_i \). In this way one finds an equation from which \( \delta \xi^i \) can be calculated.

The construction of a Lagrangian invariant under coordinate-dependent group transformations requires the introduction of a set of gauge fields \( a = a^\mu_i A_i dx^\mu \), \( \rho = \rho^i \mu V_i dx^\mu \), \( p = p^i \mu A_i dx^\mu \), \( v = v^i \mu V_i dx^\mu \), associated respectively with the generators \( V_i \) and \( A_i \). Hence \( \rho + a \) is the usual linear connection for the gauge group \( G \), and the corresponding covariant derivative is given by:
\[ D = d + f(\rho + a) \] (14)
and its transformation law under \( g \in G \) is
\[ g : (\rho + a) \rightarrow (\rho' + a') = g(\rho + a) g^{-1} - \frac{1}{f}(dg) g^{-1} \] (15)
where \( f \) is a constant which, as it turns out, gives the strength of the universal coupling of the gauge fields to all other fields.

We now consider the Lie algebra valued differential form
\[ e^{-\xi^i A_i} [d + f(\rho + a)] e^{\xi^i A_i} = p + v. \] (16)
The transformation laws for the forms \( p(\xi, d\xi) \) and \( v(\xi, d\xi) \) are easily obtained. In fact, using (11), one finds
\[ p' = h_1 p(h_1)^{-1} \] (17)
\[ v' = h_1 v(h_1)^{-1} + h_1 d(h_1)^{-1}. \] (18)

The equation (17) shows that the differential forms \( p(\xi, d\xi) \) are transformed linearly by a group element of the form (11). The transformation law is the same as by an element of \( H \), except that now this group element \( h_1(\xi_0, \xi) \) is a function of the variable \( \xi \). Therefore any expression constructed with \( p(\xi, d\xi) \) which is invariant under the subgroup \( H \) will be automatically invariant under the entire group \( G \), the elements of \( H \) operating linearly on \( \xi \), the remaining elements non-linearly.

### B. Supersymmetric Stelle-West Formalism

The basic idea of the Stelle-West formalism is founded on the non-linear realizations in anti de Sitter space. The supersymmetric extension of this formalism is based in the non-linear realizations of supersymmetry in anti de Sitter space. The formalism consider as \( G \) the graded Lie algebra
\[ [P_A, P_B] = -im^2 J_{AB} \]
\[ [J_{AB}, P_C] = i(\eta_{AC} P_B - \eta_{BC} P_A) \]
\[ [J_{AB}, J_{CD}] = i(\eta_{AC}J_{BD} - \eta_{BC}J_{AD} + \eta_{BD}J_{AC} - \eta_{AD}J_{BC}) \]

\[ [J_{AB}, Q_\alpha] = i(\gamma_{AB})_{\alpha\beta} Q_\beta \]

\[ [P_\alpha, Q_\alpha] = -\frac{i}{2}m(\gamma_\alpha)_{\alpha\beta} Q_\beta \]

\[ [Q_\alpha, Q_\beta] = -2(\gamma^\alpha)_{\alpha\beta} P_A - 2m(\gamma^AB)_{\alpha\beta} J_{AB} \quad (19) \]

having as generators \( Q_\alpha, P_\alpha \) and \( M_{AB} \). It has as a subalgebra \( H \) that of the de Sitter group \( SO(3,2) \) with generators \( P_\alpha \) and \( M_{AB} \). This, in turn, has as subalgebra \( L \) that of the Lorentz group \( SO(3,1) \) with generators \( M_{\alpha \beta} \).

An element of \( G \) can be uniquely represented in the form

\[ g = e^{\Xi Q} h = e^{\Xi Q} e^{-i\xi^A P_A} l \quad (20) \]

where \( h \in H \) and \( l \in L \). On can define the effect of a group element \( g_0 \) on the coset space \( G/H \) by

\[ g_0 g = e^{\Xi Q} h' = e^{\Xi Q} e^{-i\xi^A P_A} l' \quad (21) \]

or

\[ g_0 e^{\xi Q} = e^{\Xi Q} h_1 \quad (22) \]

\[ h_1 e^{-i\xi^A P_A} = e^{-i\xi^A P_A} l_1 \quad (23) \]

\[ l_1 l = l'. \quad (24) \]

Clearly \( h_1 = h_1(g_0, \chi) \) and \( l_1 = l_1(g_0, \chi, \xi) \).

