MONTE CARLO TECHNIQUES

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ABSTRACT

The course of "Monte Carlo Techniques" will try to give a general overview of how to build up a method based on a given theory, allowing you to compare the outcome of an experiment with that theory.

Concepts related with the construction of the method, such as, random variables, distributions of random variables, generation of random variables, random-based numerical methods, will be introduced in this course.

Examples of some of the current theories in High Energy Physics describing the $e^+e^-$ annihilation processes (QED, Electro-Weak, QCD) will also be briefly introduced.

A second step in the employment of this method is related to the detector. The interactions that a particle could have along its way, through the detector as well as the response of the different materials which compound the detector will be quoted in this course.

An example of detector at LEP era, in which these techniques are being applied, will close the course.

1. INTRODUCTION

Many times an experimental physicist has the problem of how to compare the outcome of an experiment with the prediction of a theory. Sometimes the features of the experiment and/or the uncomplexity of the theory allows a direct comparison of the analytical expression of the theory with the experimental distributions.

The current High Energy Physics panorama, although from the formal point of view looks simpler (less theories and fewer number of free parameters to describe them), becomes more and more complex in its implementation. Larger analytical representations force to consider series expansions to some degree of accuracy. The experimental apparatus also raise in complexity since the aim is to collect as much information as possible and analyze it quickly.

The detectors, made out of many components with different purposes, are not as perfect as one would like. Usually materials integrating a detector have fluctuations in their behavior, for instance, with respect to the "passage" of a particle through them. In addition, most of the detectors do not have a complete acceptance, i.e. not all the possible particles produced in a process will go through active parts of the detector.

This situation leads to the search of some tools allowing us to reproduce similar effects from the theory to those that a real experiment could produce.

The randomness with what the real data are produced, to finally fill up the distributions predicted by theories, provides the foundations of the tool we look for. Indeed, the solution comes from the possibility of generating random numbers, that, under some constraints, shall allow us to reproduce the predictions of the theories.

Normally in High Energy Physics, the observables of an experiment are the particles that have been produced as result of a collision of two particles (for instance, $e^+e^-$...
collision, \( n^-p \) collision, etc). What we call "an event" is the set of particles defined by their fourmomenta produced as result of the collision. A sample of such events is what we compare with a collection of "Monte Carlo events" which were generated according with the theory to test.

When the outcome of the experiment is not perfect, because the detector, i.e. when the detector does not see all the events produced, or does not see all particles of every event, or the measurement of the momentum and energy of the particles is made with some error, etc, the comparison with the predictions of the theory is harder, but still possible by simulating the behavior of particles created from the theory, when passing through different components of the detector. Since detectors are made out of materials, there are particle-matter interactions which should be simulated.

This course will consist of two main items:

1) The foundations to make any simulation in which the generation of random numbers and operations with them are necessary.

2) The simulation of the physical process (Theory) and the detection process (Detector) making use of known properties of matter.

2. **Grounds**

I do not want to give an exhaustive review of random processes but a general feeling and few tools we shall need to apply to the problem of simulation in High Energy Physics (H.E.P.).

In H.E.P., relativistic and quantic effects become essential and this last introduces a basic uncertainty in the observables. In other words, the problem of given an initial state, have the final state, perfectly determined has no deterministic solution and has to be answered in terms of probabilities: "probability of finding a given final state configuration. Here comes out the stochastic nature of real physical processes in which the studies are made based on probabilities. [1] As an example consider the H.E.P. process \( e^+e^- \rightarrow \mu^-\mu^- \) (studied in a later chapter) and suppose that only the diagram with \( Y \) exchanged in the s channel contributes to it. If we make an experiment searching for final states with only \( \mu^+ \) and \( \mu^- \) and we consider the angular distribution of the \( \mu^- \) in the center of mass frame of the colliding \( e^+e^- \), it comes out an \( A + B \cos^2 \Theta \) distribution. This is one of the typical random distributions that we can find in H.E.P. It says what is the probability of finding a \( \mu^- \) produced in a given angular range. If you repeat several times the same experiment, you will end up with similar distributions.

2.1 **Random variables, samples**

The variables taking values which can not be predicted in advance are called random variables. [2] The \( \cos \Theta \) variable in the previous example was a random variable. The predictions of theories are normally given in terms of distributions of random variables or equivalently in terms of probabilities of finding given values in given intervals. In that sense we can define the probability density function \( g(x) \) as the probability of finding \( x' \) in the range \( x \) and \( x + dx \)

\[
g(x)dx = P(x'; x \leq x' \leq x + dx)
\]  
(2.1)
A function $y$ of a random variable $x$, $y = y(x)$ is a random variable if it is continuous and derivable. That is, the random nature of variables does not change with transformations.

This feature of random variables will be useful at the time in which we shall try to reproduce distributions of a given theory, since we shall use mechanisms to generate special kind of random distributions as are the uniform, Gaussian, or Poisson distributions and we shall need to transform them to others.

Let's call sample $S_k(g)$ to a distribution of values of a random variable $x$ distributed according to a probability density function $g(x)$. In practice, a sample $S_k(g)$ may be thought as the observed result of an experiment $k$.

### 2.2 Cumulative distribution function

Let a distribution of random variables $x$ be described by $g(x)$ and defined in the interval $[a, b]$, then we define the cumulative distribution function of $x$ as

$$ G(x) = \int_a^x g(x') \, dx' = P(a \leq x' \leq x) $$

(2.2)

That is the probability of finding $x'$ in the range $[a, x]$. This function (also called distribution function of $x$) has the property of being monotonically non-decreasing, $|3|$ taking values from "0" to "1". The last assertion is true since we have defined $g(x)$ as a probability density function, otherwise it can always be normalized inside the definition range.

This function $G(x)$ has the property to transform non-uniform variations in the $x$ variable into uniform variations in the $G(x)$ variable

$$ d \, G(x) = g(x) \, dx $$

(2.3)

Consider the distribution of values of the $x$ variable shown in fig. 1, given by

$$ g(x) = a \sin x ; \quad 0 \leq x < \pi $$

(2.4)

![Fig.1 Representation of the probability density function $g(x)$ and its cumulative distribution function $G(x)$.

The cumulative function is

$$ G(x) = \int_0^x a \sin x' \, dx' = a(1 - \cos x) $$

(2.5)

By inspection of fig. 1, we can see that if we have a random variable $x$ distributed according $g(x)$, the variable $y = G(x)$ is uniformly distributed in the range $(0, 2a)$ [4].

When we have several random variables $x_i$, $i = 1, n$ we define the joint distribution function $G$ by

$$ G(x_1, \ldots, x_n) = P(x_1 \leq x_1 \text{ and, } \ldots, \text{ and } x_n \leq x_n) $$

(2.6)

The random variables $x_1, \ldots, x_n$ are independent when

$$ P(x_1' \leq x_1 \text{ and } \ldots \text{ and } x_n' \leq x_n') = \prod_{i=1}^n P(x_i' \leq x_i) $$

(2.7)
If $g(x_1 \ldots x_n)$ is differentiable in each variable, the joint density function is given by

$$g(x_1 \ldots x_n) = \frac{\partial^n g(x_1 \ldots x_n)}{\partial x_1 \ldots \partial x_n}$$

(2.8)

2.3 Generation of random variables

Here we give some methods to generate random numbers. In fact what we shall see are not properly random numbers, in the sense that we formulate some algorithm to generate sequences of numbers which have properties very close to theses of random numbers. Usually this numbers are called pseudo-random. Obviously, a random number can not be generated through any kind of algorithm, since one will be able from a given number to predict the next one*). Nevertheless, for the purpose of simulation, pseudo-random numbers satisfy the necessary requirements.

2.3.1 Random variables uniformly distributed

A uniform distribution of random variables $x$, is that having a constant probability density function

$$g(x) = \text{constant}$$

(2.9)

There are several techniques to produce uniform distributions of random variables in computers. The most commonly used are the so called "congruential procedures", since they are very fast in generating and they can be easily analyzed. The attractiveness of the congruential method described below [5] lies in their general susceptibility to theoretical analysis and in the ability of replace multiplications and divisions by shift and add instructions. These, called multiplicative congruential procedures, take the form:

$$x_{n+1} = \alpha x_n + \beta \text{ (mod m)}$$

(2.10)

where $x_0, \alpha, \beta$ and $m$ are parameters to be defined. When $\beta = 0$ one speaks about a "pure multiplicative congruential" procedure. Many suggestions have been made for the proper choice of the parameters [6], but for simplicity we choose $\alpha = 2^{(P/2)+1} + 3$, $\beta = 0$ and $m = 2^P$ for binary computers with word length of $P$-bits. In a computer such as the VAX 780 or the IBM 3081, with a word length of 32 bits, the choice would be $\alpha = 2^{17} + 3$, $\beta = 0$ and $m = 2^{32}$.

