Generalized Random Phase Approximation and Gauge Theories

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Abstract

Mean-field treatments of Yang-Mills theory face the problem of how to treat the Gauss law constraint. In this paper we try to face this problem by studying the excited states instead of the ground state. For this purpose we extend the operator approach to the Random Phase Approximation (RPA) well-known from nuclear physics and recently also employed in pion physics to general bosonic theories with a standard kinetic term. We focus especially on conservation laws, and how they are translated from the full to the approximated theories, demonstrate that the operator approach has the same spectrum as the RPA derived from the time-dependent variational principle, and give - for Yang-Mills theory - a discussion of the moment of inertia connected to the energy contribution of the zero modes to the RPA ground state energy. We also indicate a line of thought that might be useful to improve the results of the Random Phase Approximation.

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I. INTRODUCTION AND MOTIVATION

Recently, variational calculations based on Gaussian wave functionals have stirred some interest when applied to Yang-Mills theory [1, 2, 3, 4, 5, 6, 7]. A problem constantly encountered is the fact that Gaussian wave functionals are not gauge invariant, i.e. they are not annihilated by the Gauss law operator (except for the case of electrodynamics). Therefore one faces the situation, that the mean-field treatment (to which the Gaussian wave functionals correspond) does not respect a symmetry of the Hamiltonian. This is a situation commonly encountered in nuclear physics where one often even wants to break as many symmetries as possible in order to store a maximum amount of correlations in a wave function of a very simple form. We therefore look to nuclear physics for a possible remedy of this problem.

One possibility is the introduction of a projector which projects the Gaussian wave functional onto the subspace of gauge invariant functionals [1, 2, 3, 4, 5, 6]. Another possibility is the Random Phase Approximation. In this framework one studies not the ground state but excited states, and one finds that - if the mean-field ground state has a broken symmetry - spurious excitations exist. Under certain conditions these spurious excitations decouple, however, and we have the pleasant situation that our excitations are as good as they would be if we had started from a gauge invariant mean-field state.

The computation of the excited states of Yang-Mills theory is especially interesting for a variety of reasons: First, one has an alternative point of view from which to shed light onto the confinement problem, compared to the usual approaches that try to compute the linear potential between static quarks. Second, one has quite accurate lattice data for some of the low lying glueballs for both two and three spatial dimensions, e.g. [8, 9]. This allows to judge the approximations that usually occur in the process of analytical calculations. On the other hand, there is quite an amount of information that is very hard to obtain from the lattice, like masses of excited glueballs, and also regularities and patterns in the glueball spectrum which can only be observed but not explained within the lattice framework. Third, in recent years the glueball spectrum has also become a matter of interest for theories that claim to have a very non-trivial connection to Yang-Mills theory like e.g. supergravity theories [10, 11, 12].

There are basically two approaches to the generalized Random Phase Approximation
(gRPA). The first one starts from the *time-dependent variational principle* of Dirac \[13, 14, 15\]. The variational parameters are decomposed into one part that solves the static equations of motion and one “small” fluctuating part. In this context, the nature of gRPA as a *harmonic approximation* becomes clear; the general procedure is outlined in app. A since the equivalence to the second approach to gRPA has up to now only been shown for the fermionic case \[13\]. The second approach is based on the formalism of creation and annihilation operators, well-known in nuclear physics \[16\], and has recently been applied also in the context of pion physics \[17, 18, 19\]. In contrast to these latter investigations, we consider here *generic* bosonic field theories; they are only restricted by the requirements that the kinetic energy shall have a standard form, and the remainder of the Hamiltonian shall be expressible as a polynomial in the field operators.

The structure of the paper is as follows: We begin by introducing (in sec. II) the Hamiltonian formulation of generic bosonic theories which will be the subject of investigation in the main part of this paper. In sec. III we rewrite the usual canonical formulation in terms of creation and annihilation operators thereby bridging the gap between canonical and many-body treatment. Especially, we will give an expression of the Hamiltonian for a generic bosonic system (with a standard kinetic energy term) in terms of creation/annihilation operators. This is followed by an alternative form of the Schrödinger equation useful in the context of gRPA. We then introduce the two crucial approximations needed to obtain the gRPA equations from the Schrödinger equation. The second of these approximations can - under certain conditions - be rephrased in terms of the so-called quasi-boson approximation which will allow for a simpler formulation. We then discuss the normal mode form of the Hamiltonian, and see what kind of difficulties appear if the gRPA equations have zero mode solutions. In sec. IV we will discuss how conservation laws translate from the full theory to the theory approximated by gRPA, when zero mode solutions are implied, and also the connection to symmetry breaking in the mean-field treatment is indicated. We will also draw attention to the fact that the character of the symmetry under consideration can change from a non-Abelian to an Abelian symmetry. In sec. V we demonstrate the equivalence of the gRPA formulation based on the time-dependent variational principle to the operator approach employed in this paper; this equivalence has so far been demonstrated only in the case of fermionic systems \[13\]. In sec. VI we discuss further how the normal mode form of the Hamiltonian is altered if the gRPA equations have zero mode solutions, and the special
role that is played by the moment of inertia. We then restrict the further discussion from
generic bosonic theories to SU(N) Yang-Mills theory, where we discuss both the moment of
inertia in general, and also give an explicit computation of the leading terms in a perturba-
tive expansion. We end the section with a discussion of the possibility of interpreting the
moment of inertia as the static quark potential. In sec. VII we give a critical evaluation of
the generalized Random Phase Approximation and give an outlook to further applications.
At the end of the paper a number of appendices is given. In app. A we give a short ac-
count of the gRPA as derived from the time-dependent variational principle in so far as it
is needed for the purposes of this paper. In app. B we discuss shortly Yang-Mills theory in
Weyl gauge, and how it fits into the general framework of bosonic theories as discussed in
the main body of the text. In app. C a number of explicit expressions and computations can
be found, in app. D the proof for bosonic commutation relations among gRPA excitation
operators is given, and in app. E the proof of a theorem used in sec. III can be found.
The important issue of renormalization will not be addressed in this paper. This topic in the
context of the gRPA is subject to further investigation. In the approach to the gRPA that
is based on the time-dependent variational principle a discussion of renormalization issues
for the $\phi^4$ theory has been given in 14, 15.

II. HAMILTONIAN FORMALISM

In this section the Hamiltonian formalism for fairly general bosonic theories is introduced.
Since there are good references to the subject, e.g. 20, 21, we will be very brief here.
The basic variables in the Hamiltonian approach consist of the the field operators $\phi_i$ and
the canonical momenta $\pi_i$. In order to allow for sufficient generality we will use super-
indices which can - besides the position coordinate $x$ - also contain spatial indices or internal
(e.g. color) indices. The Einstein summation convention is adopted, implying sums over all
discrete and integrals over all continuous variables. These basic variables satisfy the basic
commutation relations

$$[\phi_i, \pi_j] = i\delta_{ij}. \quad (1)$$

We will work in the Schrödinger representation, i.e. we will not work with abstract states
but with a field representation of the states (analogous to the position representation in
quantum mechanics); the objects to be considered are thus wave functionals $\psi[\phi]$, given by

$$
\psi[\phi] = \langle \phi | \psi \rangle,
$$

(2)

for the state $|\psi\rangle$. The states $|\phi\rangle$ are eigenstates of the field operators $\phi_i$. The usage of $|\phi\rangle$ as basis states implies that we work in the position representation, i.e. in the remainder of the paper we will realize $\phi$ multiplicatively and $\pi$ as a derivative operator:

$$
\langle \phi | \phi_i | \psi \rangle = \phi_i \psi[\phi] \quad \text{and} \quad \langle \phi | \pi_i | \psi \rangle = \frac{1}{i} \frac{\delta}{\delta \phi_i} \psi[\phi],
$$

(3)

or more pragmatically we will read in all formulas where the field operator $\phi$ appears this only as a multiplicative $\phi$ and $\pi$ as $\delta/(i\delta\phi)$.

Since we work in the Schrödinger picture the operators considered are all time-independent. The states, however, satisfy the Schrödinger equation

$$
i \partial_t |\psi(t)\rangle = H |\psi(t)\rangle,
$$

(4)

where $H$ is the Hamiltonian of the system. In the main section we will be mostly interested in stationary states, i.e. states that can be written as $|\psi(t)\rangle = e^{-iEt}|\psi(0)\rangle$. These are then eigenstates of the Hamiltonian

$$
H |\psi\rangle = E |\psi\rangle.
$$

(5)

III. FORMULATION OF MANY-BODY LANGUAGE FOR GENERIC BOSONIC THEORIES

A. Creation and Annihilation Operators

In this section we will apply and generalize the operator approach used in nuclear physics \[16, 22, 23, 24, 25, 26, 27, 28, 29, 30\] and recently also in pion physics \[17, 18, 19\]. We draw the connection between the canonical treatment and the many-body language as was already briefly indicated in \[31\]. Since for our purposes it is sufficient, and in order to allow close comparison to gRPA as obtained from the time-dependent variational principle, cf. app. \[\square\] we will consider Hamiltonians of the form

$$
H = \frac{1}{2} \pi_i^2 + V[\phi].
$$

(6)
\( V[\phi] \) is a functional of the field operator, in the following referred to as ‘potential’.

Since the basis of this approach is the stationary Schrödinger equation, we start therefore from the most general time-independent Gaussian state \((i, j)\) are super-indices):

\[
\psi[\phi] = \mathcal{N} \exp \left( - (\phi - \bar{\phi})_i \left( \frac{1}{4} G^{-1} - i \Sigma \right)_{ij} (\phi - \bar{\phi})_j + i \bar{\pi}_i (\phi - \bar{\phi})_i \right),
\]

where \( \mathcal{N} \) is a normalization constant. A Gaussian state allows the explicit construction of creation and annihilation operators as linear combinations of \( \phi_i \) and \( \pi_i \) that satisfy the basic relations

\[
a_i \psi[\phi] = 0 \text{ and } [a_i, a_j^\dagger] = \delta_{ij},
\]

where the latter relation fixes the normalization. Using eq. (7) as reference state to be annihilated by \( a_i \) one obtains as explicit expressions for \( a^\dagger, a \):

\[
a^\dagger_i = U_{ij} \left\{ \left( \frac{1}{2} G^{-1} + 2i \Sigma_{jk} \right) (\phi - \bar{\phi})_k - i (\pi - \bar{\pi})_j \right\}, \quad (9)
\]

\[
a_i = U_{ij} \left\{ \left( \frac{1}{2} G^{-1} - 2i \Sigma_{jk} \right) (\phi - \bar{\phi})_k + i (\pi - \bar{\pi})_j \right\}, \quad (10)
\]

where \( U \) is (implicitly) defined via the relation

\[
U_{ij} U_{jk} = G_{ik}
\]

and could also be called the square root of \( G \). A short excursion on the existence and the implicit assumption of reality of \( U \) is in order here: Since \( G_{ik} \) is a real symmetric matrix (matrix is used in the generalized sense s.t. also continuous indices are allowed) one can always diagonalize it. Therefore one can also always write down a \( U \) as given above. However, \( G \) has to satisfy another condition, namely all of its eigenvalues have to be strictly positive, since otherwise one will run into two kinds of problems: If \( G \) has a zero eigenvalue \( G^{-1} \) does not exist and the matrix element of \( \langle \pi^2 \rangle \) will be infinite. This is certainly undesirable, and will be assumed not to be the case in the following. Moreover, if \( G \) has a negative eigenvalue, \( \psi[\phi] \) is not even normalizable, since in the direction of the negative eigenvalue \( \psi[\phi] \) will increase exponentially for increasing values of \( \phi \). Thus, all eigenvalues of \( G \) must be strictly positive, therefore \( U \) can be chosen to be real.