If \( g_0 - 1 \) and \( h_1 - 1 \) are infinitesimal, \( (22), (23) \) imply

\[ e^{-\Xi Q} (g_0 - 1) e^{\Xi Q} - e^{-\Xi Q} e^{\xi Q} = h_1 - 1 \quad (25) \]

\[ e^{i\xi^A P_A} (h_1 - 1) e^{-i\xi^A P_A} = e^{i\xi^A P_A} e^{-i\xi^A P_A} = l_1 - 1. \quad (26) \]

We consider now the following cases: If \( g_0 = l_0 \in L; \) \( (22), (23) \) give

\[ e^{\Xi Q} = l_0 e^{\Xi Q} l_0^{-1} \quad (27) \]

\[ h_1 = l_1 = l_0 \quad (28) \]

\[ e^{-i\xi^A P_A} = l_0 e^{-i\xi^A P_A} l_0^{-1}. \quad (29) \]

Both \( \chi \) and \( \xi \) transform linearly. If, on the other hand, we know only that \( g_0 = h_0 \in H \), in particular, if

\[ g_0 = e^{-i\rho^A P_A} \quad (30) \]

is a pseudo-translation, \( (22) \) gives

\[ e^{\Xi Q} = h_0 e^{\Xi Q} h_0^{-1} \quad (31) \]

while \( (23) \) gives

\[ h_0 e^{i\xi^A P_A} = e^{-i\xi^A P_A} l_1(h_0, \xi). \quad (32) \]

In this case \( \chi \) transforms linearly, but the transformation law \( (23) \) of \( \xi \) under pseudo-translations is inhomogeneous and non-linear. Infinitesimally

\[ e^{i\xi^A P_A} (e^{-i\rho^B P_B} - e^{-i\xi^A P_A} \delta e^{-i\xi^A P_A} = l_1 - 1. \quad (34) \]

Finally, if

\[ g_0 = e^{\Xi Q} \quad (35) \]

is a supersymmetry transformation, one must use \( (22) \) and \( (23) \) as they stand. Observe, however, that \( (23) \) has the same form as \( (23) \) except for the fact that \( h_1 \) is a function of \( \chi \) while \( h_0 \) is not. Therefore, the transformation law for \( \xi \) under a supersymmetry transformation has the same form as that under a de Sitter transformation but, with parameters which depend in a well defined way on \( \chi \).

An explicit form for the transformation law of \( \xi^A \) under an infinitesimal AdS boost can be obtained from \( (22) \).

The result is

\[ \delta \xi^A = -\rho^A + \frac{\left(z \cosh z - 1\right) - \left(\rho^B \xi_B \xi^A\right)}{\xi^2} \quad (36) \]

where \( z = m\sqrt{\left(\xi^A \xi_A\right)} = m\xi^A \).

The transformation of \( \xi^A \) under an infinitesimal Lorentz transformation \( l_0 = e^{i\kappa^{AB} J_{AB}} \) is

\[ \delta \xi^A = \kappa^{AB} \xi_B \quad (37) \]

and, under local supersymmetry transformation \( (23), \xi^A \) transforms as

\[ \delta \xi^A = -i \left(1 + \frac{i}{6} m \chi \right) \chi \gamma^A \chi \]

\[ + i \left(\frac{z \cosh z - 1}{\sinh z}\right) \left(\delta_B^A - \frac{\xi_B \xi^A}{\xi^2}\right) \left(1 + \frac{i}{6} m \chi \right) \chi \gamma^B \chi \]

\[ - 2i m \left(1 + \frac{i}{6} m \chi \right) \chi \gamma^A \chi \xi_B. \quad (38) \]

Using \( (25) \) with \( g_0 = l_0 = \Xi Q \), one finds that

\[ \delta \chi = \varepsilon - \frac{i}{6} m \left(5 \chi \chi + \chi \Gamma^A \Gamma^A\right) \varepsilon + \frac{1}{9} m^2 \left(\chi \chi\right) \varepsilon \quad (39) \]
Working in first order formalism, the gauge fields vierbein $e^A$, spin connection $\omega^{AB}$ and gravitino $\psi$ are treated as independent. The key observation is that $(e^A, \omega^{AB}, \psi)$, considered as a single entity, constitute a multiplet in the adjoint representation of the AdS supergroup. That is, we can write:

$$A = \frac{1}{2} i \omega^{AB} J_{AB} - i e^A P_A + \bar{\psi} Q$$

(41)

where $A$ is the gauge field of the AdS supergroup, $P_A, J_{AB}, Q^a$ being the generators of the AdS boosts. Then, based on these, we can define the corresponding non-linear connections $(V^a, W^{ab}, \Psi)$ from (10):

$$\frac{1}{2} i W^{AB} J_{AB} - i V^A P_A + \bar{\Psi} Q$$

(42)