For the analysis of the properties of the sequences of numbers generated with algorithms above, I remit the reader to the specialized literature. For us, it should be enough to know that it satisfies the minimum requirements of big periodicity and very small correlation between succeeding terms. In practice, the sequence $\{X_n / m\}$ is taken to be uniformly distributed over the range $(0, 1)$.

CERN program libraries [7] have programs to generate random numbers. For instance the

*) Von Neumann: "Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin'.
routine RANDM2 (V107) is a uniform pseudo-random generator based in the combination of a
multiplicative congruential procedure and a shift register generator. The performance in
the IBM is very good and has a period of about \(5 \times 10^{18}\) numbers.

### 2.3.2 Non-uniform distributions

Once we are able to generate random uniform distributions a wide variety of non-uniform
distributions can be generated. The method will be "direct" if the cumulative distribution
function, \(y = G(x)\), of a given distribution has inverse

\[
X = G^{-1}(y)
\]  

(2.11)

In that case, through a random uniform generation of the variable \(y\), we get the desired
distribution.

Consider the example given in paragraph 2.2. The cumulative distribution function was
\(G(x) = a(1 - \cos x)\). This function has inverse

\[
x = G^{-1}(y) = \arccos \left(1 - \frac{y}{a}\right); \quad 0 \leq y \leq 2a
\]  

(2.12)

Then if \(y\) is a random variable uniformly distributed over the range \([0, 2a]\), the variable
\(x = G^{-1}(y)\) should be distributed as: \(a \sin x\).

Unfortunately, not all the non-uniform distributions we want to generate, have a cumula-
tive distribution function easily invertible.

A general method of obtaining a random sequence, is the "acceptance-rejection" method,
that in general is not efficient and needs two different sequences of random numbers uniformly
distributed:

Let \(\{X_o\}\) and \(\{Y_o\}\) be two independent sequences of random numbers uniformly
distributed, then the sequence

\[
\{ x; \quad g(x) \geq y , \quad x \in \{ X_o \} , \quad y \in \{ Y_o \} \}
\]  

(2.13)

is \(g(x)\)-random distributed. Let us suppose that \(g(x)\) takes values in \([a, b]\), then, the
efficiency of the method can be defined as the ratio

\[
E = \frac{\int_a^b g(x) \, dx}{(b-a)(g_{\max} - g_{\min})}
\]  

(2.14)

where \(g_{\max}\) and \(g_{\min}\) are the maximum and minimum values of \(g(x)\) in its definition range.

Another method that can be very efficient under certain conditions is the "Importance
sample" method. If \(g(x)\) is the distribution to be generated, the method consists in finding
a function \(f(x)\) approximating \(g(x)\), with the cumulative distribution function \(F(x)\) having
inverse

\[
X = F^{-1}(y)
\]  

(2.15)

In that case we can take following procedure to generate \(g(x)\)
1°) take y uniformly generated over the range \([F_{\text{min}}, F_{\text{max}}]\) and compute \(x = F^{-1}(y)\). Then \(x\) is \(f(x)\)-random distributed.

2°) take \(\omega\) uniform over the range \([0, (g/f)_{\text{max}}]\)

3°) Reject \(x\) if \(\omega > g(x)/f(x)\), otherwise take it.

The sequence of remaining numbers \(\{x\}\) after repeating many times the above steps is \(g(x)\)-random distributed.

The method is more efficient when the ratio \(\omega(x) = g(x)/f(x)\) is nearly constant or has a slow variation with \(x\). Care must be taken in regions where \(g(x)\) is very small or "0" when compared with \(f(x)\).

Some other methods are still available to generate non-uniform random distributions, as for instance stratified sampling. It will be quoted in next paragraphs when talking on Monte Carlo integration.

Fig.2. Scheme of the importance sample procedure of generating random distributions

2.4 Numerical methods random-based. Monte Carlo integration

There are a few methods based in random numbers that allow computation of integrals, derivatives, etc. Among these, we are mainly interested in Monte Carlo integration, since it is needed in the computation of quantities, such as cross sections. An excellent review is given in ref [2].

2.4.1 Expectation and variance of a function

The expectation of a function \(f(x)\) is defined as the average of the function:

\[
E(f) = \int f(x) \, g(x) \, dx
\]

where \(g(x)\) is the probability density function of variable \(x\). In particular, if \(f(x) = x\), its expectation is

\[
E(x) = \int x \, g(x) \, dx
\]

The variance of a function, \(f(x)\), is defined as the quadratic deviation of \(f\) from its expectation

\[
V(f) = E(f - E(f))^2 = \int [f(x) - E(f)]^2 g(x) \, dx
\]

In the case of a function of several variables, its expectation is defined as

\[
E(f) = \int \ldots f(x_1 \ldots x_n) \, g(x_1 \ldots x_n) \, dx_1 \ldots dx_n
\]

where \(g(x_1 \ldots x_n)\) is the joint probability density function describing the distribution of
variables $x_1 \ldots x_n$.

2.4.2 Law of large numbers

The law of large numbers says that if $f$ is a function of the random variable $x$ and
\[ \{ x \} \]
is uniformly distributed in the interval $[a, b]$, and $V(x)$ is finite, then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = E(f) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
\] (2.20)

This law constitutes the fundation of the Monte Carlo integration method. The integral
\[
\int_{a}^{b} f(x) \, dx
\]
can be approached with the sum of $n$ random numbers $f(x_i)$. The method is powerful in spite it has a slow convergence to the value of the integral.

2.4.3 Central limit theorem

The sum of a large number of independent random variables is asymptotically normal-distributed, no matter how the individual random variables are distributed.

A normal distribution is characterized by giving the mean value, $\mu$, and the variance, $\sigma^2$
\[
N(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\] (2.21)

Since each term in a Monte Carlo integration is a random variable from the same distribution, the estimate
\[
\bar{I} = \frac{1}{n} \sum_{i=1}^{n} f(x_i)
\] (2.22)
will be asymptotically normal-distributed, with mean equal to the true value of the integral and variance
\[
V(\bar{I}) = \frac{1}{n} V(f)
\] (2.23)
since the standard error on $\bar{I}$ is the square root of $V(\bar{I})$, this gives the well known result that Monte Carlo estimates converge as $1/\sqrt{n}$.

2.4.4 Variance reducing techniques

In this paragraph I give an overview of the most current techniques to improve the convergence of the Monte Carlo integration method. In the previous paragraph we have seen that the uncertainty in the integral, $S(\bar{I})$, varies as $\sqrt{V(f)/n}$. The uncertainty can be improved, either increasing the number of trials, $n$, or decreasing the value of the variance $V(f)$. Some of the methods used to decrease the variance are briefly described below.

Stratified sampling

This technique consists in integrate the function in several subregions $\{ j \}$ of the total integration region and compute partial sums in each subregion $\{ j \}$, where $n_j$ points
are taken

\[ \bar{I} = \sum_{j} \frac{V_j}{n_j} \sum_{i=1}^{n_j} f(x_i) \]  \hspace{1cm} (2.25)

The variance is formed through sums over each region weighted proportionally to the volume of region \( j \) and inversely to \( n_j \)

\[ V(\bar{I}) = \sum_{j} \frac{V_j}{n_j} \left[ \int_{[j]} f^2(x) \, dx - \sum_{j} \frac{1}{n_j} \left| \int_{[j]} f(x) \, dx \right|^2 \right] \]  \hspace{1cm} (2.26)

A particular case, is the uniform stratification, in which the integration region is divided into equal volumes and the chosen number of points is equal in each.

Importance sample

This method relies in the possibility of approximating a function \( f(x) \) with another function \( g(x) \), easy enough as to be invertible or to have a generator reproducing it. Mathematically, if \( f \) is the function we want to integrate, the method corresponds to a change of variable

\[ x \rightarrow g(x) \]

\[ f(x) \, dx \rightarrow f(x) \, \frac{dG(x)}{g(x)} \]  \hspace{1cm} (2.27)

such that points, instead being chosen uniformly distributed in \( x \) are chosen from a \( g(x) \)-distributed sequence. Then \( f \) is inversely weighted by \( g(x) = dG(x) / dx \). The variance is now \( V(f / g) \) which can be small if \( g \) is close to \( f \).

The method has problems when \( g(x) \) is very small or "0" when compared with \( f(x) \).

Control variates

This method similar to the previous one, looks also for some function \( g(x) \) that approximates \( f(x) \), but this time the function is subtracted rather than divided

\[ \int f(x) \, dx = \int (f(x) - g(x)) \, dx + \int g(x) \, dx \]  \hspace{1cm} (2.28)

If \( g(x) \) is known over the whole integration range, the only uncertainty comes from the integral of \( f - g \) which will have a smaller variance. Moreover, zeros of \( g \) will not produce singularities in \( f - g \). Another advantage is that \( g \) needs not to be analytically inverted.