Since \( a_i, a^\dagger_i \) are just given via linear combinations of \( \phi \) and \( \pi \), one can invert these relations to obtain \( \phi, \pi \) in terms of \( a, a^\dagger \) and the parameters of the Gaussian wave functional we started
with:

\[
\phi_i = \tilde{\phi}_i + U_{ij}(a_j + a_j^\dagger),
\]

(12)

\[
\pi_i = \tilde{\pi}_i + 2i\left\{\left(\frac{1}{4}G^{-1}_{ik} - i\Sigma_{ik}\right)U_{kj}a_j^\dagger - \left(\frac{1}{4}G^{-1}_{ik} + i\Sigma_{ik}\right)U_{kj}a_j\right\},
\]

(13)

One should note that the dependence of the canonical operators on the parameters of the wave functional we choose is only seeming, since the creation and annihilation operators depend implicitly on these parameters as well. If one inserts eqs. (9), (10) into eqs. (12), (13) one obtains an identity \(\phi = \phi, \pi = \pi\). In a practical sense, however, we have transferred information that is contained in the wave functional to the operators, since in the following the only property of \(\psi[\phi]\) that we will use for practical computations is that \(a_i\psi[\phi] = 0\). All parameter dependence that usually comes about by calculating matrix elements now enters the formulas via normal ordering.

Since we have now a representation of the canonical operators in terms of \(a, a^\dagger\) (referred to in the following as \(c/a\) representation), all operators permissible in a canonical system can be expressed in terms of \(a, a^\dagger\), especially the Hamiltonian which for obvious reasons is central to the following calculations.

**B. Form of Hamiltonian in the c/a Representation**

We have required the Hamiltonian to have a certain structure [cf. eq. (11)]. The Hamiltonian there resolves naturally into a kinetic energy \(T = \frac{1}{2}\pi^2\) and a potential term \(V[\phi]\) which is a functional of \(\phi\) only. It is very useful to normal-order these expressions to make further progress. In the course of this, one makes the useful observation that one can write
the kinetic energy as:

\[
\frac{1}{2} \pi_i^2 = \left\{ \frac{1}{2} \left( \bar{\pi}_i^2 + \frac{1}{4} \text{Tr}(G^{-1}) + 4 \text{Tr}(\Sigma G \Sigma) \right) \right\} \\
+ \left\{ \left( \frac{\delta}{\delta \bar{\pi}_k} \right) \left( \frac{1}{2} \bar{\pi}_i^2 \right) \left( \frac{i}{2} U_{k_1 j_1}^{-1} \left( a_{j_1}^\dagger - a_{j_1} \right) + 2 \Sigma_{k_1 l_1} U_{l_1 j_1} \left( a_{j_1}^\dagger + a_{j_1} \right) \right) \right\} \\
+ \left\{ \left( \frac{\delta}{\delta G_{k_1 k_2}} \right) \frac{1}{2} \left( \bar{\pi}_i^2 \right) \left( U_{k_1 j_1} U_{k_2 j_2} \left( a_{j_1}^\dagger a_{j_2}^\dagger + a_{j_1} a_{j_2} + 2 a_{j_1}^\dagger a_{j_2} \right) \right) \right\} \\
+ \left\{ \left( \frac{\delta}{\delta \Sigma_{k_1 k_2}} \right) \frac{1}{2} \left( \bar{\pi}_i^2 \right) \left( U_{k_1 j_1} U_{k_2 j_2} \frac{i}{4} \left( a_{j_1}^\dagger a_{j_2}^\dagger - a_{j_1} a_{j_2} \right) \right) \right\} \\
+ \left\{ \frac{1}{4} \left( G_{j_1 j_2}^{-1} + \frac{i}{2} \left( U^{-1} \Sigma U - U \Sigma U^{-1} \right) \right)_{j_1 j_2} \right\},
\]

where \( \text{Tr} \) is a trace over the super-indices. A similar result can be found for the potential part (except that it is independent of \( \Sigma \) and \( \bar{\pi} \)). In app. [E] we will demonstrate that the potential can be decomposed into c/a operators s.t. the prefactors can be written as functional derivatives of the expectation value of the potential between Gaussian states, and the c/a operators always appear in a fixed structure\(^1\). In the following we will restrict ourselves to the contributions to \( V \) with up to four c/a operators, since the terms containing higher numbers of c/a operators do not contribute\(^2\) to the gRPA matrices that will be introduced.

\(^1\) e.g. if in the potential there are terms that contain two creation operators, or two annihilation operators, or one creation/one annihilation operator, they can always be written as factor \( \times (a_i^\dagger a_j^\dagger + a_i a_j + 2 a_i^\dagger a_j) \), and similarly for all other terms that contain a fixed sum of creation and annihilation operators. For the kinetic terms things are a bit different, but we have given the decomposition of the only kinetic term allowed in eq. (14).

\(^2\) That this is true can be seen by usage of Wick’s theorem and the so-called second gRPA approximation \( \langle a \rangle = 0 \) that will be introduced in sec. III D.
in eq. (20):

\[ V[\phi] = \langle V[\phi] \rangle + \left( \frac{\delta}{\delta \phi_k} \langle V[\phi] \rangle \right) U_{k_1 j_1} \left( a_{j_1}^\dagger + a_{j_1} \right) \]

\[ + \left( \frac{\delta}{\delta G_{k_1 k_2}} \langle V[\phi] \rangle \right) U_{k_1 j_1} U_{k_2 j_2} \left( a_{j_1}^\dagger a_{j_2}^\dagger + a_{j_1} a_{j_2} + 2a_{j_1}^\dagger a_{j_2} \right) \]

\[ + \frac{1}{3} \left( \frac{\delta}{\delta \phi_k} \frac{\delta}{\delta G_{k_2 k_3}} \langle V[\phi] \rangle \right) U_{k_1 j_1} U_{k_2 j_2} U_{k_3 j_3} \]

\[ \times \left( a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}^\dagger + 3a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3} + 3a_{j_1}^\dagger a_{j_2} a_{j_3} + a_{j_1} a_{j_2} a_{j_3} \right) \]

\[ + \frac{1}{6} \left( \frac{\delta}{\delta G_{k_1 k_2}} \frac{\delta}{\delta G_{k_3 k_4}} \langle V[\phi] \rangle \right) U_{k_1 j_1} U_{k_2 j_2} U_{k_3 j_3} U_{k_4 j_4} \]

\[ \times \left( a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}^\dagger a_{j_4}^\dagger + 4a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3} a_{j_4} + 6a_{j_1}^\dagger a_{j_2} a_{j_3} a_{j_4} + 4a_{j_1} a_{j_2} a_{j_3} a_{j_4} + a_{j_1} a_{j_2} a_{j_3} a_{j_4} \right) \] (15)

By adding the expressions eq. (14) (15) together, one observes that the Hamiltonian has a very simple schematic structure:

\[ H = \langle H \rangle + \left( \frac{\delta}{\delta \phi_k} \langle H \rangle \right) U_{k_1 j_1} \left( a_{j_1}^\dagger + a_{j_1} \right) \]

\[ + \left( \frac{\delta}{\delta \pi_{k_1}} \langle H \rangle \right) \left( \frac{i}{2} U_{k_1 j_1}^{-1} \left( a_{j_1}^\dagger - a_{j_1} \right) + 2 \Sigma_{k_1} U_{k_1 j_1} \left( a_{j_1}^\dagger + a_{j_1} \right) \right) \]

\[ + \left( \frac{\delta}{\delta G_{k_1 k_2}} \langle H \rangle \right) U_{k_1 j_1} U_{k_2 j_2} \left( a_{j_1}^\dagger a_{j_2}^\dagger + a_{j_1} a_{j_2} + 2a_{j_1}^\dagger a_{j_2} \right) \]

\[ + \left( \frac{\delta}{\delta \Sigma_{k_1 k_2}} \langle H \rangle \right) U_{k_1 j_1} U_{k_2 j_2} \left( a_{j_1}^\dagger a_{j_2}^\dagger - a_{j_1} a_{j_2} \right) \]

\[ + \frac{1}{4} \left( G_{j_1 j_2}^{-1} + \frac{i}{2} \left( U^{-1} \Sigma U - U \Sigma U^{-1} \right) \right)_{j_1 j_2} 2a_{j_1}^\dagger a_{j_2} \]

\[ + \frac{1}{3} \left( \frac{\delta}{\delta \phi_k} \frac{\delta}{\delta G_{k_2 k_3}} \langle H \rangle \right) U_{k_1 j_1} U_{k_2 j_2} U_{k_3 j_3} \]

\[ \times \left( a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}^\dagger + 3a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3} + 3a_{j_1}^\dagger a_{j_2} a_{j_3} + a_{j_1} a_{j_2} a_{j_3} \right) \]

\[ + \frac{1}{6} \left( \frac{\delta}{\delta G_{k_1 k_2}} \frac{\delta}{\delta G_{k_3 k_4}} \langle V \rangle \right) U_{k_1 j_1} U_{k_2 j_2} U_{k_3 j_3} U_{k_4 j_4} \]

\[ \times \left( a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}^\dagger a_{j_4}^\dagger + 4a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3} a_{j_4} + 6a_{j_1}^\dagger a_{j_2} a_{j_3} a_{j_4} + 4a_{j_1} a_{j_2} a_{j_3} a_{j_4} + a_{j_1} a_{j_2} a_{j_3} a_{j_4} \right) \] (16)
One should note that in all save the last term, the derivatives are taken of \( \langle H \rangle \) whereas in the last term the derivative is taken of \( \langle V \rangle \). The importance of this will become clear in sec. V B.

Up to now, we have taken an arbitrary Gaussian as a reference state: we have defined our creation/annihilation operators relative to that state - nothing else. Eq. (16) is exactly the same as \( H \) given in eq. (6). However, it is obvious from eq. (16) that the Hamiltonian greatly simplifies if we choose a specific reference state: a state that is a stationary point of the energy functional \( \langle H \rangle \) under variation of the parameters \( \bar{\phi}, \bar{\pi}, G, \Sigma \) - in other words, a state that satisfies the Rayleigh-Ritz variational principle\(^3\).

### C. Alternative Form of Schrödinger equation

In the following we want to generalize the RPA known for many-body physics to the quantum field theory described by the Hamiltonian of eq. (6). For this purpose we assume that the exact (excited) state \( |\nu\rangle \) can be created from the exact vacuum \( |0\rangle \) by an operator \( Q^\dagger_\nu \), i.e.

\[
|\nu\rangle = Q^\dagger_\nu |0\rangle.
\]

If \( E_\nu \) denotes the corresponding eigenvalue of \( H \) we have

\[
HQ^\dagger_\nu |0\rangle = E_\nu Q^\dagger_\nu |0\rangle.
\]

If we denote the vacuum energy by \( E_0 \), we can write this equivalently with a commutator:

\[
[H, Q^\dagger_\nu] |0\rangle = (E_\nu - E_0) Q^\dagger_\nu |0\rangle.
\]

We may multiply both sides of the equation with an arbitrary operator \( \delta Q \), and obtain an expectation value by multiplying from the left with \( \langle 0| \):

\[
\langle 0| \delta Q [H, Q^\dagger_\nu] |0\rangle = (E_\nu - E_0) \langle 0| \delta Q Q^\dagger_\nu |0\rangle.
\]

By subtracting zero on both sides we obtain an equation that only contains expectation values of commutators:

\[
\langle 0| [\delta Q, [H, Q^\dagger_\nu]] |0\rangle = (E_\nu - E_0) \langle 0| [\delta Q, Q^\dagger_\nu] |0\rangle.
\]

---

\(^3\) One should note that we require the same in the time-dependent approach, when we decompose the time-dependent parameters \( \bar{\phi}(t), \bar{\pi}(t), G(t), \Sigma(t) \) into a static part and small fluctuations, cf. app. A. The equations for the static part are identical to the Rayleigh-Ritz equations.
Using this equation as a starting point to derive the (generalized) RPA is known in nuclear physics as the \textit{equations of motion method} \cite{10,32}.