If $G = OSP(1,4)$ and $H = SO(3,2)$, the gauge fields $V^A$ form a square $4 \times 4$ matrix which is invertible and can be identified with the vierbein fields. In the same way we have that $W^{AB}$ is a connection and that $\Psi$ can be identified with the Rarita-Schwinger field. From (42) one can obtain the fields $V^A, W^{AB}, \Psi$ in terms of the fields $e^A, \omega^{AB}, \psi$. The results are given in equations (81), (83) and (84) of ref. [4].

The corresponding transformation laws for $V^a, W^{ab}, \Psi$ can be obtained from (17), (18). In fact, one can verify that, under the AdS supergroup, the non-linear connections transform as:

$$\bar{\Psi} Q = h_1 (\bar{\Psi} Q) (h_1)^{-1}$$

(43)

$$-i V^a P_a = h_1 (-i V^a P_a) (h_1)^{-1}$$

(44)

$$\frac{1}{2} i W^{ab} J_{ab} = h_1 \left( \frac{1}{2} i W^{ab} J_{ab} \right) (h_1)^{-1} + h_1 d(h_1)^{-1}.$$  

(45)

C. Supergravity invariant under the AdS group

Within the supersymmetric extension of the Stelle-West formalism, the action for supergravity with cosmological constant can be rewritten as

$$S = \int \varepsilon_{abcd} R^{ab} V^c V^d + 4 \bar{\Psi} \gamma_5 \gamma_a D \Psi V^a$$

(46)

which is invariant under the supersymmetric extension of the AdS group. From such equations we can see that the vierbein $V^a$ and the gravitino field transform homogeneously according to the representation of the AdS superalgebra but, with the nonlinear group element $h_1 \in H$.

The corresponding equations of motion are obtained by varying the action with respect to $\xi^a, \chi, e^a, \omega^{ab}, \psi$. The field equations corresponding to the variation of the action with respect to $\xi^a$ and $\chi$ are not independent equations. Following the same procedure of Ref. [15], we find that equations of motion for supergravity genuinely invariant under Super AdS are:

$$2\varepsilon_{abcd} \bar{R}^{ab} V^c + 4 \bar{\Psi} \gamma_5 \gamma_a \rho$$

(47)

$$\Theta^a = 0$$

(48)

$$8 \gamma_5 \gamma_a \rho V^a - 4 \gamma_5 \gamma_a \Psi \Theta^a = 0$$

(49)

where

$$\Theta^a = \Theta^a - \frac{i}{2} \bar{\Psi} \gamma^a \Psi$$

(50)

$$\bar{R}^{ab} = R^{ab} + 4 \alpha^2 \epsilon_{abc} V^b V^c + \alpha \bar{\Psi} \gamma^{ab} \Psi = 0$$

(51)

$$\rho = \delta \Psi - i \alpha \gamma^a \Psi V^a.$$  

(52)

D. Supergravity and the Poincaré group

Taking the limit $m \rightarrow 0$ in equations (24), (73), (75), (76), (81), (83) and (84) one can see that: (i) the superalgebra (19) take the form of the superalgebra of Poincaré

$$[P_A, P_B] = 0$$

$$[J_{AB}, P_C] = i (\eta_{AC} P_B - \eta_{BC} P_A)$$

$$[J_{AB}, J_{CD}] = i \epsilon_{ABCD} K_D + i \epsilon_{ABCD} K_{D'}.$$
\[
[J_{AB}, J_{CD}] = i (\eta_{AC} J_{BD} - \eta_{BC} J_{AD} + \eta_{BD} J_{AC} - \eta_{AD} J_{BC})
\]

where now
\[
[J_{AB}, Q_{\alpha}] = i (\gamma_{AB})_{\alpha\beta} Q_{\beta}
\]
\[
[P_{A}, Q_{\beta}] = 0
\]
\[
[Q_{\alpha}, \overline{Q}_{\beta}] = -2 (\gamma^{A})_{\alpha\beta} P_{A}.
\]