In general, the two last methods need a good knowledge of the function \( f(x) \) in all the integration range.

3. APPLICATIONS IN HIGH ENERGY PHYSICS

We have seen how to generate distributions of numbers according to some given function. Now, we can go a step further and describe how to produce "events" according to a theory and later, how the constituents of the events (particles), can be detected.
3.1 Simulation of theories

We describe, now, some of the theories we deal with from the point of view of our interest, i.e., preparing the way for the simulation. Some of the problems related with the lack of knowledge or with the extremely complicated representation of the theory, will be quoted. Problems such as radiative corrections, or fragmentation and hadronization in QCD, become very important as the available energies at PETRA or PEP in $e^+e^-$ or at the SPS in $pp$, introduce sizeable effects.

3.1.1 QED and Electroweak

Let us look to the process

$$e^+e^- \rightarrow \mu^+\mu^- \quad (3.1)$$

that can be mediated by $\gamma$ or by the neutral vector boson $z_0$. The lowest order cross section is given by

$$\frac{d\sigma_{\gamma}}{dW_{\mu}} = \frac{\alpha^2}{4\pi} \left[ W_1(s) (1 + \cos^2 \theta) + W_2(s) \cos \theta \right] \quad (3.2)$$

where $s$ is the center of mass energy squared, $\theta$ is the angle between $e^+$ and $\mu^+$ and $W_1, W_2$ are parameters depending on $s$, the mass $M_2$ and width $T_2$ of the neutral boson and the vector and axial coupling constants $C_V, C_A$. Only diagrams in fig. 3 contribute to the above cross section. Higher order corrections, as bremsstrahlung and virtual corrections, must be included when $\sqrt{s}$ is close to the $Z_0$ mass. A treatment of an almost $\alpha^3$ order corrections to the $\mu^-\mu^+$ pair production and in general fermion-pair production can be found in Ref [8]. In it, the interference between diagrams in figures 3 and 4 and the whole contribution from bremsstrahlung diagrams in fig. 5, are included.

Here I only quote the procedure to generate events without entering in details.

To generate events, two regions should be considered: The soft bremsstrahlung region and the hard bremsstrahlung region. Infrared divergences in the soft region are cancelled out with the vertex corrections and box diagrams. (Fig 4a, c, d).

The separation between soft and hard regions in made through the parameter $K_0$, which is the photon momentum, choosed to be 1% of the beam energy, $E_{\text{beam}} = \sqrt{s} / 2$.

Since the cross section can be computed in both regions

$$\sigma_{\text{Total}} \quad \text{by numerical integration}$$

$$\sigma_{\text{Total}} \quad \text{by analytical integration}$$
we can choose at random in what region to generate an event. That can be done by generating a random variable $x'$ uniformly distributed in $(0, 1)$. Then if

$$0 < x' < \sigma_{\text{soft}} \frac{\sigma_{\text{soft}}}{\sigma_{\text{Total}}} \frac{\sigma_{\text{hard}}}{\sigma_{\text{Total}}}$$

we are in the soft region, otherwise, we are in the hard region.

**Fig. 4 Virtual corrections to the process**

$$e^+e^- \rightarrow \mu^+\mu^- (Y)$$

**Fig. 5 Bremsstrahlung diagrams contributing to the cross section in the process**

$$e^+e^- \rightarrow \mu^+\mu^- (Y)$$

In the soft region, we can use as approximate cross section the lower order cross section in eq. 3.2, and neglect the momentum of the soft photon. This allows a very fast and easy generation. Only two variables have to be generated: the angles $(\theta, \Psi)$ between $e^+$ and $\mu^+$. $\Psi$ is uniformly distributed in $(0, 2\pi)$ and $\cos \theta$ is distributed according to equation 3.2, in the interval $(-1, 1)$. The momentum of the outcoming $\mu^+$'s is fixed,

$$p_\mu = \sqrt{E_{\text{beam}}^2 - m_\mu^2}.$$

In the hard bremsstrahlung region it is not longer possible neglect the photon. Then, the final state is composed by 3 particles, namely, $Y$, $\mu^+$ and $\mu^-$, characterized by their fourmomenta (12 variables), that should satisfy the energy-momentum conservation constraints (4 equations). There are then 5 independent variables to be generated according to the cross section in the hard bremsstrahlung region. This cross section can be denoted by the initial and final state radiation amplitudes

$$|M_I + M_F|^2 = |M_I|^2 + |M_F|^2 + 2 \operatorname{Re}(M_I^\dagger M_F)$$

(3.4)
In order to generate events, it is possible to use, as approximate cross section, the sum of two positive terms: the initial state and the final state radiation cross sections. Then, the exact value of the cross section is in the region

$$0 \leq d\sigma_{\text{exact}} \leq 2d\sigma_{\text{appr}}$$  \hspace{1cm} (3.5)

The procedure to generate events, using the importance sampling method, can be the following:

1) Generate variables $\mathcal{N}_\mu'$, $\mathcal{N}_Y'$, $K$ according to approximate cross section (procedure described in reference [9]).

2) Generate a number $\omega'$ uniformly distributed in $(0, 2)$.

3) Accept $\mathcal{N}_\mu'$, $\mathcal{N}_Y'$, $K$ as good event if $\omega' \leq d\sigma_{\text{exact}} / d\sigma_{\text{appr}} (\mathcal{N}_\mu', \mathcal{N}_Y', K)$.

4) Repeat points 1, 2, 3 until an event is accepted.

3.1.2 QCD

QCD is well known from the computational point of view up to $\mathcal{O}^2_s$ order. The possible parton configurations in $e^+e^-$ annihilations, are

$$e^+e^- \rightarrow q\bar{q}, q\bar{q}E, q\bar{q}EE, q\bar{q}qqq$$  \hspace{1cm} (3.6)

Although at the parton level, the problem of generating events is rather similar to the previous one, there are additional complications due, mainly, to the unknown fragmentation process. The fragmentation process, has to be modeled with some reasonable scheme predicting the production of particles in the final state. Here, as an example, the Feynman – Field approach is adopted to fragment partons.

To simplify this generator, only the QED initial state radiative corrections are included.

In the generation of this kind of events, the following scheme is taken:

1. Generation of the initial state photon

2. Production of 2, 3 or 4 partons

3. Fragmentation

Generation of the initial state photon

At this point our problem is similar to that of the $\mu$-pairs, with an additional complication: Our final state at the quark level is a mixture of $u\bar{u}$, $d\bar{d}$, $s\bar{s}$, $c\bar{c}$ and $b\bar{b}$ pairs. We must choose, at random, the kind of flavor to produce, according with its charge squared and then generate the radiative $\gamma$. The procedure to generate the radiative $\gamma$ is described in reference [10]. A value $K_{\text{min}}$, is chosen to separate soft and hard bremsstrahlung regions, in such a way that the photon energy can be neglected below $K_{\text{min}}'$ and another value $K_{\text{max}}$ to fix the maximum photon energy in the hard bremsstrahlung region.

In the soft region the cross section can be computed by integration of

$$\frac{d\sigma_{\text{rad}}}{d\mathcal{N}_1...\mathcal{N}_n} = \frac{d\sigma^e(s)}{d\mathcal{N}_1...d\mathcal{N}_n} \left(1 + S_A + S_\mu + S_\gamma + S_{\text{had}}\right)$$  \hspace{1cm} (3.7)
where $\delta$-corrections come from the soft bremsstrahlung, vertex and vacuum polarization correction, $q_1 \ldots q_n$ is a convenient set of independent variables defining the final state and $\sigma^0(s)$ is the hadronic cross-section at energy $\sqrt{s}$.

In the hard region the total cross section can be computed from

$$
\frac{d\sigma^\text{rad}}{dK^2} \propto K \left( \frac{m_{s'}^2}{4K^2} + \frac{m_{s'}^2}{4K^2} - s' \right) - 1 \right) \sigma^0(s')
$$

where $K$ and $\sum_i$ are the photon energy and solid angle and

$$
s' = q^2 = 4 \frac{\sqrt{s}}{2} (\frac{\sqrt{s}}{2} - K)
$$

$$
K = P_+ - K
$$

$P_+$, $P_-$ denote the four momenta of the $e^+$, $e^-$.

Integration of equation 3.8 in the $\sum_i$ variables leads to

$$
\frac{d\sigma^\text{rad}}{dK} = \frac{\alpha}{\pi} \sigma^0(s') (\frac{n}{n_s^2} - 1) \left[ 1 + (\frac{s'}{s^2})^2 \right] \frac{1}{K}
$$

With the help of equations 3.7 and 3.9, after integration, we choose the region. In the soft region, we use $d\sigma^0(s)/dq_1 \ldots dq_n$ to generate partons. In the hard region, we generate first the energy $K$ and then, the solid angle of the photon by using equations 3.8 and 3.9 and finally the partons with $d\sigma^0(s')/dq_1 \ldots dq_n$.