\textbf{D. First and Second gRPA Approximations}

The generalized Random Phase Approximation consists now of approximating eq. (21) in order to obtain a solvable set of equations. Two obvious candidates for approximations suggest themselves: \textit{first}, the excitation operator \(Q^\dagger_\nu\), \textit{second}, the vacuum state. In the remainder of the paper, we will call the approximation concerning the first topic the \textit{first gRPA approximation}, the approximation concerning the vacuum state the \textit{second gRPA approximation} (even though this may be doubtful linguistically). The second gRPA approximation is actually easier to state, therefore we start with it: \textit{approximate in all expressions involving vacuum expectation values of commutators the true vacuum state by the reference state of the creation/annihilation operators}, which in this context we will usually refer to as \textit{mean-field vacuum} and which will be denoted by \(\langle \rangle\). Later on, we will be even more restrictive and require the reference state to really be a stationary state of the Rayleigh-Ritz principle as was indicated before, but we will state explicitly from when on this additional restriction will be necessary. As already noted, the \textit{first gRPA approximation} deals with the class of allowed operators. In nuclear physics it is quite reasonable to assume that the lowest excited state above a Hartree-Fock ground state consists of a particle-hole excitation. This results in the so-called Tamm-Dancoff approximation. In the generalized RPA one assumes that one has a correlated ground state, s.t. not only the creation of a hole and a particle, but also their destruction is a possible excitation. In Yang-Mills theory - the theory we will be ultimately interested in - it is by far not so clear what structure the lowest excitation will have; we have thus taken the two following guiding principles (an argument similar to our second principle can be found in \cite{17,19})

1. one of the main differences between fermionic and bosonic systems is that in the latter there exist single-particle condensates; thus, one has at least to extend the \textit{ansatz} for the excitation operator by linear terms allowing for fluctuations. Furthermore, since we can compare to the gRPA as derived from the time-dependent variational principle, we will see that the \textit{ansatz} for \(Q^\dagger_\nu\) to be proposed leads to equations of motion that are identical to those derived from the time-dependent variational principle, thereby
verifying the ansatz to be the correct one.

2. The principal goal of the Random Phase Approximation is the restoration of symmetries that are violated at the mean-field level. We give here a short outline, drawing from concepts that will be introduced further below, in order to motivate from this property of symmetry restoration the form to be allowed for the excitation operators. The first point is that the second gRPA approximation can be replaced (under certain conditions) by the so-called quasi-boson approximation (QBA). There, it will turn out that gRPA equations can be written as

$$[H_B, Q^\dagger_{B\nu}] = (E_\nu - E_0)Q^\dagger_{B\nu}$$

where $O_B$ indicates that the QBA has been used. One the other hand, for a symmetry generator (in our case the Gauss law operator) $\Gamma$ which is a one-body operator (i.e. an operator that can be written as a linear combination of the operators $a, a^\dagger, aa, a^\dagger a^\dagger, a^\dagger a$), one can derive that

$$[H, \Gamma] = 0 \rightarrow [H_B, \Gamma_B] = 0.$$  (23)

If the mean-field vacuum is annihilated by $\Gamma$ this is a trivial statement, since in this case $\Gamma_B \equiv 0$. However, if the symmetry is violated on the mean-field level, one obtains a non-trivial result, namely that $\Gamma_B$ is a solution of the gRPA equations with zero excitation energy provided that the class of excitation operators contains $\Gamma_B$. Since all gRPA excitations are orthogonal this would be a very desirable state of affairs: effectively the spurious excitations caused by the symmetry violation on the mean-field level would not affect the physical excitations we are interested in.

To put it all in a nut-shell: one looks at the Gauss law operator, determines its structure (cf. app. C2), and keeps the class of excitation operators so large that the Gauss law operator belongs to this class. All the concepts and claims will become clear and will be proved in what follows. After this rather long motivation, we will consider excitation operators of the following form:

$$Q^\dagger_{\nu} = \frac{1}{2} \left( X^\nu_{mi} a^\dagger_{m} a^\dagger_{i} - Y^\nu_{mi} a_{m} a_{i} \right) + \left( Z^\nu_{m} a^\dagger_{m} - Z^\nu_{m} a_{m} \right),$$

where $X, Y, Z, \tilde{Z}$ are called amplitudes. From the form of the Gauss law operator one would have expected a few more terms; the reasons for leaving them out will be explained in
With the ansatz \( \{a_i, a_i^\dagger, a_j, a_i^\dagger a_j, a_i, a_j^\dagger\} \) we derive from eq. (21) a closed set of equations for the amplitudes \( X^\nu, Y^\nu, Z^\nu, \tilde{Z}^\nu \). To this end, the (up to now arbitrary) operators \( \delta Q \) will be appropriately chosen, together with the second gRPA approximation. We choose \( \delta Q \) in such a way that the different amplitudes are extracted individually on the RHS of eq. (21). It will be useful to consider \( \delta Q \in \{a_i, a_i^\dagger, a_j, a_i^\dagger a_j, a_i, a_j^\dagger\} \), and compute the commutators \([\delta Q, Q^\nu]\), approximating the real vacuum by the mean-field vacuum according to the second gRPA approximation. We obtain:

\[
\langle [a_j, Q^\nu]\rangle = \tilde{Z}^\nu_j \quad \langle [a_j^\dagger, Q^\nu]\rangle = Z^\nu_j \nonumber
\]

\[
\langle [a_n^\dagger a_j^\dagger, Q^\nu]\rangle = \frac{1}{2} Y^\nu_{\{jn\}} \quad \langle [a_n a_j, Q^\nu]\rangle = \frac{1}{2} X^\nu_{\{jn\}} \nonumber
\]

\[
\langle [a_n^\dagger a_j, Q^\nu]\rangle = 0,
\]

where we have used the notation \( \langle \ldots \rangle = \langle \ldots \rangle \) and introduced the abbreviation \( X_{\{ij\}} \) defined as \( X_{\{ij\}} = X_{ij} + X_{ji} \), and correspondingly for \( Y_{\{ij\}} \). With the \( \delta Q \)s defined above we now have to compute the LHS of eq. (21). For this purpose we introduce a number of matrices

\[
A_{njmi} = \langle [a_n^\dagger a_j^\dagger, [H, a_m^\dagger a_i^\dagger]]\rangle, \quad D_{njm} = \langle [a_n^\dagger a_j^\dagger, [H, a_m]]\rangle, \nonumber
\]

\[
B_{njmi} = \langle [a_n^\dagger a_j^\dagger, [H, a_m a_i]]\rangle, \quad E_{nm} = \langle [a_n^\dagger, [H, a_m^\dagger]]\rangle, \nonumber
\]

\[
C_{njm} = \langle [a_n^\dagger a_j^\dagger, [H, a_m^\dagger]]\rangle, \quad F_{nm} = \langle [a_n^\dagger, [H, a_m]]\rangle,
\]

and study their properties under interchange of labels. This will allow to reduce the number of independent entries of the LHS of eq. (21). The basic tools for this study are the Jacobi identity and the second gRPA approximation, i.e. \( a\rangle = 0 \). We also use frequently that \( [a, a] = [a^\dagger, a^\dagger] = 0 \). It turns out that the matrices \( A \) and \( E \) are symmetric in all indices, whereas \( F \) is a hermitean matrix, \( F^*_{nm} = F_{nm} \). In order to establish that \( B \) is also hermitean in the sense \( B_{njmi}^* = B_{minj} \) (whereas it is symmetric under interchange \( n \leftrightarrow j, m \leftrightarrow i \)) we need the second gRPA approximation since only upon usage of this approximation \( \langle [H, [a_n^\dagger a_j^\dagger a_n a_j]]\rangle \) will be zero generally.

Apart from one matrix, all other matrices that arise from inserting the different \( \delta Q \)s into the LHS of eq. (21) are trivially related to the matrices \( A, \ldots, F \) introduced above; this one non-trivial matrix is

\[
\langle [a_n^\dagger, [H, a_m a_i]]\rangle. \tag{27}
\]

It is related to the matrix \( D \), cf. eq. (26), via

\[
\langle [a_n^\dagger, [H, a_m a_i]]\rangle = \delta_{in} \Lambda_m + \delta_{nm} \Lambda_i + D^*_{min}, \tag{28}
\]

13
with $\Lambda_{i,m} = \langle [H,a_{i,m}] \rangle$. In sec. III E the second gRPA approximation will be replaced by the so-called *quasi-boson approximation*. In that formulation, it will be obvious that we have to neglect the terms $\Lambda$. From the explicit expression for the expectation values, given further below in eq. (38), we see that, at the stationary mean-field point$^4$, $\Lambda$ is indeed zero. We note two points:

- The gRPA matrix [essentially the LHS of eq. (21)] is hermitian iff $\Lambda = 0$.

- The second gRPA approximation and the quasi-boson approximation to be introduced below are compatible iff $\Lambda = 0$; such a constraint does not appear in nuclear physics as is easily comprehensible in two different ways: First, the matrix under consideration (as well as the matrices $C, D$) are non-zero only if the Hamiltonian contains terms with in total three creation/annihilation operators, i.e. terms that violate particle-number conservation. Such terms are not allowed in the usual nuclear physics framework, and thus there is no possibility for the above consistency condition to arise$^5$. Second, if we replace (as we will do in sec. III E) the operators containing two ordinary boson creation operators by a new boson operator, in the case of bosonic theories we have still to keep the original boson, in contrast to the RPA treatment of fermion systems, where it is not necessary to retain the original fermions once one has the formulation in terms of bosonic operators at hand;

This whole discussion allows us now to write down a first form of the gRPA equations:

\[
\begin{pmatrix}
\frac{1}{2} B^*_{ijk;kl} & -\frac{1}{2} A^*_{ij;kl} & D^*_{ij;k} & -C^*_{ij;k} \\
-\frac{1}{2} A_{ij;kl} & B_{ij;kl} & -C_{ij;k} & D_{ij;k} \\
\frac{1}{2} D_{kl;i} & -\frac{1}{2} C_{kl;i} & F^*_{ik} & -E^*_{ik} \\
-\frac{1}{2} C_{kl;i} & \frac{1}{2} D_{kl;i} & -E_{ik} & F_{ik}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} X^\nu_{\{ij\}} \\
\frac{1}{2} Y^\nu_{\{kl\}} \\
\tilde{Z}^\nu_k \\
Z^\nu_i
\end{pmatrix}
= \Omega^\nu
\begin{pmatrix}
1 & -1 & -1 & -1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} X^\nu_{\{ij\}} \\
\frac{1}{2} Y^\nu_{\{ij\}} \\
\tilde{Z}^\nu_i \\
Z^\nu_i
\end{pmatrix}
\tag{29}
\]

with $\Omega^\nu = E^\nu - E_0$. The factors $\frac{1}{2}$ associated with some of the matrices $C, D$ shouldn’t lead one to conclude that the gRPA matrix on the LHS is not hermitian. In fact, as we will see later on, we can write the gRPA equations in a very compact form resulting from a

$^4$ i.e., that point in parameter space where the energy expectation value is stationary with respect to variations of $\vec{\phi}, \vec{\pi}, G, \Sigma$; for $\Lambda$ to be zero it is necessary that $\frac{\delta}{\delta \phi_i} (H) = 0$, and $\frac{\delta}{\delta \pi_i} (H) = 0$.

$^5$ In the bosonic theories under consideration here, these terms with three creation/annihilation operators appear in the Hamiltonian usually due to a condensate.
hermitian Hamiltonian (at the stationary point of the mean-field equations $\frac{\delta}{\delta \phi_i} \langle H \rangle = 0$, and $\frac{\delta}{\delta \pi_i} \langle H \rangle = 0$).