(ii) the transformation laws of \(\xi^{A}\) under an infinitesimal Poincarè translation, under an infinitesimal Lorentz transformation, and under a local supersymmetry transformation are given respectively by
\[
\delta\xi^{A} = -\rho^{A}
\]
\[
\delta\xi^{A} = \kappa^{AB} \xi^{B}
\]
\[
\delta\xi^{A} = -\epsilon \gamma^{A} \chi;
\]
where \(\rho^{A}, \kappa^{AB} = -\kappa^{BA}\) and \(\epsilon\) are the infinitesimal parameters corresponding to Poincarè translations, Lorentz rotations and supersymmetry respectively.

(iii) the transformation laws of \(\chi\) under an infinitesimal Poincarè translation, under an infinitesimal Lorentz transformation, and under a local supersymmetry transformation are given respectively by
\[
\delta\chi = 0
\]
\[
\delta\chi = \frac{1}{2} \kappa^{AB} \gamma_{AB} \chi
\]
\[
\delta\chi = -\epsilon.
\]

In this limit \(G\) is the Poincarè supergroup and \(H = SO(3,1)\); and the fields vierbein \(V^{A}\), the connection \(W^{AB}\), and the Rarita-Schwinger field \(\overline{\Psi}\) are given by
\[
V^{A} = e^{A} + D\zeta^{A} + i (2\overline{\psi} + D\overline{\chi}) \gamma^{A} \chi
\]
\[
W^{AB} = \omega^{AB}
\]
\[
\overline{\Psi} = \overline{\psi} + D\overline{\chi}
\]

The corresponding components of the curvature two-form are now
\[
T^{A} = D V^{A}
\]
\[
R^{A}_{B} = d \omega^{A}_{B} + \omega^{A}_{C} \omega^{C}_{B}.
\]

III. SUPERGRAVITY IN \((2 + 1)\) FROM SUPERGRAVITY IN \((3 + 1)\)

A. Supergravity in \((3 + 1)\)

The limit \(m \to 0\) of the action (60) is obviously the action for \(N = 1\) Supergravity in \((3 + 1)\)-dimensions:
\[
S = \int_{A} e^{ABC} R^{AB} V^{C} V^{D} + 4 \omega \gamma^{A} D \psi^{A}
\]
which is genuinely invariant under the Poincarè group. In fact, \(d = 3 + 1, N = 1\) supergravity is based on the Poincarè supergroup, whose generators \(P_{A}, J_{AB}, Q^{A}\) satisfy the Lie-superalgebra (53). Using this algebra and the general form for gauge transformations on \(A\)
\[
\delta A = -D\lambda = d\lambda - [A, \lambda],
\]
with
\[
\lambda = \frac{1}{2} \kappa^{AB} J_{AB} - \rho^{A} P_{A} + \overline{\psi} Q,
\]
we obtain that \(e^{A}, \omega^{AB}\), and \(\psi\), under local Lorentz rotations, transform as
\[
\delta e^{A} = \kappa^{AB} e^{B}; \quad \delta \omega^{AB} = -D \kappa^{AB}; \quad \delta \psi = -\frac{1}{2} \kappa^{AB} \gamma_{AB} \psi;
\]
under local Poincarè translations, transform as
\[
\delta e^{A} = D \rho^{A}; \quad \delta \omega^{AB} = 0; \quad \delta \psi = 0;
\]
and under local supersymmetry transformations, as
\[
\delta e^{A} = -2 \epsilon \gamma^{A} \psi; \quad \delta \omega^{AB} = 0; \quad \delta \psi = D \epsilon.
\]

This means that the vierbein \(V^{A}\) transforms, under the Poincarè supergroup, as
\[
\delta V^{A} = \kappa^{AB} V^{B};
\]

The space-time supertorsion \(T^{A}\) is given by
\[
\hat{T}^{A} = T^{A} - \frac{1}{2} \gamma^{A} \psi,
\]
where
\[
T^{A} = D V^{A}.
\]

It is direct to verify that the action (66) is invariant under (69), (70), (71), (54), (55), (56), (57), (58), (59), (61), (62).
B. Dimensional Reduction