Production of 2, 3 or 4 partons

The evolution of the QCD matrix elements with the energy is known up to $\alpha_s^2$. Then we can generate 2, 3 or 4 partons with $d\sigma^0(s')/dq_1 \ldots dq_n$ as was outlined above. This integrated cross section has the form:

$$
\sigma^0(s') = \sigma_0 \left[ 1 + \frac{3}{4} C_F \frac{d_s}{n} + K \frac{d_s}{n} \right]
$$

(3.10)

Imposing "jet criteria resolution", we can construct the 3 and 4 jet production cross section to be finite

$$
\sigma^3_{\text{jet}} = \sigma_{\text{3 jet born}} + \sigma_{\text{3 jet virt}} + \sigma_{\text{4 soft}}
$$

(3.11)

$$
\sigma^4_{\text{jet}} = \sigma_{\text{4 jet born}} - \delta_A \text{soft}
$$

(3.12)

The part of the total cross section which is neither $\sigma^3_{\text{jet}}$ nor $\sigma^4_{\text{jet}}$ is called $\sigma^2_{\text{jet}}$

$$
\sigma^2_{\text{jet}} = \sigma - \sigma^3_{\text{jet}} - \sigma^4_{\text{jet}}
$$

(3.13)

A fraction of the final states with 3 and 4 partons can contribute to $\sigma^2_{\text{jet}}$ and $\sigma_3$ jet when one of the partons has very low energy or is very close to some other parton, such that they fail the jet resolution criteria (see figure 6).
We choose the Sterman-Weinberg variables $\delta, \varepsilon$ to define the distinction level between close jets and the minimum energy required to define a jet.

\[
\varepsilon = \min \left\{ \frac{E_i}{s^2} \right\}; \quad E_i = \text{Energy of parton } i
\]

\[
\delta = \min \left\{ \frac{\theta_{ij}}{2} \right\}; \quad \theta_{ij} = \text{Angle between parton } i \text{ and } j
\]

Fig. 6 Example of 3 and 4 partons failing the jet resolution criteria on the Sterman-Weinberg variables $\varepsilon, \delta$ and passing to the category of 2 and 3 jets.

a) failing $\varepsilon$. b) failing $\delta$.

Under these assumptions we can compute the total cross section for the production of 2, 3 and 4 jets independently, with radiative corrections included by numerical integration, using them as weight to choose how many jets to produce. [11].

Fragmentation

The process through what a parton produces other particles is called fragmentation. Fragmentation can not yet be calculated in QCD and therefore some parametrization for the production of final state particles, must be made [12]. Fragmentation should reproduce two experimental facts.

1. Reproduction of energy distributions of particles in a jet
2. Reproduction of the spread of particles in a jet.

The Feynman-Field fragmentation model describes the hadron production through the fundamental transition

\[
q_a \rightarrow M(q_a, \bar{q}_b) + q_b
\]

in which the incident quark $q_a$ fragments into a meson $M$ and a residual quark $q_b$. The process is repeated many times until the energy of the last residual quark is not enough to yield a new meson by picking up a pair $q\bar{q}$ from the sea, in which case the process is stopped. Two distributions characterize the formation of mesons in the process. The first is the fraction of the quark energy carried by the meson, described by a probability function $f(z)$, where
can be interpreted as the fraction of the energy of the quarks carried out by the meson. The second is the transversal spread of mesons described by an exponential probability function,

\[ S(p_T) = \exp\left(-\frac{p_T^2}{2\sigma_q^2}\right) \]  

(3.17)

where \( \sigma_q \) is a parameter to be fixed. The probability function \( f(z) \) is normalized such that \( \int_0^1 f(z) \, dz = 1 \) and it is conventionally parametrized as

\[ f(z) = 1 - \alpha_F' + 3\alpha_F' (1 - z)^2 \]

(3.18)

although special shapes can be given, as is normaly the case for heavy quarks.

The description of the basic process (3.15) is given by the function

\[ D(z) = \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{dz} \]

(3.19)

which is normalized such that

\[ \int dz \, D(z) = \langle N(\tau_\bar{\tau}) \rangle \]

(3.20)

where \( \langle N(\tau_\bar{\tau}) \rangle \) is the average number of \( \tau_\bar{\tau} \)'s produced in the reaction.

\( D(z) \) and \( f(z) \) are related by the integral equation

\[ D(z) = f(z) + \int_z^1 d\eta \left[ f(1 - \eta) \, D(z / \eta) \right] \]

(3.21)

There are still two parameters coming from experimental data giving the rate of pseudo-scalar to vector mesons, \( a_v \), and the rate of strange particles production, \( a_s \). The standard values given to the free parameters of the model are \( a_v = 0.5 \), \( a_s = 0.4 \), \( \sigma_q = 300 \text{ MeV} \) and \( \alpha_F' = 0.77 \).

The Feynman-Field model must be extended to fragment gluons. Here we adopt the extension proposal by Ali et al. in which the gluon is splitted in a \( q\bar{q} \) pair according to the Altarelli-Parisi splitting function for \( g \rightarrow q\bar{q} \)

\[ \frac{dp}{dz} \sim [z^2 + (1 - z)^2] \]

(3.22)

and then every quark is fragmented independently.

### 3.2 The detector simulation

A basic need for experimentalists is the ability of measuring properties of particles appearing in the final state of a physical process. This need is fulfilled with detectors, i.e., different materials with some convenient behaviour depending upon the particles crossing through. We can distinguish between two big sets of materials. Active and passive materials. Active, are those producing some kind of signal which is possible to pick up through some
interfase. Examples are the scintillator materials, gases with properties as the ability of being ionized or producing light when excited with particles passing through, or even gases in metastable state, like liquid Hydrogen in bubble chambers. There are many properties of materials that we can profit of. Passive materials are those which do not produce a direct signal to observe, but can help to some active material or can constitute a bulk for some particle difficult to cross or are the support of the detector components.

The detector simulation has two major parts: The simulation, as good as possible, of all the interactions, that a particle could have along the detector materials, like multiple scattering, nuclear interactions, coulomb interactions, etc., as well as, the subsidiary interactions of particles produced before. In here the development of the electromagnetic and hadronic shower constitute the goldstone of the simulation. The simulation of the signals coming from the active materials and further storage of them, usually in digital form, is the other major item.

3.2.1 Simulation of showers

By shower development we designate a set of phenomena involving transport and interactions of particles in media. A particle going through a medium has a path without interacting until it has some kind of interaction producing other particles which can, as well, have some free path to further interact and repeat the previous process until it ends up with a cloud of very low energy particles, most of them photons, that are absorbed by the media or are conveniently driven to some device to collect a signal.

The process of simulation of showers, will be performed with some basic ingredients: the simulation of the free path of particles, the simulation of the interactions and the accounting of energy that particles give to matter.

In relation with the relevant kind of interaction, we distinguish between "Electromagnetic and hadronic showers".

3.2.2 Electromagnetic showers

Two basic processes dominate the energy losses of an electron or positron when traversing matter: collision and radiation. The first has two possible effects; the atom is left in a excited state or is ionized. If in excited state, the atom emits a low energy photon to recover the ground state. If ionized, the ejected electron, has most of the time small energy which is locally deposited. However some time the orbital electron gets enough energy to be regarded as a secondary particle called Delta ray. In the case of radiation, the energy spectrum of the radiated photon goes from "0" to the maximum energy of the electron or positron. The collision process dominates at low energy, whereas radiation does at high energy. The radiated photon can interact with matter. Three photo-processes dominate, depending on the energy of the photon and the nature of the medium: pair production or materialization, compton scattering and photo-electric effect. The first dominates at high energy whereas compton scattering does at lower energy. These two processes give back electrons to the system which can again radiate photons repeating and multiplying the process which is globally called "Electromagnetic cascade shower". The photo-electric effect as well as the Coulomb scattering of electrons introduce some perturbation in the shower. The shower lateral spread comes mainly from Coulomb scattering and Compton scattering. The net effect
in the development of the shower is an increase in the number of particles and a decrease in their average energy reaching energies low enough, so that collision losses become predominant.

![Diagram of various particle interactions including Compton, Möller, Bhabha, Annihilation, Bremsstrahlung, Pair production.](image)

Fig. 7 Diagrams representing basic processes treated in the EGS code

In this paragraph, I will describe, briefly, one of the codes that, so far, is recognized to simulate with great accuracy the development of Electromagnetic showers. It is called EGS (Electron-Gamma Simulation).