E. The Quasi-Boson Approximation

Sometimes, it is useful to have the second gRPA approximation at hand without having to take expectation values. This can be done with the help of the quasi-boson approximation. The name has its roots in nuclear physics [16, 22], where two fermions are combined into one boson. This works only approximately\(^6\), so that the result was christened a quasi-boson. Interestingly enough, we can do the same in a bosonic system: we replace a two boson operator (like $aa$ or $a^\dagger a^\dagger$) by a new operator that again has bosonic commutation relations.

We construct the new boson-pair operators $B, B^\dagger$ s.t. the commutation relations are identical to the mean-field expectation values of commutators containing still the pair of original boson operators:

\[
[a_m a_i, a_j] = 0 \quad \rightarrow \quad [B_{mi}, a_j] = 0
\]

\[
[a_m a_i, a_j^\dagger] = \delta_{ij} a_m + \delta_{mj} a_i \quad \rightarrow \quad \langle [a_m a_i, a_j^\dagger] \rangle = 0 \quad \rightarrow \quad [B_{mi}, a_j^\dagger] = 0
\]

\[
[a_m^\dagger a_i^\dagger, a_j] = -\delta_{ij} a_m^\dagger - \delta_{mj} a_i^\dagger \quad \rightarrow \quad \langle [a_m^\dagger a_i^\dagger, a_j] \rangle = 0 \quad \rightarrow \quad [B_{mi}^\dagger, a_j] = 0.
\]

The only non-vanishing commutator involving $B, B^\dagger$ originates from $[a_m a_i, a_n^\dagger a_j^\dagger]$:

\[
\left[ \frac{1}{\sqrt{2}} a_m a_i, \frac{1}{\sqrt{2}} a_n^\dagger a_j^\dagger \right] = \frac{1}{2} \left( \delta_{in} \delta_{mj} + \delta_{ij} \delta_{mn} + \delta_{im} a_j^\dagger a_m + \delta_{mn} a_j^\dagger a_i + \delta_{ij} a_n^\dagger a_m + \delta_{mj} a_n^\dagger a_i \right)
\]

\[
\rightarrow [B_{mi}, B_{nj}^\dagger] = \frac{1}{2} \left( \delta_{in} \delta_{mj} + \delta_{ij} \delta_{mn} \right).
\]

We therefore replace

\[
\frac{1}{\sqrt{2}} \left( a_m a_i \right) \leftrightarrow B_{mi}
\]

\[
\frac{1}{\sqrt{2}} \left( a_m^\dagger a_i^\dagger \right) \leftrightarrow B_{mi}^\dagger.
\] (30)

The operators $B, B^\dagger$ have the following commutation relations

\[
[a_j, B_{mi}] = [a_j, B_{mi}^\dagger] = [B_{mi}, B_{nj}] = 0
\] (31)

and

\[
[B_{mi}, B_{nj}^\dagger] = \frac{1}{2} (\delta_{in} \delta_{mj} + \delta_{ij} \delta_{mn}).
\] (32)

\(^6\) i.e. the ordinary bosonic commutation relations are only fulfilled if we take the mean-field expectation value of the commutator of the boson pairs, and thus in some sense employ the second gRPA approximation, by using the mean-field instead of the exact vacuum state.
With these new operators, the excitation operators

\[ Q_{\nu}^\dagger = \frac{1}{2} \left( X_{\nu} a_m^\dagger a_i^\dagger - Y_{\nu} a_m a_i \right) + \left( \tilde{Z}_{\nu} a_m^\dagger - Z_{\nu} a_m \right) \]  \hspace{1cm} (33)

become:

\[ Q_{B\nu}^\dagger = \frac{1}{\sqrt{2}} \left( X_{\nu} B_m^\dagger a_i^\dagger - Y_{\nu} a_m B_m a_i \right) + \left( \tilde{Z}_{\nu} a_m^\dagger - Z_{\nu} a_m \right). \]  \hspace{1cm} (34)

At this point a short comment on how one constructs quasi-boson approximations is in order, cf. also eq. (35) below. Usually we will have to consider only one- and two-body operators, but the procedure should work for higher-body operators as well:

(i) first, one writes down the operator \( O \) under consideration as a polynomial of \( a, a^\dagger \) operators. Then one tries to extract the coefficients of the different powers in analogy to eqs. (25), (26) via (a) taking multiple commutators of \( O \) with up to two \( a, a^\dagger \) operators, and then (b) taking vacuum expectation values of these multiple commutators employing the second gRPA approximation.

(ii) The quasi-boson approximation \( O_B \) of this operator \( O \) is then given as a polynomial of \( B, B^\dagger, a, a^\dagger \) operators. The coefficients of the different powers are extracted by taking multiple commutators of \( O_B \) with \( a, a^\dagger, B, B^\dagger \) without taking expectation values. One then requires that the coefficients of \( O_B \) determined in this way are identical to those of \( O \) determined in (i) if one replaces in the multiple commutators of (i) \( O \) by \( O_B \) and terms of the structure \( aa \) by \( \sqrt{2}B \) and \( a^\dagger a^\dagger \) by \( \sqrt{2}B^\dagger \) as indicated in eq. (30).

This is precisely the procedure that led to eq. (34). Using the procedure outlined, one can in fact express the QBA \( O_B \) of a general one-body operator \( O \) in terms of mean-field vacuum expectation values of commutators:

\[ O_B = \langle O \rangle - \frac{1}{\sqrt{2}} \langle [a_i^\dagger a_j^\dagger, O] \rangle B_{ij} + \frac{1}{\sqrt{2}} \langle [a_i a_j, O] \rangle B_{ij}^\dagger - \langle [a_i^\dagger, O] \rangle a_i + \langle [a_i, O] \rangle a_i^\dagger. \]  \hspace{1cm} (35)

Incidentally, one can at this point easily verify -by writing out the commutators - the claim made above: the QBA of a symmetry generator (which is generally hermitian) of one-body type that annihilates the mean-field vacuum \textit{vanishes}.

Now we will follow the above procedure to obtain the Hamiltonian in quasi-boson approximation. The coefficients mentioned are nothing but the matrices \( A, ..., F \). We therefore
have the requirements

\begin{align}
A_{njmi} &= \langle [a_n^i a_j^m, [H, a_m^i a_j^m]] \rangle \quad \overset{!}{=} \quad [\sqrt{2} B_{nj}^i, [H_B, \sqrt{2} B_{mj}^i]],
B_{njmi} &= \langle [a_n^i a_j^m, [H, a_m^i a_j^m]] \rangle \quad \overset{!}{=} \quad [\sqrt{2} B_{nj}^i, [H_B, \sqrt{2} B_{mj}^i]],
C_{njm} &= \langle [a_n^i a_j^m, [H, a_m^i a_j^m]] \rangle \quad \overset{!}{=} \quad [\sqrt{2} B_{nj}^i, [H_B, a_m^i]],
D_{njm} &= \langle [a_n^i a_j^m, [H, a_m^i a_j^m]] \rangle \quad \overset{!}{=} \quad [\sqrt{2} B_{nj}^i, [H_B, a_m^i]],
E_{nm} &= \langle [a_n^i, [H, a_m^i]] \rangle \quad \overset{!}{=} \quad [a_n^i, [H_B, a_m^i]],
F_{nm} &= \langle [a_n^i, [H, a_m^i]] \rangle \quad \overset{!}{=} \quad [a_n^i, [H_B, a_m^i]],
\end{align}

where \( H_B \) denotes the Hamiltonian in quasi-boson approximation. At this point it seems in order to discuss a question that turned up in connection with eq. (27): There we saw that one of the double commutators was not directly connected to one of the matrices \( A, ..., F \) but that some extra terms appeared, the \('\Lambda\)-terms'. If we consider the same double commutator in the quasi-boson approximation, we obtain

\[ \langle [H, [a_n^i, a_m^i a_j^m]] \rangle \rightarrow [H_B, [\sqrt{2} B_{mj}^i \text{ or } a_n^i \text{ or } a_m^i]] = 0. \]  

(37)

Thus the quasi-boson approximation implies the vanishing of \( \Lambda \). An explicit expression can be given for \( \Lambda \) as well (cf. app. C1)

\[ \Lambda_i = \left( \frac{\delta}{\delta \phi_k} \langle H \rangle \right) U_{ki} + \left( \frac{\delta}{\delta \pi_k} \langle H \rangle \right) \left( \frac{1}{2i} U_{ki}^{-1} + 2 \Sigma_{ki} U_{ki} \right), \]

(38)

which vanishes if we choose the parameters of the reference state s.t. the energy is stationary under their variation (at least w.r.t. \( \phi, \pi \)). We obtain a consistency condition between the second gRPA approximation and the quasi-boson approximation: the latter is equivalent to the former only at the stationary point - at least w.r.t the condensates \( \phi, \pi \) - of \( \langle H \rangle \). Thus, in the following we will always assume that the state (which we often call the mean-field vacuum) relative to which our creation/annihilation operators are defined (cf. eq. (17)) has its parameters chosen s.t. it satisfies the Rayleigh-Ritz principle - that this restriction would become necessary was already indicated when the second gRPA approximation was introduced. With this qualification, we can give the (now hermitian) Hamiltonian in quasi-boson approximation:

\[ H_B = E_{MF} - \frac{1}{4} \left( A_{n_1 j_1 n_2 j_2} B_{n_1 j_1} B_{n_2 j_2} + A_{n_1 j_1 n_2 j_2}^* B_{n_1 j_1}^* B_{n_2 j_2}^* \right) + \frac{1}{2} B_{n_2 j_2 n_1 j_1} B_{n_1 j_1}^* B_{n_2 j_2}^* \]

\[ - \frac{1}{2} \left( C_{n_1 j_1 n_2 a_n^i} a_{n_2} + C_{n_1 j_1 n_2 a_n^i}^* a_{n_2}^* \right) + \frac{1}{2} \left( D_{n_2 j_2 n_1 j_1} a_{n_2} + D_{n_2 j_2 n_1 j_1}^* a_{n_2}^* \right) \]

\[ + \frac{1}{2} \left( E_{n_1 j_1 n_2 a_n^i} a_{n_2} + E_{n_1 j_1 n_2 a_n^i}^* a_{n_2}^* \right) + F_{n_2 j_2 n_1 j_1} a_{n_2}^* a_{n_2}, \]

(39)
with \( E_{MF} = \langle H \rangle \). We have achieved writing the approximated Hamiltonian as a quadratic form, which can always be diagonalized. Using this Hamiltonian \( H_B \) one can see that the gRPA equations eq. (29) can be written in the transparent form\(^7\)

\[
[H_B, Q^\dagger_{B\nu}] = \Omega_{\nu} Q^\dagger_{B\nu}.
\]

(40)

This form of the gRPA equations demonstrates that we have performed the approximation to the dynamics (Hamiltonian) and to the excitation operators consistently, since the form of the equations is identical to the form of the Schrödinger equation eq. (19) with the exact Hamiltonian and the exact excitation operators.

F. Restriction on Excitation Operators, part II

Having introduced all the concepts of gRPA, it is time to look again at some terms that appear neither in \( Q^\dagger_{\mu} \) nor in \( Q^\dagger_{B\mu} \).

1. \( Q^\dagger_\nu \) does not contain an \( a^\dagger a \) term. The superficial reason for this is that the corresponding amplitude cannot be extracted in a way similar to the other amplitudes \( X, Y, Z, \tilde{Z} \), since due to the second gRPA approximation

\[
\langle |[\delta Q, a^\dagger a]| \rangle = \langle |\delta Q a^\dagger a| \rangle \quad = 0 = \langle |a^\dagger a \delta Q| \rangle = 0.
\]

(41)

This point will become clearer in the context of the QBA, cf. item 3 below.