The dimensional reduction process, as well as the notation, is similar to those used in refs. \[10\] and \[11\]. Latin indices \(a, b, c, \cdots\) = 0, 1, 2 and capital latin indices \(A, B, C, \cdots\) = 0, 1, 2, 3 denote (2 + 1) and (3 + 1) internal (gauge) indices respectively. They are raised and lowered by the Minkowski metrics

\[
\eta_{ab} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (75)

and

\[
\eta_{AB} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (76)

In the dimensional reduction the first three values of \(A, B, C, \cdots\) will denote the corresponding (2 + 1) internal indices \(a, b, c, \cdots\), i.e. \(A = (a, 3), B = (b, 3), C = (c, 3), \cdots\). We shall use the antisymmetric symbol \(\varepsilon^{ABCD}\) with \(\varepsilon^{0123} = 1\) and in (2 + 1)-dimensions \(\varepsilon^{abc} = \varepsilon^{abc3}\), so that \(\varepsilon^{012} = 1\).

Following the procedure of ref. \[10\] we carried out a dimensional reduction of the Poincaré generators of the (3 + 1)-dimensional theory and, correspondingly, of the space-time dimensions that, from the (3 + 1)-dimensional action \[10\] and the algebra \[11\], lead to the (2 + 1)-dimensional action. With such reductions from the (3+1) gauge transformations \[10\], with the identifications of this Table of dimensional reduction are mapped onto:

\[
\delta \xi^a = \kappa^a_b \delta \xi^b; \quad \delta \varepsilon^a = \kappa^a_b \varepsilon^b; \quad \delta \omega^{ab} = -D \kappa^{ab}; \quad \delta \psi = -\frac{1}{2} \kappa^{ab} \gamma_{ab} \psi; \quad \delta \xi^3 = -\rho^a; \quad \delta \varepsilon^a = D \rho^a; \quad \delta \omega^{ab} = 0; \quad \delta \psi = 0; \quad \delta \gamma^3 = 0
\] (79)

where the \(\gamma^i\)’s with multiple indices are antisymmetrized products of gamma matrices, which for \(d\)-dimensions satisfy the relationship \[20\]

\[
\gamma^i_1 \gamma^i_2 \cdots \gamma^i_k \gamma^{i_{k+1}} \cdots \gamma^{i_d} = \alpha \varepsilon^{i_1 i_2 \cdots i_d} \gamma^{i_{k+1}} \cdots \gamma^{i_d + 1}
\] (77)

with

\[
\alpha = \frac{1}{(d - k)} (-1)^{\frac{k(k-1)}{2} + \frac{d(d-1)}{2}}. \quad (78)
\]

It is direct to verify that the (3 + 1)-gauge transformations \[10\], \[11\], with the identifications of this Table of dimensional reduction are mapped onto:

\[
\delta \xi^a = \kappa^a_b \delta \xi^b; \quad \delta \varepsilon^a = \kappa^a_b \varepsilon^b; \quad \delta \omega^{ab} = -D \kappa^{ab}; \quad \delta \psi = -\frac{1}{2} \kappa^{ab} \gamma_{ab} \psi; \quad \delta \xi^3 = -\rho^a; \quad \delta \varepsilon^a = D \rho^a; \quad \delta \omega^{ab} = 0; \quad \delta \psi = 0; \quad \delta \gamma^3 = 0
\] (80)

i.e. onto the correct (2 + 1)-dimensional gauge transformations. In particular, the quantities that are set to a constant in the Table consistently have vanishing gauge transformations. In the same way we have

\[
R^{AB} = \begin{pmatrix}
R_{ab} & R^{a3} \\
R^{3b} & R^{33}
\end{pmatrix} = \begin{pmatrix}
R_{ab} & 0 \\
0 & 0
\end{pmatrix}
\] (82)

\[
\omega^{AB} = \begin{pmatrix}
\omega^{ab} & \omega^{a3} \\
\omega^{3b} & \omega^{33}
\end{pmatrix} = \begin{pmatrix}
\omega^{ab} & 0 \\
0 & 0
\end{pmatrix}
\] (83)
\[ V^A = \begin{pmatrix} V^a \\ V^3 \end{pmatrix} = \left( e^a + D\xi^a + i(\overline{\psi} + D\chi)\gamma^a \chi \right) \] (84)

\[ \Psi = \psi + D\chi \] (85)

where \( D\xi^a = d\xi^a + \omega^a_{\phantom{a}c} \xi^c; \ D\chi = d\chi - \frac{1}{2} \xi^{ab} \gamma_{ab} \chi \).