The simulation of the electromagnetic shower is decomposed into a simulation of the transport and interactions of a single particle. The information about the particles produced at the interaction is stored in the top of a stack initiated with the primary particle of the shower. The basic strategy is to transport the particle in the top of the stack until an interaction takes place, or until its energy drops below some given cutoff energy, or until it enters a particular space region. In the later two cases the particle is taken out of the stack and simulation resumes with the next top particle.

**Transport**

Let $\sigma_x$ be the total cross section per molecule of a particle in a medium, then, the mean free path, $\lambda$, is given by
\[ \tilde{\lambda} = \frac{A}{N_a \rho \sigma_T} \]  

(3.23)

where \( A, N_a, \) and \( \rho \) are the molecular weight, Avogadro’s number and density respectively.

In general the mean free path may change as the particle moves from one medium to another, or when it loses energy. The number of mean free paths crossed when going from a point \( x_0 \) to \( x \) will be

\[ N_{\lambda} = \int_{x_0}^{x} \frac{dx}{\tilde{\lambda}(x)} \]  

(3.24)

It can be shown that the probability of a particle to traverse \( N_{\lambda} \) mean free paths before interaction is

\[ G(N_{\lambda}) = e^{-N_{\lambda}} \]  

(3.25)

Obviously this expression gives the possibility of sampling random values of \( N'_{\lambda} \) since the function \( G \) is distributed in \([0, 1]\)

\[ N'_{\lambda} = -\ln G' \]  

(3.26)

This value can be replaced in equation (3.24) to obtain the location of next interaction point.

Transport of photons

The only effects accounted for photons in EGS are: Pair production, Compton scattering and photoelectric processes. The cross sections for the above processes are finite and small enough that all interactions can be simulated. The photon travels along a straight line with constant energy between interactions. Assuming that the medium in which the simulation takes place is composed of a finite number of homogeneous materials with constant density, then the integral 3.24 becomes

\[ N_{\lambda} = \sum_{j=1}^{i-1} \left( \frac{x_j - x_{j-1}}{\lambda_j} \right) + \frac{x - x_{i-1}}{\lambda_i} \]  

(3.27)

where \( x \in (x_{i-1}, x_i) \) and \( x_0, x_1, \ldots \) are the boundary distances between materials with \( \lambda \) constant.

Simulation of photon transport is then as follows: First, pick up the number of mean free paths to the next interaction, \( N_{\lambda} \), using 3.26. Then perform the following steps

1) Compute \( \lambda \) at the current location
2) Let \( t_1 = \lambda N_{\lambda} \)
3) Compute \( d \), distance to next boundary along the photon direction
4) Let \( t_2 = \) the smaller of \( t_1 \) and \( d \). Transport by distance \( t_2 \)
5) Subtract \( t_2/\lambda \) from \( N_{\lambda} \). If the result is zero (this happens when \( t_2 = t_1 \)), then, the interaction takes place. Jump out of the loop to treat the interaction
6) Here we arrive when $t_2 = d$. Thus, a boundary was reached. If the new region is a different material go to step 1, otherwise to step 2.

If any of the particles resulting in the interaction is again a photon, and is still inside the medium where tracking is possible a new number $\lambda$ is chosen and steps from 1. to 6. are performed.

**Transport of charged particles**

Several problems appear in the transport simulation of electrons and positrons. The relevant interactions considered are elastic Coulomb scattering off the nucleus, inelastic scattering off the atomic electrons, bremsstrahlung production and positron annihilation into photon pairs. The first three processes have a very large cross section resulting in the production of a cloud of low energy particles which is not practical to simulate in a discrete way. For this reason we shall apply cut-off energies to distinguish between continuous and discrete interactions. In this way when in a given interaction the energy of the electron produced is above a given cut-off AE or the energy of the photon is above AP, we consider them as discrete events, otherwise they are treated in a continuous manner. All this particles in the continuum give rise to energy losses and direction changes of the electron between discrete interactions. The energy losses are due to soft interactions with the atomic electrons (ionization loss) and to the emission of soft bremsstrahlung photons. The changes in direction are mostly due to multiple Coulomb scattering from the nucleus, with some contribution coming from soft electron scattering.

The energy losses and the changes of direction along the path of electrons and positrons complicate the transport simulation, since the cross section changes along the path and the paths are no longer straight lines. The fact that the discrete electron total cross section decreases with decreasing energy allows the following trick which is used to account for the change of $\lambda$ along the path: We introduce a fictitious interaction such that the total cross section is constant along the path

$$\sigma_{T,\text{fict}}(x) = \sigma_{T,\text{real}}(x) + \sigma_{T,\text{fict}}(x) = \text{constant} = \sigma_{T,\text{real}}(x_0) \quad (3.28)$$

The location of the "next interaction" is then sampled using, as before, equations 3.26 and 3.24, along with the total fictitious cross section $\sigma_{T,\text{fict}}(x)$. When the interaction point is reached, we generate a random number between 0 and 1 and if it is greater than $\sigma_{T,\text{real}}(x)/\sigma_{T,\text{real}}(x_0)$, then, the interaction is fictitious and the transport continues from that point without interaction. Otherwise the interaction is real.

The remaining problem is the change of direction between interaction points due to multiple scattering. We shall give here a general explanation. The transport of the electron between interactions is divided into smaller steps which are supposed to be straight lines. The multiple scattering is accounted for by changing the electron direction at the end of each step. The angle between the initial and final directions is sampled from the appropriate distribution and the azimuthal angle is chosen randomly.

**Interactions**

Here we shall describe the general procedure to decide what interaction we deal
with at a given interaction point.

The probability that a given type of interaction occurs is proportional to its cross section. Suppose that all possible types of interactions at a given point are numbered from 1 to \(n\). Let's divide the interval \([0,1]\) into \(n\) ordered subintervals \(i\), \((a_{i-1}, a_i)\) such that

\[
a_{i-1} = \frac{1}{n}
\]

Then, we decide that the interaction is of type \(i\), if by choosing a random number \(r \in [0, 1]\), it satisfies

\[
a_{i-1} < r \leq a_i
\]

Once the type of interaction has been decided, the next step is to determine the parameters of the produced particles. (In general the four-momentum of particles). Suppose that the final state is characterized by, say, \(n\) parameters \(q_1, q_2, \ldots, q_n\). The differential cross section will have an expression of the form

\[
d^n\sigma = g(\vec{q}) \ d^n q
\]

with a total cross section given by

\[
\sigma^- = \int g(\vec{q}) \ d^n q
\]

The function

\[
f(\vec{q}) = \frac{g(\vec{q})}{\sigma^-}
\]

is normalized to 1 and has the properties of a probability density function. This may be sampled by using some of the methods given in section 2.3 and then obtain the parameters \(q_1, q_2, \ldots, q_n\). The previous parameters define the particles produced as result of the interaction. We store them in the stack and we follow with the simulation taking the particle in the top of the stack.

The discrete interactions which are treated in detail in the EGS code are

- Photons interacting (Compton scattering, pair production and photoelectric effect)
- Electrons or positrons interacting (Multiple scattering, annihilation, bhabha scattering, \(\mathcal{W}\)\-\(\mathcal{W}\) scattering and bremsstrahlung).

To finish this paragraph we shall comment on the continuous electron energy loss. As we said, the continuous loss is the result of interactions in which the energy transfer to secondary particles is not enough to put them above the discrete transport energy thresholds. The secondary particles are either soft bremsstrahlung photons, or atomic electrons which absorb some energy. The mean total continuous energy loss per unit length is

\[
- \frac{dE}{dx} \text{Continuous} = - \frac{dE}{dx} \text{Soft} \text{Bremsstrahlung} + \frac{dE}{dx} \text{Sub-cutoff Atomic electron}
\]
where $-$, + denotes electron or positron.

In this code it is supposed that when an electron is transported by a given distance, its loss of energy due to sub-cutoff secondaries is equal to the distance travelled times the mean loss per unit length as evaluated using previous equation. Actually, energy loss over a transported distance is subjected to fluctuations and gives rise to a Landau distribution (Not implemented yet).

3.2.3 Hadronic showers

The simulation of hadronic showers, although has some similarity with the one considered in the previous section, has different features, due mainly to the variety of particles and interactions involved in it. In here, we understand the hadronic shower as the development of interactions in cascade with matter, carried mainly by hadrons, and with the starting particle being a hadron. The transport of particles is carried step by step computing the interaction probability in each, based on the total cross section.

There are many processes involved in the development of hadronic showers: on one side, we have the energy loss, $\gamma$ -ray emission and multiple scattering of the particle along its way, on the other side, we have the nuclear scattering of this particle in media with problems concerning to the final state of the collisions, such as, the generation of multiplicity distributions, energy distributions, angular distributions, etc. In addition, if as a result of the nuclear scattering, there is production of neutral pions, these will decay to photons, starting electromagnetic cascades.