2. \( Q^\dagger_\nu \) does not contain an \( aa^\dagger \) term. This is due to the fact that \( aa^\dagger \) is distinguished from \( a^\dagger a \) only by a constant. However, this constant would lead to a non-vanishing expectation value of \( Q^\dagger_\nu \) between mean-field vacua, in view of the second gRPA approximation a situation certainly not desirable for an excitation operator (all other terms contained in \( Q^\dagger_\nu \) vanish between mean-field vacua !). One could also argue on a formal level that the way the generalized RPA is derived here does not allow to determine any constant parts of \( Q^\dagger_\nu \), so if we cannot determine a constant, we shouldn’t put it into our ansatz in the first place.

\(^7\) One inserts eq. (39) and eq. (34) into eq. (40) and compares the coefficients of the different operators \( a, a^\dagger, B, B^\dagger \). This then reproduces eq. (29).
3. Here we want to have a look at the fact that $a^\dagger a$ does not appear in our excitation operator from the perspective of the QBA. The excitation operator is linear in all the different boson creation/annihilation operators that we construct as one can see in eq. (34). Thus, the question arises whether we can construct a boson operator from $a^\dagger a$ similar to how we construct one from $a^\dagger a^\dagger$. The problem is immediately apparent: if we consider $C_{mi} = a_m^\dagger a_i$, then its adjoint has the same structure as $C_{mi}$, since $C_{mi}^\dagger = a_i^\dagger a_m = C_{im}$; if we compute the commutator $[C_{ij}, C_{kl}^\dagger] = \delta_{jk} a_i^\dagger a_l - \delta_{li} a_k^\dagger a_j$ we see that its mean-field expectation value is zero. Therefore we cannot construct boson operators with the correct commutation relations from $a^\dagger a$ in the same way as we did for $a^\dagger a^\dagger$, and thus they don’t appear in the excitation operators $Q^\dagger_\nu$. In nuclear physics this is well-known. There one can see that the quasi-boson operators correspond to a creation of a particle-hole pair (or its annihilation), whereas the operator that creates and annihilates a particle is translated into an operator that creates and annihilates a quasi-boson.

G. Normal Mode Form

In app. D we consider the commutation relations of the normal mode operators. There we show that if all excitation energies $\Omega_\nu$ are distinct and non-zero, the normal modes have the usual bosonic commutation relations:

$$[Q_{B\mu}, Q_{B\nu}^\dagger] = \delta_{\mu\nu}. \quad (42)$$

Using this fact, we can conclude from eq. (40)

$$H_B = E_{RPA} + \sum_\nu \Omega_\nu Q_{B\nu}^\dagger Q_{B\nu}. \quad (43)$$

The sum over $\nu$ extends over all positive semi-definite $\Omega_\nu$. The constant $E_{RPA}$ can be determined as usual, cf. [16], namely by requiring

$$\langle H_B \rangle = E_{MF}. \quad (44)$$

Using

$$\langle Q_{B\nu}^\dagger Q_{B\nu} \rangle = \frac{1}{2} \sum_{mi} \left| \frac{1}{2} Y_{\nu(mi)} \right|^2 + \sum_i \left| Z_i^\nu \right|^2, \quad (45)$$
with $Y^\nu_{\{mi\}} = Y^\nu_{mi} + Y^\nu_{im}$ this gives

$$E_{RPA} = E_{MF} - \sum_\nu \Omega_\nu \left( \frac{1}{2} \sum_{mi} \left| Y^\nu_{\{mi\}} \right|^2 + \sum_i \left| Z^\nu_i \right|^2 \right). \quad (46)$$

A similar expression can be obtained in nuclear physics [16] but without the appearance of $\left| Z^\nu_i \right|^2$. The changes brought about by excitations with zero excitation energy will be considered in sec. [VI].

### IV. CONSERVATION LAWS

In this section we want to demonstrate that - under certain conditions - conservation laws from the full theory translate to conservation laws in the gRPA treatment. The proof follows the lines of the fermionic case as given in [22]. For the conservation laws to hold in the approximated theory, it is mandatory that the reference state minimizes the expectation value of the Hamiltonian, since then the terms of the Hamiltonian linear in $a, a^\dagger, a^\dagger a^\dagger$ vanish, cf. eq. (16). One should also keep in mind that the terms of $H$ linear in $a, a^\dagger$ have to vanish anyway, since otherwise a treatment using the QBA is not valid as was discussed before.

The fact that conservation laws translate into the approximated theory has to be taken with a grain of salt, however, since it turns out (and will be discussed in sec. [VII]) that the character of symmetries may change. Non-Abelian symmetries are usually reduced to Abelian symmetries which is not surprising since the gRPA is basically a small-fluctuation approximation which does not probe the group manifold.

#### A. General Observation

With these qualifications, let us now compute the commutator of $H_B$, cf. eq. (39), with a general one-body operator $\Gamma_B$

$$\Gamma_B = \Gamma_0 + \left( \Gamma_{10}^{n_1} a^\dagger_{n_1} + \Gamma_{01}^{n_1} a_{n_1} \right) + \sqrt{2} \left( \Gamma_{20}^{n_1 n_2} B^\dagger_{n_1 n_2} + \Gamma_{02}^{n_1 n_2} B_{n_1 n_2} \right), \quad (47)$$

where the coefficients have been chosen, s.t. the full operator before QBA reads

$$\Gamma = \Gamma_0 + \Gamma_{10}^{n_1} a^\dagger_{n_1} + \Gamma_{01}^{n_1} a_{n_1} + \Gamma_{20}^{n_1 n_2} a^\dagger_{n_1} a^\dagger_{n_2} + \Gamma_{02}^{n_1 n_2} a_{n_1} a_{n_2} + \Gamma_{11}^{n_1 n_2} a^\dagger_{n_1} a^\dagger_{n_2}. \quad (48)$$
Then the commutator gives

\[
[H_B, \Gamma_B] = -\frac{1}{\sqrt{2}} B_{n_1 j_1} \left( C_{n_1 j_1 m_1} \Gamma_{m_1}^{10} + D_{n_1 j_1 m_1} \Gamma_{m_1}^{01} + A_{n_1 j_1 m_1} \Gamma_{m_1}^{20} + B_{n_1 j_1 m_1} \Gamma_{m_1}^{02} \right)
\]

\[
+ \frac{1}{\sqrt{2}} B_{n_1 j_1}^{*} \left( C_{n_1 j_1 m_1} \Gamma_{m_1}^{01} + D_{n_1 j_1 m_1} \Gamma_{m_1}^{10} + A_{n_1 j_1 m_1} \Gamma_{m_1}^{02} + B_{n_1 j_1 m_1} \Gamma_{m_1}^{20} \right)
\]

\[
- a_{n_1} \left( E_{n_1 m_1} \Gamma_{m_1}^{10} + F_{n_1 m_1} \Gamma_{m_1}^{01} + \Gamma_{m_1 m_2}^{20} C_{m_1 m_2 n_1} \Gamma_{m_1}^{02} \right)
\]

\[
+ a_{n_1}^{*} \left( E_{n_1 m_1}^{*} \Gamma_{m_1}^{01} + F_{n_1 m_1}^{*} \Gamma_{m_1}^{10} + \Gamma_{m_1 m_2}^{02} C_{m_1 m_2 n_1}^{*} \Gamma_{m_1}^{20} \right).
\] (49)

Let us now concentrate on the first line to make the principle clear; we use the definition of \(A, B, C, D\) as mean-field expectation values of double commutators involving the full Hamiltonian, cf. eq. \(\text{[18]}\). Then it is clear that - using also eq. \(\text{[18]}\) - one can rewrite it as

\[
B_{n_1 j_1} \left( \langle [a_{n_1}^{\dagger}, a_{j_1}^{\dagger}, [H, a_{m_1}]] \rangle \Gamma_{m_1}^{10} + \langle [a_{n_1}^{\dagger}, a_{j_1}^{\dagger}, [H, a_{m_1}]] \rangle \Gamma_{m_1}^{01} \right)
\]

\[
+ \langle [a_{n_1}^{\dagger}, a_{j_1}^{\dagger}, [H, a_{m_1}^{\dagger}, a_{m_2}]] \rangle \Gamma_{m_1 m_2}^{20} + \langle [a_{n_1}^{\dagger}, a_{j_1}^{\dagger}, [H, a_{m_1} a_{m_2}]] \rangle \Gamma_{m_1}^{02} \right)
\]

\[
= B_{n_1 j_1} \left( \langle [a_{n_1}^{\dagger}, a_{j_1}^{\dagger}, [H, \Gamma]] \rangle - \Gamma_{m_1 m_2} \langle [a_{n_1}^{\dagger}, a_{j_1}^{\dagger}, [H, a_{m_1} a_{m_2}]] \rangle \right). \] (50)

The last term \((*)\) is proportional to \(H^{02}\), i.e. that term in the Hamiltonian that multiplies two annihilation operators, and is zero in the case we are considering here, i.e. our reference state minimizes the expectation value of the Hamiltonian. The procedure can be repeated for the other three terms in \([H_B, \Gamma_B]\). Eventually, one finds

\[
[H_B, \Gamma_B] = -\frac{1}{\sqrt{2}} \langle [a_{n_1}^{\dagger}, a_{j_1}^{\dagger}, [H, \Gamma]] \rangle B_{n_1 j_1} + \frac{1}{\sqrt{2}} \langle [a_{n_1} a_{j_1}, [H, \Gamma]] \rangle B_{n_1 j_1}^{*}
\]

\[
- \langle [a_{n_1}, [H, \Gamma]] \rangle a_{n_1} + \langle [a_{n_1}, [H, \Gamma]] \rangle a_{n_1}^{\dagger} + \text{terms that vanish at the stationary point}. \] (51)

Two points should be noted:

1. In eq. \(\text{[35]}\) we have given a formula of the QBA of a general one-body operator; if \([H, \Gamma]\) is a one-body operator, then by comparing eq. \(\text{[51]}\) to eq. \(\text{[35]}\) we conclude that \([H_B, \Gamma_B]\) is the QBA of this operator (apart from a possible mean-field expectation value).

2. The stationarity condition \((H^{20} = H^{10} = 0)\) has been used several times; if we are not at the stationary point, some of the terms neglected above do not vanish, i.e. \([H, \Gamma] = 0\) does not imply \([H_B, \Gamma_B] = 0\) away from the stationary point.
We want to assume that we have chosen the reference state s.t. indeed $H^{20} = H^{10} = 0$. Then eq. (51) simplifies to

$$[H_B, \Gamma_B] = 0.$$

(52)

Comparing this with the gRPA equations as given in eq. (40) we see that - since $\Gamma_B$ is exactly of the form of $Q^1_B\nu$ - that $\Gamma_B$ is a solution of the gRPA equations with excitation energy zero, i.e. it is a zero mode.

### B. Change of Symmetry Character

It seems that by this construction we have precisely obtained what we wanted: the spurious excitation made possible by the deformed mean-field state is a solution by itself, and does not influence the other - physical - solutions inappropriately. However, this result has to be taken with a bit of caution: as has been indicated before, the character of the symmetry may change. This can be seen as follows: Compute the commutator of the QBA $Q_B, P_B$ of two arbitrary one-body operators $Q, P$. Using eq. (35) it is very simple to obtain

$$[Q_B, P_B] = \langle [Q, P] \rangle.$$

(53)

In the case of Yang-Mills theory, we obtain for the commutator of the Gauss law operators

$$[\Gamma^a_B, \Gamma^b_B] = \langle [\Gamma^a, \Gamma^b] \rangle = i f^{abc} \langle \Gamma^c \rangle.$$

(54)

At the stationary point, however, we have for the bosonic theories under consideration $\vec{\pi} = \Sigma = 0$, since $\langle \pi \pi \rangle$ is quadratic in both $\vec{\pi}$ and $\Sigma$, cf. eq. (A7). Thus, in the case of Yang-Mills theory, we obtain

$$[\Gamma^a_B, \Gamma^b_B] = 0,$$

(55)

using the expression given in eq. (C10) for $\langle \Gamma^\alpha \rangle$. In other words, the QBA has reduced the non-Abelian $SU(N)$ symmetry to an Abelian $U(1)^{N^2-1}$ symmetry. This is a phenomenon also well known from perturbation theory.