From equation (77) we see that, for \( d = 4 \) and \( k = 3 \),

\[ \gamma^{ABC} = -\varepsilon^{ABCD} \gamma_D \gamma^5 \] (86)

which allows one to write the action for \((3+1)\)-dimensional supergravity in the form

\[ S^{4D} = \int \varepsilon_{ABCD} \left( R^{AB} V^C V^D + \frac{1}{3!} \overline{\psi} \gamma^{ABCD} V^D D\Psi \right). \] (87)

By substituting the content of the Table of dimensional reduction and (82) into the action (87) one gets

\[ S^{4D} = \int \left( 2\varepsilon_{aabc} R^{ab} V^c + \frac{3}{2} \varepsilon_{aabc} \overline{\psi} \gamma^{abc} D\Psi \right) dx^3. \] (88)

Using (84), (85) and the identity \( \gamma_{ab} = -i\varepsilon_{abc} \gamma^c \) we find that the first term is

\[ 2\varepsilon_{aabc} R^{ab} V^c dx^3 = (2\varepsilon_{aabc} R^{ab} e^c + 2\varepsilon_{aabc} R^{ab} \xi^c \]

\[ -4R^{ab} \overline{\psi} \gamma_{ab} \chi - 2 R^{ab}(D\chi)\gamma_{ab} \chi dx^3. \] (89)

Using (17): \( \gamma^{abc} = -\varepsilon^{abc} I \) and the identities \( D\chi = \frac{1}{2} R^{ab} \gamma_{ab} \chi; \overline{\chi} = -\overline{\psi} \gamma_{ab} \chi \) we find that the second term is

\[ 4 \frac{3!}{3!} \varepsilon_{aabc} \overline{\psi} \gamma^{abc} D\Psi dx^3 = \left( 4 \frac{3!}{3!} \varepsilon_{aabc} \overline{\psi} \gamma^{abc} D\Psi \]

\[ +4D(\overline{\psi} D\psi) + 4R^{ab} \overline{\psi} \gamma_{ab} \chi + 2 R^{ab}(D\chi)\gamma_{ab} \chi dx^3. \] (90)

By substituting (89) and (90) in (88) we obtain

\[ S^{4D} = \int \left( 2\varepsilon_{aabc} R^{ab} e^c + \frac{3}{2} \varepsilon_{aabc} \overline{\psi} \gamma^{abc} D\Psi \right. \]

\[ +2\varepsilon_{aabc} R^{ab} \xi^c + 4D(\overline{\psi} D\psi) \right) dx^3. \]

Using the Bianchi identity \( D R^{ab} = 0, \varepsilon_{abc} e^{abc} = -3! \) and (77) with \( d = 3 \) and \( k = 3 \), we find that the action for \((2+1)\)-supergravity is given by

\[ S^{4D} \rightarrow S^{3D} = \int \varepsilon_{abc} R^{ab} e^c + 4\overline{\psi} D\psi + \text{surface term}. \] (91)

which proves that the dimensional reduction from \((3+1)\)-dimensional supergravity to \((2+1)\)-supergravity is possible.

IV. COMMENTS

We have shown that the successful formalism proposed in refs. [10] and [11] can be extended to the supersymmetric case. That is, \((3+1)\)-dimensional supergravity can be dimensionally reduced to supergravity in \((2+1)\)-dimensions following the method of refs. [10] and [11].

Finally we can say that supergravity genuinely invariant under the Poincaré supergroup [3], [4] is a natural context to connect, preserving the invariance under the Poincaré supergroup, such a theory with \((2+1)\)-dimensional supergravity.

It is interesting to note that all the terms containing \( \xi^a, \chi \) disappear from the action and that \( e^a, \psi \) can be interpreted as the space-time dreibein and gravitino, and yet the theory is invariant under the Poincaré supergroup, contrary to what happens in \((3+1)\)-dimensions.

The absence of the \( \xi^a, \chi \) variables of (91) and the interpretation of \( e^a, \omega^{ab} \) and \( \psi \) as gauge fields makes of (91) an action that can be conceived as a Chern-Simons three form.

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