In this section, we describe one of the codes (GHEISHA, [14]), which is being successfully used in the MARK-J experiment at PETRA, to simulate the passage of particles through the detector, and that is being used in the specific design of some of the components of the L3 detector at LEP.

Energy Losses

The ionization energy losses of charged particles in matter are carried using the notation of Sternheimer: The most probable energy loss is

$$ - \frac{dE}{dx} = \frac{D}{\beta^2} \left[ G + 2 \ln \frac{\beta^2}{1 - \beta^2} - 2 \beta^2 - 2 S(\beta) \right] $$ (3.35)

where $\beta$ is the velocity of a particle traversing an absorber with atomic number, atomic weight and mean excitation potential $A$, $Z$ and $I$, respectively. $D$ and $G$ are functions of $A$, $Z$ and $I$. $S(\beta)$ is the density effect.

The energy loss of a particle for a given length is subject to fluctuations. Under some conditions the energy loss distribution is close to a Gaussian distribution, under other conditions is close to a Landau distribution. In GHEISHA we use a Gaussian distribution when $W/E_{\text{max}} \gg 1$ and Landau when $W/E_{\text{max}} \ll 1$ where

$$ W = 0.0001536 \; \frac{Z}{A} x / \beta^2 $$ (3.36) and

$$ E_{\text{max}} = 2m_e \beta^2 / (1 - \beta^2) $$ (3.37)
In the region where \( W / E_{\text{max}} \sim 1 \) we use for liquids and plastics Landau distributions and for metals Gaussian fluctuations. (Let us note than in the EGS code there were not such fluctuations).

**\( \delta \)-rays**

The \( \delta \)-rays are the electrons coming from the ionization of atoms is matter. The distance they will cover is approximated by

\[
R_e = 0.71 \frac{E^{1.72}}{\rho} \text{ (cm)}
\]  

(3.38)

where \( E \) is given in MeV and \( \rho \) in g/cm\(^3\). The simulation of \( \delta \)-rays is made in the following way. First we give the minimum range \( R_{\text{min}} \) fixed normally by the position resolution of the detector. Then we compute \( E_{\text{min}} \) by inverting the formula above. The number of \( \delta \)-rays is Poisson distributed around \( \langle N \rangle_E \) given by

\[
\langle N \rangle_E = \frac{W}{E_{\text{min}}}
\]  

(3.39)

and the energies of the electrons are generated according to

\[
P(E) \, dE = \frac{W \, dE}{E^2}
\]  

(3.40)

The emission angle \( \Theta \) with respect to the primary particle direction is given by

\[
\cos^2 \Theta = \frac{E}{E_{\text{max}}}
\]  

(3.41)

The energy of \( \delta \)-rays is subtracted from the energy deposit of the ionizing particle. It should be noted that the contribution from \( \delta \)-rays is only meaningful in light materials.

**Multiple scattering**

For identification of muons, we normally use thick absorbers to stop hadrons which could be misidentified as muons. The use of absorbers, nevertheless, produces an increase in the angular divergence and lateral position of a particle traversing it. The simulation of the multiple small collisions that a particle has through a medium is described here. The average of the total deflexion squared angle in a step of length \( z \) is given by [15]

\[
\langle \Theta^2 \rangle = \left( \frac{\alpha}{p \beta} \right)^2 \frac{z}{x_0} \left[ 1 + \frac{1}{9} \log_{10} \left( \frac{z}{x_0} \right) \right]^2
\]  

(3.42)

where \( \alpha = 0.015 \text{ GeV} \), \( x_0 \) is the radiation length of the material and \( p \) is the momentum.

Then the simulation of multiple scattering in a small step is carried by, first generating
two numbers A and B distributed according to a two-dimensional Gaussian with r.m.s. of 1 each. The angles $\Delta \theta, \Delta \varphi$ in fig. 8 are

$$\sin(\Delta \theta) = \sqrt{\theta_x^2} \cdot B$$
$$\sin(\Delta \varphi) = \sqrt{\theta_y^2} \cdot A$$

and the displacements $\Delta x$ and $\Delta y$

$$\Delta x = \sin(\Delta \theta) \cdot \Delta z / 2$$
$$\Delta y = \sin(\Delta \varphi) \cdot \Delta z / 2$$

The step size in the tracking procedure is rather small $\sim 0.1$ to 0.2 cm. The decrease of momentum due to the ionization energy loss is applied after the step has been carried out.

Nuclear scattering, Transport

The transport of particles produced in the hadronic shower is carried in a similar way to what was done in the previous chapter. Depending on the material, momentum and type of incoming particle we have an interaction probability given by

$$P(x) = 1 - \exp(x / \lambda)$$

where $x$ is the length of the step and $\lambda$ is the nuclear collision length given by

$$\lambda = \frac{A}{N_\sigma}$$

To see if there is interaction or not we sample in the distribution

$$x' = \lambda \ln \left\{ 1 - p' \right\}$$

where $p'$ is random uniformly distributed in $(0, 1)$. If $x' \leq x$ then an interaction has occurred and we should choose the kind of interaction to perform, otherwise there is no interaction and we transport the particle a length $x$ from the previous point.

Interactions

As quoted before, the bulk of interactions in hadronic showers are with nuclei. There are four major classes of interactions of hadrons with nuclei (see figure 9), depending on the status of the nucleus after the interaction:

a) coherent elastic
b) coherent inelastic
c) Incoherent quasielastic
d) Incoherent inelastic

The first two are well understood from the theoretical point of view and a generalization of the hadron-nucleon interactions can be applied to them. Formally the nucleon with $A = 1$ has to be replaced with $A > 1$. The incoherent scattering is more complicated and involves the modeling of the intranuclear cascade. As a further step we must take care of Nuclear
Fig. 9 Interactions hadron-nucleus  a) coherent elastic, b) coherent inelastic, c) incoherent quasielastic, d) incoherent inelastic

evaporation, stopping particles and nuclear fission.

The elastic and inelastic cross sections hadron-proton are tabulated in GHEISHA, as a function of the momentum for \( \Pi^+, \Pi^-, K^-, K^+, K_L^0, p \) and \( \bar{p} \). For all the other cross sections on protons we use the charge independent approximation for the forward elastic scattering amplitudes according to the naive quark parton model and write them as function of the quark scattering amplitudes. By using the available data we fit the quark scattering amplitudes as a function of the momentum and determine the corresponding cross sections of all unmeasured or poorly measured reactions. The cross sections for target neutrons are assumed to be the same as for protons.

It is commonly accepted that diffraction and absorption cross sections for the scattering on heavy nuclei can be parametrized as

\[
\sigma_d = \sigma_{d0} A^{\alpha_d} \\
\sigma_a = \sigma_{a0} A^{\alpha_a}
\]

(3.50) (3.51)

where \( A \) is the atomic number. The formulas are based in the optical model, that assumes that at high energy the nucleus structure can be approximated described by an average potential. The real and imaginary part of the potential gives the diffraction and absorption cross sections. We identify the absorption cross section with the coherent elastic scattering whereas the coherent inelastic and incoherent scattering contribute to the absorption cross section.

The parameters \( \sigma_{d0}, \sigma_{a0}, \alpha_d, \alpha_a \) are determined by fits to the \( A \)-dependence of cross sections coming from the available experimental data. Nevertheless the program uses the
experimental data when it is available. In the very low energy region, \( p \leq 0.5 \text{ GeV/c} \), the above parametrization has problems. This region is where the nucleus gets into an excited state and emits heavy particles like \( p, d, t \) etc (Nuclear reactions become dominant). The formulas 3.50 and 3.51 can no longer describe the cross sections in this region. To avoid these problems is quite wise to measure all the elastic and inelastic cross sections of all the materials used in an experimental set-up and insert these values in the program.

For the cross sections of the scattering on molecules and compounds, shadowing effects may play an important role and a direct scaling based on the atomic number of the constituents may not be right. The right definition of the effective atomic number lies somewhere between the average atomic number and the sum of the atomic numbers of all the constituents.

**Generation of secondary particles**

The generation of secondary particles in inelastic reactions hadron-nucleon is carried out by the special Koba-Nielsen-Olesen (KNO) formula

\[
P(n) = \frac{1}{\langle n \rangle} \frac{n}{\langle n \rangle} \exp \left[ -\frac{\pi}{4} \left( \frac{n}{\langle n \rangle} \right)^2 \right]
\]  

(3.52)

where \( \langle n \rangle \) is the average number of particles produced at a given energy. A fit to the experimental data in pp interactions gives for \( \langle n \rangle \),

\[
\langle n \rangle = 3.626 + 0.666 \, x + 0.337 \, x^2 + 0.118 \, x^3
\]

(3.53)

where \( x = \ln (\sqrt{s} - 2m_p) \).