### C. Possibilities for Improvement

One main drawback of the gRPA is the fact that non-Abelian symmetries may be reduced to Abelian symmetries. There have been a couple of investigations in nuclear physics, e.g.
and other quantum field theories (where the fields were in the fundamental representation) whether one can regard the gRPA as a certain order in a systematic expansion\(^8\), symbolically

\[
\text{complete result} = \text{mean field} + \text{gRPA} + \text{higher orders}
\]

(56)

A first hint on how one may construct such higher orders for one-body operators like the Gauss law operator can be obtained from the following observation: if we calculate (as we have done above) the commutator of two quasi-boson approximated Gauss law operators, we obtain the mean-field value of the full commutator; apparently, in the expansion alluded to above, taking a commutator reduces the order in the expansion by one (for a similar observation in nuclear physics, cf. e.g. \([16, 34]\)). If we construct a next order of say the Gauss law operator, taking the appropriate commutator should give the quasi-boson approximation of the commutator, the latter being again a one-body operator. In formulas, if we write\(^9\)

\[
\Gamma \quad \Rightarrow \quad \langle \Gamma \rangle + \Gamma_B + \Gamma_{2B} + \text{higher orders}
\]

(57)

we have

\[
[\Gamma^a_B, \Gamma^b_B] = \langle [\Gamma^a, \Gamma^b]\rangle
\]

(58)

and \(\Gamma_{2B}\) would be correct if it would reproduce

\[
[\Gamma^a_B, \Gamma^b_{2B}] + [\Gamma^a_{2B}, \Gamma^b_B] = [\Gamma^a, \Gamma^b]_B.
\]

(59)

Such a \(\Gamma_{2B}\) can indeed be constructed. If one requires that \(\Gamma_{2B}\) reproduces the same commutators as \(H_B\) in \(\text{eq. (36)}\) [with \(H\) obviously replaced by \(\Gamma\) on the LHS of \(\text{eq. (36)}\)] one obtains \(\Gamma_{2B}\) as

\[
\Gamma_{2B} = 2\Gamma^{11}_{kl} b^\dagger_{kj} B_{ij} + \Gamma^{20}_{ji j_2} a^\dagger_{j_1} a^\dagger_{j_2} + \Gamma^{02}_{ji j_2} a_{j_1} a_{j_2} + \Gamma^{11}_{kj} a^\dagger_{k} a_j
\]

(60)

where the coefficients \(\Gamma^{ab}_{ij}\) are defined in \(\text{eq. (48)}\).

For the calculation of the commutator \([\Gamma^a_B, \Gamma^b_{2B}]\) we use a technique very similar to the one

---

\(^8\) Within the path integral approach one can show that the mean-field approximation and the RPA correspond to the first and second, respectively, order in the loop expansion \(33\).

\(^9\) One should note that this is a slight deviation from the conventions in the rest of the paper, e.g. in \(\text{eq. (36)}\) the mean-field order \(\langle O \rangle\) is assigned to \(O_B\).
used in sec. [IV A] by rewriting the coefficients of $\Gamma_{2B}^b$ as double commutators (note that the coefficients of $\Gamma_{2B}^b$ are called $^b\Gamma_{jk}^{11}$ etc; note also that - as in the remainder of the paper - we use $\Gamma_{\{ij\}} = \Gamma_{ij} + \Gamma_{ji}$)

\[ ^b\Gamma_{kj}^{11} = \langle [a_j^\dagger, [\Gamma^b, a_k]] \rangle \quad ; \quad ^b\Gamma_{\{j1,j2\}}^{02} = -\langle [a_j^\dagger, [\Gamma^b, a_{j2}^\dagger]] \rangle \quad ; \quad ^b\Gamma_{\{j1,j2\}}^{20} = -\langle [a_j, [\Gamma^b, a_{j2}]] \rangle \quad (61) \]

Using these results, one can rewrite

\[
[\Gamma_B^a, \Gamma_{2B}^b] = a_{j1} \langle [a_{j1}^\dagger, [\Gamma^b, (a^\Gamma_{10} \Gamma_{j2}^b a_{j2}^\dagger + a^\Gamma_{j2}^b a_{j2})]] \rangle - a_{j1} \langle [a_{j1}, [\Gamma^b, (a^\Gamma_{10} \Gamma_{j2}^b a_{j2}^\dagger + a^\Gamma_{j2}^b a_{j2})]] \rangle
+ \frac{1}{\sqrt{2}} B_{j1,j2} \langle [a_{j1}^\dagger, [\Gamma^b, (a^\Gamma_{02} \Gamma_{k1,k2}^a a_k a_{k2})]] \rangle - \frac{1}{\sqrt{2}} B_{j1,j2} \langle [a_{j1}, [\Gamma^b, (a^\Gamma_{02} \Gamma_{k1,k2}^a a_k a_{k2})]] \rangle
= [\Gamma_B^b, \Gamma_{2B}^a] \quad (62a)
\]

\[-a_{j1} \langle [a_{j1}^\dagger, [\Gamma^a, \Gamma^b]] \rangle + a_{j1} \langle [a_{j1}, [\Gamma^a, \Gamma^b]] \rangle \quad (62b)
- \frac{1}{\sqrt{2}} B_{j1,j2} \langle [a_{j1}^\dagger, [\Gamma^a, \Gamma^b]] \rangle + \frac{1}{\sqrt{2}} B_{j1,j2} \langle [a_{j1}, [\Gamma^a, \Gamma^b]] \rangle \quad (62c)\]

Since $[\Gamma^a, \Gamma^b]$ just again gives a one-body operator, we can compare eqs. (62a), (62b) to eq. (33), and see that it is indeed the QBA of $^{10} B_{\{a, b\}}$, i.e. $[\Gamma^a, \Gamma^b]_{2B}$. Thus we have fulfilled our goal: we have constructed an extension of $\Gamma_B^a$, which improves the commutation relations as was desired.

At this point one should keep in mind the following: We have constructed $\Gamma_{2B}$ by requiring that $\langle [a_j^\dagger a_j, [\Gamma, a_k]] \rangle = [\sqrt{2B_{ij}}, [\Gamma_{2B}, a_k]]$. This gave zero as the coefficient of $a^\dagger B$. Had we chosen instead to require $\langle [a_j^\dagger a_j, a_k] \rangle = [a_k, [\Gamma_{2B}, \sqrt{2B_{ij}}]]$ we would have obtained a non-zero coefficient for $a^\dagger B$. This incompatibility of the two construction principles goes back basically to the problem that the commutation relations for $a, B$ are constructed to work with one-body operators (as was noted in the context of $H_B$ where both coefficients are required to be equal. This can be achieved by going to the stationary point since the offending terms all vanish there; here we cannot follow the same route since the state is already specified).

The ambiguity is here in some sense resolved, since the latter construction does not lead to the aspired commutation relations\(^{11}\).

\(^{10}\) Note again that in this section the expectation value $\langle [\Gamma^a, \Gamma^b] \rangle$ is not incorporated into $[\Gamma^a, \Gamma^b]_{2B}$.

\(^{11}\) There can be circumstances where even in the latter construction the correct commutation relations can be produced. We have investigated the case where the symmetry generators are the Gauss law operators, and found two instances where the correct commutation relations were produced even in the presence of the $aB^\dagger, a^\dagger B$-terms. These examples were: $G^{-1} \propto \mathbb{1}$ - this is not a very likely mean-field vacuum since there is no correlation between different points in space - and the case where $DU^{-1} = 0$. For the definition
V. P-Q FORMALISM

In sec. III G we have learned that the usual gRPA procedure encounters problems when the gRPA equations have zero mode solutions; in sec. IV we have learned that in the case we are interested in - namely a mean-field vacuum that breaks certain symmetries - zero modes appear necessarily. Therefore, it seems that we may have a problem here. This problem, however, was already encountered in nuclear physics, and solved in [22]. We present the problem and its solution for the simplest case available, namely the simple harmonic oscillator. We then give an alternative formulation of the gRPA in QBA using - instead of creation/annihilation operators - operators with canonical commutation relations. This formulation will furthermore allow a simple comparison between the gRPA equations derived from the time-dependent variational principle and the operator approach.

A. Resolution of Zero-Mode Problem

In the nuclear physics literature it is well-known [22] that the appearance of zero-modes, despite being welcome in our case, leads to all sorts of problems, especially that the solutions of the gRPA equations no longer form a complete set\textsuperscript{12}. Marshallek \textit{et al}. [22] however pointed out that the problems one has are actually an artifact of the creation/annihilation operator formalism. We want to give a very primitive illustration of their point: One can write down a quantum mechanical oscillator in two equivalent forms (we have set the mass of $U$, cf. eq. (11). But this requirement on $U^{-1}$ implies that $G^{-1}$ has zero modes, and is thus excluded in the gRPA framework, cf. sec. III A. Thus, for practical purposes, at least in Yang-Mills theory, the required commutation relations resolve all ambiguities in the construction of $\Gamma_{2B}$. \textsuperscript{12}

In nuclear physics this is a problem since one there cannot prove that the Hamiltonian can be written as a sum of $Q^\dagger_\nu Q_\nu$. Completeness is lost in the following way: We see - in eq. (D1) - that from every solution of the gRPA equations we can form another solution by considering the adjoint operator, therefore the solutions to the gRPA equations always come in pairs. In the case of zero modes, this is not necessarily true any more, since now the 'excitation operators' that we find may be hermitian; thus, by considering the adjoint, we do not get a new solution. In fact this situation is the usual case for zero modes, as we have seen in sec. IV the excitation operators for the zero modes are just the symmetry generators, which are usually hermitian. In our discussion in app. D we hit upon the problem from a different direction; we see that only if there are no zero modes in the spectrum we can show that the excitation operators can be considered as independent oscillators.
equal to one for simplicity):

\[ H = \frac{p^2}{2} + \frac{\omega^2}{2} x^2 \]  \hspace{1cm} (63)

\[ = \omega (a^\dagger a + \frac{1}{2}), \]  \hspace{1cm} (64)

where \( \omega \) denotes the frequency of the harmonic oscillator. If we now send the frequency to zero, something seemingly strange happens:

\[ \frac{p^2}{2} + \frac{\omega^2}{2} x^2 \xrightarrow{\omega \to 0} \frac{p^2}{2}, \]  \hspace{1cm} (65)

\[ \omega (a^\dagger a + \frac{1}{2}) \xrightarrow{\omega \to 0} 0. \]  \hspace{1cm} (66)

If one now looks at the transformation from canonical to c/a operators, one begins to see more clearly:

\[ a = \sqrt{\frac{\omega}{2}} x + i \frac{1}{\sqrt{2\omega}} p \xrightarrow{\omega \to 0} i \frac{1}{\sqrt{2\omega}} p, \]  \hspace{1cm} (67)

\[ a^\dagger = \sqrt{\frac{\omega}{2}} x - i \frac{1}{\sqrt{2\omega}} p \xrightarrow{\omega \to 0} -i \frac{1}{\sqrt{2\omega}} p \]  \hspace{1cm} (68)

and if we take this small-\( \omega \) behavior into account, there is actually no paradox\(^\text{13}\), since

\[ \frac{\omega}{2} a^\dagger a \xrightarrow{\omega \to 0} \frac{\omega}{2} \left( \frac{1}{\sqrt{\omega}} \right)^2 p^2 = \frac{p^2}{2}. \]  \hspace{1cm} (69)

This is actually the solution of the problems involving zero modes: one puts the system into an external field, s.t. there are no zero modes any more, since the external field\(^\text{14}\) breaks

\(^{13}\) One should note that, in this limit \( \omega \to 0 \), no new information can be obtained by considering the adjoint of \( a^\dagger \).

\(^{14}\) The external field can in fact be engineered just to do that, and this works as follows: We solve the gRPA equations and determine all modes available. The operator belonging to the zero mode will be known beforehand - it is just the quasi-boson approximation of the (known) symmetry generator, which is usually a hermitian operator, and so will be the operator associated with the zero mode. One then constructs a canonically conjugate variable, s.t. \([q, p] = i\) and \(q\) commutes with all other normal mode creation/annihilation operators, and adds a term \(\omega^2 q^2\) to the Hamiltonian; this will lift the zero mode to a finite frequency. By this construction, only the zero mode will be lifted and nothing strange will happen to the other modes. Two points should be mentioned: First, the construction of a canonically conjugate coordinate to a such a generator is only possible in the gRPA context, since we have seen in sec.\(^\text{15}\) that the symmetry we have in mind is usually reduced to an Abelian symmetry; if we would still have the non-Abelian symmetry this would not be possible (the problems with the construction of ‘angle operators’ in nuclear physics are well-known, for a recent discussion cf. e.g. \cite{36}). As a second point, one should mention that in a quantum field theory there might be problems with breaking gauge symmetries in intermediate steps. Since we have indicated here only a formal construction, it is not easy to see where these kinds of problems might surface.
all symmetries. One diagonalizes the Hamiltonian into a set of harmonic oscillators using canonical coordinates, and at the end sends the external field to zero. Then the would-be zero modes indeed become zero modes with the correct operator representation, namely $p^2$, and the non-zero modes can be rewritten using creation and annihilation operators. As a last comment on this topic, one should keep in mind the following: since we already have some prejudice about the zero mode operators (as the symmetry generators), we cannot just choose their mass to be one as we did in the example. We will rather have to allow for a mass tensor, and this will be dealt with in section $\S VI$.

### B. Equivalence of Operator and TDVP Approach to gRPA

We have seen that at least three formulations (using the tdvp, the two gRPA approximations, and the quasi-boson approximation) of the generalized Random Phase Approximation of bosonic systems are available. In this paragraph we want to add one more, since it simplifies the treatment of those Hamiltonians we set out to consider, cf. eq. (6). Since we start from the gRPA formulation using quasi-boson operators, the whole treatment is only valid for the mean-field vacuum representing a solution of the Rayleigh-Ritz variational principle (at least w.r.t. variations of $\bar{\phi}$ and $\bar{\pi}$). In this section we assume that the mean-field vacuum is a solution of the Rayleigh-Ritz variational principle w.r.t. all parameters $\bar{\phi}, \bar{\pi}, \Sigma, G$. We consider linear combinations of $a, a^\dagger$ and $B, B^\dagger$, s.t.

$$q_m = \frac{1}{\sqrt{2}}(a^\dagger_m + a_m) \quad p_m = \frac{i}{\sqrt{2}}(a^\dagger_m - a_m)$$
$$Q_{mi} = \frac{1}{\sqrt{2}}(B^\dagger_{mi} + B_{mi}) \quad P_{mi} = \frac{i}{\sqrt{2}}(B^\dagger_{mi} - B_{mi}).$$

They fulfill the usual canonical commutation relations:

$$[q_m, p_n] = i\delta_{nm}; \quad [Q_{mi}, P_{nj}] = \frac{i}{2}(\delta_{mn}\delta_{ij} + \delta_{mj}\delta_{ni})$$

and all other commutators vanishing. If we now rewrite the Hamiltonian $H_B$ in terms of these new operators, we obtain a complicated expression where the real and imaginary parts of the matrices $A, ..., F$ are separated. We can now put to use that we have computed the matrices $A, ..., F$ in app. C; we have found that

1. $A, C, D$ are always real

2. $C = -D$
3. $B, E, F$ may be imaginary; however, as this imaginary part is due to $\Sigma$, and in the theories under consideration\footnote{This excludes the cranking Hamiltonian $H_c = H - \int d^3 x \omega^a(x) \Gamma^a(x)$ - where $H$ is the Yang-Mills Hamiltonian and $\omega^a(x)$ is a Lagrange multiplier, cf. e.g. \cite{5} - here, since it is not of the form given in eq. (6).} at the stationary point $\Sigma = 0$, the imaginary parts of $B, E, F$ vanish.

4. As a matter of fact, $E$ vanishes altogether at the stationary point.

5. A last simplification is that one may decompose $B = B^{11} + B^{22}$ and, using this notation, we have $A = -B^{22}$.

Using this information, the Hamiltonian simplifies considerably:

$$H_B = E' + Q_{n_1j_1} Q_{n_2j_2} \frac{1}{4} (B_{n_1j_1n_2j_2} - A_{n_1j_1n_2j_2}) + P_{n_1j_1} P_{n_2j_2} \frac{1}{4} (B_{n_1j_1n_2j_2} + A_{n_1j_1n_2j_2})$$
$$+ \sqrt{2} Q_{n_1j_1} q_{n_2} D_{n_1j_1n_2} + \frac{1}{2} F_{n_2n_1} (q_{n_2} q_{n_1} + p_{n_2} p_{n_1})$$

and can even be written in a nice matrix form:

$$H = E' + \frac{1}{2} (P_{n_1j_1} \quad p_{n_1}) \begin{pmatrix} \frac{1}{2} (B_{n_1j_1n_2j_2} + A_{n_1j_1n_2j_2}) & 0 \\ 0 & F_{n_2n_1} \end{pmatrix} \begin{pmatrix} P_{n_2j_2} \\ p_{n_2} \end{pmatrix}$$
$$+ \frac{1}{2} (Q_{n_1j_1} \quad q_{n_1}) \begin{pmatrix} \frac{1}{2} (B_{n_1j_1n_2j_2} - A_{n_1j_1n_2j_2}) & \sqrt{2} D_{n_1j_1n_2} \\ \sqrt{2} D_{n_2j_2n_1} & F_{n_2n_1} \end{pmatrix} \begin{pmatrix} Q_{n_2j_2} \\ q_{n_2} \end{pmatrix},$$

(73)

where $E'$ is a constant that has to be chosen s.t. $\langle H_B \rangle = E_{MF}$. This is a viable starting point for proving the equivalence between the gRPA formulation originating from the time-dependent variational principle and the one using the quasi-boson approximation. We will rewrite eq. (73) into a Hamiltonian that leads to the same eigenvalue equations as does the Hamiltonian of the small-fluctuation approach given in eq. (A33). From this we will conclude - since we have both times the same matrix to diagonalize - that the spectra of the Hamiltonians of the two approaches are identical, and we will see a correspondence in the eigenvectors. For this purpose we note the following:
1. We start with the term quadratic in $p, P$; there we have

$$
\frac{1}{2} (B + A)_{n_1 j_1 m_2 j_2}
= \left( \frac{1}{2} U_{n_1 m_1}^{-1} U_{j_1 1}^{-1} \right) \left( \frac{1}{2} U_{n_2 m_2}^{-1} U_{j_2 2}^{-1} \right) (G \parallel)_{m_2 i_1; m_2 i_2}
$$

(74)

and

$$
F_{n_2 n_1} = \left( \frac{1}{\sqrt{2}} U_{n_2 m_2}^{-1} \right) \left( \frac{1}{\sqrt{2}} U_{n_1 m_1}^{-1} \right) \delta_{m_2 m_1},
$$

(75)

where we have used the abbreviation

$$(G \parallel)_{ij; kl} = G_{ki} \delta_{lj} + G_{kj} \delta_{li} + G_{jl} \delta_{ki} + G_{li} \delta_{kj}.$$  

(76)

This suggests that one should introduce new momentum operators

$$
P_{m_1 i_1} = \frac{1}{2} U_{n_1 m_1}^{-1} U_{j_1 1}^{-1} P_{n_1 j_1},
$$

(77)

$$
P_m = \frac{1}{\sqrt{2}} U_{n_1 m_1} P_{n_1}.
$$

(78)

2. Now we consider the term quadratic in $q, Q$; there the observation

$$
B_{m_1 j_1}^{11} = 8 U_{n_1 k_1} U_{i_1 k_2} U_{n_2 k_3} U_{j_1 k_4} \frac{\delta}{\delta G_{k_1 k_2}} \frac{\delta}{\delta G_{k_3 k_4}} \langle T \rangle
$$

(79)

$$
B_{m_1 j_1}^{22} - A_{m_1 j_1} = 8 U_{n_1 k_1} U_{i_1 k_2} U_{n_2 k_3} U_{j_1 k_4} \frac{\delta}{\delta G_{k_1 k_2}} \frac{\delta}{\delta G_{k_3 k_4}} \langle V \rangle
$$

(80)

(where $T$ denotes the kinetic energy $\frac{1}{2} \pi_i^2$) is useful. Then one can rewrite

$$
(B - A)_{m_1 j_1} = 8 U_{n_1 k_1} U_{i_1 k_2} U_{n_2 k_3} U_{j_1 k_4} \frac{\delta}{\delta G_{k_1 k_2}} \frac{\delta}{\delta G_{k_3 k_4}} \langle H \rangle.
$$

(81)

Next, one can put to use the fact that the mean-field vacuum is a stationary point (s.p.) and thus $\delta \langle H \rangle / \delta G = 0$, the identity eq. (E19) from app. E and the fact that the kinetic energy $\langle \frac{1}{2} \pi_i^2 \rangle$ is independent of $\bar{\phi}$:

$$
F_{n_2 n_1} = \frac{1}{2} G_{n_2 n_1}^{-1}
$$

(82)
This now suggests introducing new coordinates as well:

\[ Q_{m_1n_1} = 2U_{m_1n_1}U_{j_1j_1}Q_{n_1j_1}, \]
\[ q_{m_1} = \sqrt{2U_{m_1n_1}}q_{n_1}. \]  

(83)  
(84)

3. One should note that the newly defined coordinates and momenta still fulfill the canonical commutation relations eq. (71); thus, the new definitions amount to a canonical transformation that leaves the dynamics unchanged.

We therefore end up with a Hamiltonian where the matrices in the quadratic forms are identical to those appearing in the Hamiltonian of the small-fluctuation approach, eq. [A33],

\[ H = E' + \frac{1}{2}(p \; p) \begin{pmatrix} 1 & 0 \\ 0 & (G\|) \end{pmatrix} \begin{pmatrix} p \\ p \end{pmatrix} \]
\[ + \frac{1}{2}(q \; Q) \begin{pmatrix} \frac{\partial^2 H}{\partial q\partial q} & \frac{\partial^2 H}{\partial q\partial G} \\ \frac{\partial^2 H}{\partial Q\partial q} & \frac{\partial^2 H}{\partial Q\partial G} \end{pmatrix} \begin{pmatrix} q \\ Q \end{pmatrix}, \]

(85)

since \( H = \langle H \rangle \) as defined in eqs. [A10], [A17]. As a last point, we want to show that the eigenvalue equations that result from the Hamiltonian given in eq. (85) are identical to eqs. [A31], [A32]. This is in fact quite simple; since the excitation operator \( Q_B^\dagger \) is a linear combination of \( B, B^\dagger, a, a^\dagger \) one can write it just as well as a linear combination of \( q, Q, p, P \) or \( q, Q, p, P \). Thus, we may write

\[ Q_{B_B}^\dagger = \sum_{mi} (\tilde{Q}_{mi}^\nu Q_{mi} + \tilde{P}_{mi}^\nu p_{mi}) + \sum_m (\tilde{q}_m^\nu q_m + \tilde{p}_m^\nu p_m). \]

(86)