---

**Fig. 10** Average charged kaon multiplicity per inelastic collision as a function of \( s \). [16]
The multiplicity distribution for the production of \( n_{\pi^+} \) positive pions, \( n_{\pi^-} \) negative pions, \( n_{\Lambda} \) neutral pions, \( n_{K^+} \) positive kaons, etc, can be expressed by

\[
W(n_{\pi^+}, n_{\pi^-}, n_{\Lambda}, n_{K^+}, n_{K^0}, \ldots) = P(n_{\pi^+}, n_{\pi^-}, n_{\Lambda}, n_{K^+}, n_{K^0}, \ldots) \mid n
\]

(3.54)

where the conditional probability \( P(n_{\pi^+}, n_{\pi^-}, \ldots \mid n) \) can be obtained from the isospin model and tables. Three constraints are imposed in the produced particles: 1.) charge conservation, 2.) strangeness conservation and 3.) baryon number conservation. In figure 10 we show the average charged kaon multiplicity per inelastic collision for pp collisions as a function of \( s \).

**Simulation of final state kinematics**

The generation of the four-momenta for the final state particles is based on quark or hadron-jets in soft hadron-nucleon collisions. As discussed in reference [14] it is possible to find scaling laws for the flux of energy and additive quantum numbers into a given interval of the polar angle \( \Theta \) with respect to the jet axis

\[
\frac{dE}{d\lambda} = \rho(\lambda) \frac{M}{(1 + M^2 \lambda^2)^{3/2}}
\]

(3.55)

\[
\frac{d\phi}{d\lambda} = \phi(\lambda)
\]

(3.56)

\[
\frac{ds}{d\lambda} = s(\lambda)
\]

(3.57)

where the variables are defined as follows:

\[
\lambda = \cot^2 \Theta / E_{jet}
\]

(3.58)

\[
\Delta \xi = E(\Theta) / E_{jet}
\]

(3.59)

\[
\Delta Q = \langle Q(\Theta) \rangle
\]

(3.60)

\[
\Delta S = \langle S(\Theta) \rangle
\]

(3.61)

\( \Delta \xi \), \( \Delta Q \) and \( \Delta S \) are the energy fraction, average charge and average strangeness emitted into an angular interval \( \Theta \) and \( \Theta + \Delta \Theta \). \( Q \) and \( S \) are the charge and strangeness of the fragmenting quark. And \( M \) is the mass parameter which is interpreted as the average transverse mass. The \( M \) parameter has different values for each fragmenting quark.

Choosing a momentum distribution of the quarks inside the hadron one can calculate the corresponding energy and quantum number flow distributions for pion and proton fragmentation.

To finish with the inelastic hadron-nucleon cross sections we summarize the procedure to obtain the final state particles. We stay in the c.m.s. of the colliding hadrons.

1. Calculate number of secondary particles and assign mass, baryon-number, charge and strangeness according parametrization 3.54.

2. Generate \( P_{T} \)-values and \( \varphi \)-values (azimuth angle) for all the particles, so that the transverse momenta balance exactly in the plane perpendicular to the beam axis. The
\( P_T \) distributions are generated according to \( \exp(-b P_T^2) \), where \( b \) is tabulated from experimental data.

3. Assign \( \lambda \)-values in \([0, 1 / P_T]\) according to the three fragmentation equations 3.55 to 3.57, where \( Q \) and \( S \) are the charge and strangeness of the fragmenting beam or target particle. The mass \( M \) is chosen slightly different for pions, kaons and baryons.

4. Calculate \( x \) according \( x = \lambda \cdot P_T \) and \( P_L = P_\pi \cdot x \)

5. Choose a scale factor for the longitudinal momentum conservation. This factor is normally very close to one.

**Elastic scattering and quasi-two-body reactions**

Elastic scattering and quasi-two-body reactions are generated both in the same way. The production of secondaries with masses \( M_1 \) and \( M_2 \) obeys the law

\[
\frac{d\sigma}{dt} \sim e^{-b|t|} \quad (3.62)
\]

where \( t \) is the momentum transfer in the limits between \( t_{\text{min}} \) and \( t_{\text{max}} \). The slope parameter can be parametrized as

\[
b = \begin{cases} 
4.225 + 1.795 \ln p & \text{(elastic)} \\
4.000 + 1.600 \ln p & \text{(quasi-two-body)}
\end{cases} \quad (3.63)
\]

where \( p \) is the beam momentum in the Lab. system.

The masses \( M_1 \) and \( M_2 \) are distributed like

\[
P(M) = gc \left[ g(M-M_o) \right]^{c-1} \exp \left[ - \left[ g(M-M_o) \right]^c \right] \quad (3.64)
\]

where \( M_o \) is the threshold mass

\[
M_o = \sum_{i=1}^{N} m_i \quad (3.65)
\]

and \( g, c \) are two parameters which determine the shape of the distribution.

**Intra-nuclear cascade. Nuclear evaporation**

In high energy hadron-nucleus collisions the nucleus can be considered as a dense set of nucleons. Nuclear structure, correlations between nucleons can be ignored. That comes from the fact that the wavelength of the incoming high energy hadrons is much smaller than the nuclear radius and typical times of hadronic interactions are shorter than nuclear periods.

The absorption length for protons in nuclear matter is \( \sim 2 \text{fm} \), which is smaller compared, for instance, to the Uranium diameter \( \sim 14 \text{ fm} \).

Two situations can be considered depending on the distance in which an interaction hadron-nucleon inside the nucleus yields the asymptotic final state. If this distance is less than the typical dimension of the nucleon, then the final state particles are produced
inside the nucleus with possible further interactions giving rise to a "intramuracl cascade". The final number of particles leaving the nucleus is given by

$$\langle n \rangle_A = \langle n \rangle_p A^{1/3} \tag{3.66}$$

When the typical distance for the production of the asymptotic final state is bigger than the nuclear radius, the asymptotic hadrons will be produced after the nucleus, through, perhaps, an excited hadron state. In this case the multiplicity is

$$\langle n \rangle_A = \langle n \rangle_p \tag{3.67}$$

In other words we can say that backward produced particles in the center of mass of the first collision, do make an intramuracl cascade, whereas forward produced particles do not.

The first collision is exactly simulated in the program. The forward particles produced are not furtherly affected. The backward produced particles undergo a further intramuracl cascade. The average number of particles additionaly produced is parametrized with the formula

$$\langle n_{add} \rangle = \alpha(s, A, n) (A^{1/3} - 1) \times n_b \tag{3.68}$$

where $n_b$ is the number of backward produced particles and $\alpha(s)$ is a very slow varying function of $s$. $n_{add}$ is poisson distributed.

In the phenomenology of nuclear interactions, the particles produced in the previous interactions are called shower particles (mesons and leading baryons) and grey track particles (nucleons from cascading processes).

The nuclear evaporation shows that after the interaction, the nucleus remains in a highly excited state emitting further nucleons, low mass fragments like $d$, $t$, $\alpha$-particles and also high mass fragments. These particles are called black track particles. The kinetic energy is very low ($< 20$ MeV) and the angular distribution is isotropic in the laboratory system. The number of black particles is nearly proportional to the number of grey track particles

$$\langle n_b \rangle = 1.50 + 1.35 \, n_g \tag{3.69}$$

The total amount of kinetic energy communicated, to the particles in the nuclear evaporation is in first aproximation

$$F_{eva} = F(A) \times G(E_k) \tag{3.70}$$

where $F(A)$ and $G(E_k)$ are functions of the atomic number $A$ and of the kinetic energy, $E_k$ of the primary particle.

**Nuclear fission**

Four steps can be distinguished in the fission process (suppose we have Uranium)

1°) nuclear interaction with production of 2 heavy fragments ($\tau \sim 10^{-24}$ sec.)

2°) deexcitation of the fragments by neutron emission ($\tau \sim 10^{-14}$ sec.)
3) deexcitation by photon emission \( (\tau \sim 10^{-8} \text{ sec.}) \)
4) deexcitation by \( \beta^- \)-decay \( (\tau \sim 10^{-6} \text{ sec.}) \).

The energy spectra of the emitted neutrons has been parametrized as

\[
P(E_n) = e^{-E/0.965} \sinh \sqrt{2.29 E}, E \text{ in MeV.}
\]

and the number of emitted neutrons is normal distributed around

\[
N(E_o) = 2.569 + 0.559 \ln E_o
\]

with a r.m.s. of 1.23. \( E_o \) is the energy of the incoming neutron in MeV.

The energies of the emitted photons are given by

\[
P(E_p) = 10 e^{-1.15 E_p}
\]

and the number of emitted photons

\[
N(E_o) = 9.5 + 0.6 \ln E_o
\]

with a r.m.s. of 3.

The Gheisha-Monte Carlo described so far, is sufficient for the description of showers in calorimetric experiments. All the formulas in it contained are based on experimental data when available. In some case the parameters entering in formulas describing some hidden processes had to be tuned to reproduce related distributions according with the data. Some topics, like decay of particles, have not been discussed here, although they are included in the code.

### 3.3 Response simulation

To close these lectures, the simulation of the response of a calorimetric detector will be presented, as well as some comparison with experimental data. In general a calorimeter has active and passive components. The active components can be materials with luminescent properties, or drift chambers or some other device producing some convenient signal. A scintillator is chosen as active material to study the simulation.

Several effects are accounted since particles go through the scintillator until a signal is conveniently stored in some device: 1) The conversion of absorbed energy into fluorescent light, 2) The conduction of light through scintillator and Light-guides, 3) The conversion of light into photo-electrons in the photomultiplier (P.M.), 4) Amplification inside the photomultiplier of the electronic cascade and further transformation into an electric current and 5) The current integration, pulse shaping and read-out by the analog-to-digital converter (ADC) and time-to-digital converter (TDC). All this effects must be accounted when simulating interactions since we want to compare with experimental data obtained in this way.
The read-out system to collect a signal can be described in general by an expression (see figure 11)

\[ u_2(t) = \int_0^t u_1(t - \tau) g(\tau) \, d\tau \]  

(3.75)

where \( u_1(t) \) is the input signal, \( u_2(t) \) is the output signal and \( g(\tau) \) is the response function which can be calculated or measured experimentally. The response function involves all the effects that modify the input signal until getting the output signal: fluorescence, decay of it, transport of light, attenuation, shape of devices, etc.

The amount of fluorescent light produced per unit of energy absorbed in the scintillator depends on the particle type, energy and the kind of material used as scintillator. The scintillation response can be calculated through (Birk's Law)

\[
\frac{dL_0}{dx} = \text{const.} \frac{dE}{dx} / (1 + \frac{\alpha}{2} \frac{dE}{dx})
\]  

(3.76)

where the light produced per unit of length is almost proportional to the kinetic energy loss. The parameter \( \alpha \) obtained from fits to experimental data.

The atoms and molecules of the scintillator excited at a time \( t_0 \) by a primary event have a certain lifetime. Then, the intensity of the emitted light will decrease exponentially with the time

\[
L_1(t) = \frac{L_0}{C_{f1}} e^{-\frac{t-t_0}{C_{f1}}} \cdot H(t-t_0)
\]  

(3.77)

where \( H(t) \) is the step function with value "0" for \( t < 0 \) and 1 in all the other cases. Inorganic scintillators have normally a slower fluorescent decay time (800 ~ 320 ns, NaI(Tl) ~ 250 ns, ZnS(Ag) ~ 200 ns) than organic scintillators (Anthracene ~ 32 ns, Estilbene ~ 4 ns, Liquid ~ 3 ns, plastic ~ 4 ns).

The light received at the entrance of the photomultiplier is given by

\[
L_2(t) = \varepsilon L_1(t - \frac{x}{c_{\text{scin}}} - \frac{d}{c_{\text{LG}}}) e^{-x/\lambda_{\text{att}}}
\]  

(3.78)

where \( \varepsilon \) is a constant accounting for light losses in the light guide, \( \lambda_{\text{att}} \) is the effective attenuation length combining effects like secondary excitation, light losses by diffuse reflections on surfaces, etc., \( x \) is the length of the light rays in the scintillator, \( d \) the
length of the light guide and $C_{\text{scin}} (C_{LG})$ is the velocity of light in the scintillator (Light guide).

![Diagram of the processes]

Fig. 12 Schematic view of the processes, since the scintillator is excited until a signal is collected.

After the interaction in the scintillator, the light is emitted isotropically, so that a distribution $F(x)$ of the light ray lengths $x$, must be folded into the above expression to get an effective signal at the entrance of the P.M.

Under the assumption, that the integration time constant $RC$ of the output system is much greater than the pulse duration, and that the propagation time of the electron cascade in the PM and the spread of it are much smaller than the fluorescence decay time, we can get the anode current at output of the P.M.

$$I_a(t) = \bar{A} \cdot e \cdot \bar{N} \cdot L_2(t)$$  \hspace{1cm} (3.79)

where $\bar{A}$ is the mean amplifier gain, $e$ the electron charge and $\bar{N}$ the number of electrons reaching the multiplier system of the PM. The energy $W = \Delta E / \bar{N}$ necessary to form a photo-electron is relatively high. (0.3 - 1 KeV for NaI(TL), 1-3 KeV in organic scintillators, 5-20 KeV in glass scintillators)

The anode current $I_a(t)$ can be integrated over by a $RC$ network to obtain a voltage pulse proportional to $\Delta E$. The time dependence of the output voltage is

$$V_{\text{out}}(t) = -\frac{\bar{A} \cdot e \cdot \bar{N}}{C} \int_0^t e^{-(t-t')/t_{RC}} L_2(t') \, dt'$$  \hspace{1cm} (3.80)

With time constants $t_{RC}$ long enough, the voltage pulse height is given by

$$V_o = \frac{\bar{A} \cdot e \cdot \bar{N}}{C}$$  \hspace{1cm} (3.81)

The deviation $\sigma_v$ of the total pulse light $V_o$ depends on the staticstical desviation $\sigma_N$ of the number $\bar{N}$ of photo cathode electrons and on the desviation $\sigma_A$ of the P.M. gain:

$$\left(\frac{\sigma_v}{V_0}\right)^2 = \frac{1}{\bar{N}} \left[ 1 + \left(\frac{\sigma_A}{\bar{A}}\right)^2 \right]$$  \hspace{1cm} (3.82)
The response function, \( g(t) \) may be obtained by solving the integral (3.80) or (3.79) replacing \( L_2(t) \) by (3.78) and \( L_1(t) \) by (3.77).

Once the function \( g(t) \) is known, the simulation of the output voltage can be easily performed by sampling the absorbed energy of the whole event in the scintillators connected to the same P.M., as function of the space and time \( F \), \( F \) may be converted to \( x \) and the integral (3.75) gives immediately the output voltage as function of \( t \).

Unfortunately, the response function \( g(t) \) is not possible to solve if the distribution of light ray lengths \( F(x) \) is not simple enough.

A general way to skip this problem, is to produce \( n \) light rays at the position of the energy absorption applying a time delay for each ray according (3.77), then use (3.78) for the light absorption assuming some distribution function \( F(x) \). For energy light ray \( i \), \( V_i(t) \) is calculated by numerical integration using (3.80). The total pulse shape is obtained by sampling all \( V_i(t) \) in a histogram.

The Monte Carlo method has played and is playing an important role in the specific design of the components of detectors. Monte Carlo studies were made, for instance, to decide the optimal BGO crystal size in the L3 electromagnetic calorimeter at LEP.

![Graph](image)

**Fig. 13** Pulse height distribution. Comparison between data and Gheisha Monte Carlo.
Test runs with $\gamma$, $\mu$, and $\pi$ beams have been carried at CERN to test scintillator performances and readouts for the BGO electromagnetic calorimeter. Pion runs have been made with a BGO crystal size of $9 \times 9 \times 20$ cm$^3$. A comparison of the pulse height distribution between GHEISHA Monte Carlo and data for $\pi^+$ at 200 MeV/c is shown in fig. 13.

To explain the peak at £280 ADC-channels, we had to assume a muon contamination of the beam of 5% with a momentum resolution of 0.5%. The momentum resolution for pions has been quoted to be smaller than 0.1%. The width of the peak for non-interacting particles has been fitted by selecting a sensitive trigger time of the ADC's of $1 \mu$s. The signal for electrons of 200 MeV/c momentum would lie at ADC-channel £500.

4. FINAL REMARKS

We have seen a few examples of how to apply Monte Carlo techniques to the simulation and detection of processes in H.E.P. In these applications, Monte Carlo is a very powerful technique allowing the simulation of any kind of process, provided that the underlying theory is known. The full simulation of detectors needs Monte Carlo to account for all the incidences that a particle could have along the different compounds, as well as the inherent fluctuations to the multiple interactions in the development of showers.

In Monte Carlo programs, such as EGS and GHEISHA, phenomena without a clear theory or description behind, must be parametrized based on experimental data (so it is normal the case for the cross sections hadron-nucleus). Some parameters have to be tuned up in order to reproduce experimental distributions trying to balance the efficiency and accuracy of the program. Quantities, such as cross sections, might be changed inside the limits given by the experimental uncertainties in the tuning process.

In general there is some freedom in the programs allowing the tuning according to the experimental set up. Special care must be taken in situations where fine effects are important.

* * *

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