We can now use the gRPA equations eq. [40] to derive the eigenvalue equations: we insert eq. (86) and eq. (85) into \( [H_B, Q_{B_B}^\dagger] = \Omega_\nu Q_{B_B}^\dagger \) and compare the coefficients of the various operators \( q, p, Q, P \). The resulting equations are then easily expressed as

\[ \begin{pmatrix} \frac{\partial^2 H}{\partial q\partial q} & \frac{\partial^2 H}{\partial q\partial G} \\ \frac{\partial^2 H}{\partial Q\partial q} & \frac{\partial^2 H}{\partial Q\partial G} \end{pmatrix} \begin{pmatrix} \tilde{q}_s^\nu \\ \frac{1}{2} \tilde{Q}_s^\nu \end{pmatrix} = \Omega_\nu^2 \begin{pmatrix} \tilde{q}_s^\nu \\ \frac{1}{2} \tilde{Q}_s^\nu \end{pmatrix}, \]

(87)

and

\[ \begin{pmatrix} \frac{\partial^2 H}{\partial q\partial q} & \frac{\partial^2 H}{\partial q\partial G} \\ \frac{\partial^2 H}{\partial Q\partial q} & \frac{\partial^2 H}{\partial Q\partial G} \end{pmatrix} \begin{pmatrix} \tilde{p}_s^\nu \\ \frac{1}{2} \tilde{P}_s^\nu \end{pmatrix} = \Omega_\nu^2 \begin{pmatrix} \tilde{p}_s^\nu \\ \frac{1}{2} \tilde{P}_s^\nu \end{pmatrix}, \]

(88)
where \( \tilde{Q}_s^\nu \) and \( \tilde{P}_s^\nu \) denote the symmetric part of \( \tilde{Q}_s^\nu \) and \( \tilde{P}_s^\nu \) respectively, e.g. \( (\tilde{Q}_s^\nu)_{ij} = \tilde{Q}_{ij}^\nu + \tilde{Q}_{ji}^\nu = \tilde{Q}_{(ij)}^\nu \). These equations are now identical to eqs. (A31), (A32), only the components of the vectors have acquired new names:

\[
\delta \tilde{\phi} \to \tilde{p}^\nu; \quad \delta G \to \frac{1}{2} \tilde{P}_s^\nu \quad \text{and} \quad \delta \tilde{\pi} \to \tilde{q}^\nu; \quad \delta \Sigma \to \frac{1}{2} \tilde{Q}_s^\nu.
\]

Thus, we have proven that the two different approaches to the generalized RPA, namely the small-fluctuation approach from the time-dependent variational principle, and the operator approach, give in fact the same spectrum of possible excitations. Now the comments made at the end of app. A carry over to the operator formulation of gRPA: If the stability matrix of the mean-field problem is positive, all ‘eigenvalues’ \( \Omega_\nu^2 \) are larger than zero, and thus all \( \Omega_\nu \)s are real; if we are not at a minimum of the energy with our choice of the mean-field vacuum, the stability matrix will also have negative eigenvalues, and thus we will obtain complex conjugate pairs \( \pm i|\Omega_\nu| \). As long as we have real ‘eigenvalues’ \( \Omega_\nu \) in the gRPA problem, the mean-field vacuum under consideration will be stable at least w.r.t. small fluctuations. One should note that this relation was of some importance in proving the bosonic commutation relations of the excitation operators.

VI. THE MOMENT OF INERTIA

A. General Form of Kinetic Energy

From the discussion in sec. \[\text{V.A}\] we saw that we can in general write the Hamiltonian in quasi-boson approximation as

\[
H_B = E_{\text{RPA}} + \sum_{\nu \in \nu_+} \Omega_\nu \tilde{Q}_\nu^\dagger \tilde{Q}_\nu + \sum_{\nu \in \nu_0} \frac{1}{2} \tilde{P}_\nu^2,
\]

(90)

where \( \{ \nu_+ \} \) denotes the set of modes with positive \( \Omega_\nu \) and \( \{ \nu_0 \} \) denotes the set of zero modes. However, quite often an alternative expression to eq. (90) in terms of symmetry generators is useful. For this we have to recall from sec. \[\text{IV.A}\] that if our reference state is a solution of the Rayleigh-Ritz equations, and if the symmetry generators under consideration are one-body operators, then the quasi-boson approximations \( \Gamma_B^\alpha \) of the symmetry generators commute with the quasi-boson approximation of the Hamiltonian. This implies that they are zero mode solutions of the gRPA equations. If we now assume that there are no ‘accidental
zero modes', i.e. the whole space of zero modes is spanned by the $\Gamma_B^a$s, then we can write
\[ P_{\nu}s, \nu \in \{\nu_0\}, \] as linear combinations of the $\Gamma_B^a$s, and $\sum_{\nu \in \{\nu_0\}} P_\nu^2$ becomes a quadratic form in terms of the $\Gamma_B^a$s. The advantage of this alternative expression is two-fold: First, in general the quasi-boson approximations of the symmetry operators are known. Second, the expression one obtains is (physically) more transparent, and can be more easily compared to other frameworks like e.g. the Thouless-Valatin method \[5, 36\] or the Kamlah expansion \[6\]. The general dependence of $H_B$ on the generators $\Gamma_B^a$ will then be\[16\]
\[ H_{B,zm} = \frac{1}{2} \Gamma_B^a (\mathcal{M}^{-1})^{ab} \Gamma_B^b, \tag{91} \]
where $\mathcal{M}$ is the moment-of-inertia tensor, and Einstein’s summation convention is also used for the indices $a, b$ which are employed to label the different symmetry generators $\Gamma_B^a$. We cannot (as we have done in preceding discussions where masses were set to 1) 'normalize' $\mathcal{M}$ 'away', since the normalization of $\Gamma_B^a$ is fixed by its very nature of a known operator. Furthermore, if the original symmetry generators (before the quasi-boson approximation) are generators of a non-Abelian symmetry, their normalization is fixed by the commutation relations. Thus, in this section, we will see how one can actually compute the moment-of-inertia tensor, cf. \[16\]. We have seen in sec. V A that in cases of appearances of zero modes one has to pass to a description of the oscillators in terms of canonical coordinates\[17\] and momenta (which we have done above). This actually allows to determine the moment-of-inertia tensor. We assume that we can construct a set of coordinates $\Theta_B^a$, s.t.
\[ [\Theta_B^a, \Gamma_B^b] = i \delta^{ab}, \tag{92} \]
and which commute with all other normal modes. Since we already know the form of the part of $H_B$ that is not supposed to commute with $\Theta_B$, we also require\[18\]
\[ [\Theta_B^a, H_B] = i \mathcal{M}^{ab} \Gamma_B^b. \tag{93} \]

\[ H_{B,zm} \] means 'that part of the Hamiltonian in quasi-boson approximation that only contains the zero mode operators'.

\[ \text{Let us once again mention that the reduction of the non-Abelian to an Abelian symmetry (in the Yang-Mills case) in the quasi-boson approximation seems to be essential since otherwise one cannot construct canonically conjugate 'angle' operators fulfilling the canonical commutation relations \[35\].} \]

\[ \text{We assume here that the moment-of-inertia tensor is symmetric, which is at least true for the case of Yang-Mills theory, since there } [\Gamma_B^a, \Gamma_B^b] = 0. \]
We start by making an *ansatz* of $\Theta_B^a$ as a generalized one-body operator in quasi-boson approximation\(^\text{19}\):

$$\Theta_B^a = \Theta_{Q;mi}^a Q_{mi} + \Theta_{P;mi}^a P_{mi} + \Theta_{q;m}^a q_m + \Theta_{p;m}^a p_m \quad (94)$$

whereas in general $\Gamma_B$ reads

$$\Gamma_B^a = \Gamma_{Q;mi}^a Q_{mi} + \Gamma_{P;mi}^a P_{mi} + \Gamma_{q;m}^a q_m + \Gamma_{p;m}^a p_m. \quad (95)$$

In the latter case the coefficients are obviously known. We now insert eqs. (94), (95) into eq. (93), and compare coefficients. This gives in general four equations, and allows to determine $\Theta_{Q;mi}^a, \Theta_{P;mi}^a, \Theta_{q;m}^a, \Theta_{p;m}^a$ in terms of $M^{-1}$ and the coefficients of $\Gamma_B$. If we now insert the thus determined coefficients of $\Theta_B$ into the equation that results from eq. (92) we obtain an equation of the type

$$(M^{-1})^{ac} \mathcal{N}^{cb} = i\delta^{ab}, \quad (96)$$

where $\mathcal{N}$ is a matrix given entirely in terms of the matrices $A, ..., F$ and the coefficients of $\Gamma_B$. Thus, one has to invert the matrix $\mathcal{N}$ in order to obtain the correct kinetic term for the zero modes. One should note that the logic is just the other way around from what one would expect, namely that $\Theta_B$ is defined via its commutator with $\Gamma_B$. However, for a proper definition of $\Theta_B$ we also need the fact that it commutes with all the other normal modes. These general considerations find their application to the specific case of Yang-Mills theory in secs. \textbf{VI C} \textbf{VI D} below.

**B. Energy Contributions of the Zero Modes**

We have discussed in some detail that zero modes cannot be treated via the ordinary creation/annihilation operator formalism; they have to be treated with the help of canonical coordinates and momenta. This changes their contribution to the gRPA vacuum energy. The gRPA energy in eq. (13) was fixed s.t.

$$\langle H_B \rangle = E_{MF}. \quad (97)$$

Since now $H_B$ no longer contains oscillator modes only but rather has the form

$$H_B = E_{RPA} + \frac{1}{2} \Gamma_B^a (M^{-1})^{ab} \Gamma_B^b + \sum_{\nu \in \nu} \Omega^\nu Q^\dagger_{\nu} Q_{\nu}, \quad (98)$$

\(^{19}\) The usage of the p-q formulation as in section \textbf{VB} will be useful in this context.
where \( \{\nu_+\} \) denotes the set of modes with positive \( \Omega_\nu \), eq. (46) becomes in this context

\[
E_{\text{RPA}} = E_{\text{MF}} - \frac{1}{2} \langle \Gamma^a_B (\mathcal{M}^{-1})^{ab}_B \Gamma^b_B \rangle \\
- \sum_{\nu \in \{\nu_+\}} \Omega_\nu \left( \frac{1}{2} \sum_{mi} \left| \frac{1}{2} Y_\nu^{\mu} \right|^2 + \sum_i |Z_\nu^i|^2 \right),
\]

which is again similar to the result obtained in nuclear physics \[16\]. In the special case of Yang-Mills theory to leading order in perturbation theory, we obtain an energy correction (see below) to the mean-field energy due to the zero modes\[20\]

\[
\Delta E = \frac{1}{2} G^{ab}_\Delta (x, y) \langle \Gamma^a(x) \Gamma^b(y) \rangle
\]

that is essentially identical in form to what one obtains in a second order Kamlah expansion \[6\] and also to the Thouless-Valatin correction proposed in \[5, 36\], but also with an important difference: the energy contribution which is due to the zero modes (and thus ultimately due to the deformation) is subtracted off after variation of the mean-field vacuum wave functional, and not before, as in the cases of \[5, 6, 36\]; therefore, the determination of the parameters of the mean-field vacuum is not influenced by the correction.

**C. Special Considerations for Yang-Mills Theory**

We have repeatedly emphasized that the quasi-boson formulation works properly only if the reference state fulfills the Rayleigh-Ritz equations (the parameters are such that ‘the energy functional is at its stationary point’). In Yang-Mills theory without a cranking term\[21\] this leads automatically to

\[
\bar{\pi} = 0 \quad \text{and} \quad \Sigma = 0.
\]

\[20\] In this context, one should note that \( \langle \Gamma^a_B \Gamma^b_B \rangle = \langle \Gamma^a \Gamma^b \rangle \).

\[21\] As mentioned before the cranking Hamiltonian differs from the ordinary Yang-Mills Hamiltonian by the term \( \int d^3 x \omega^a(x) \Gamma^a(x) \), where \( \omega^a(x) \) is a Lagrange multiplier.
This simplifies the expression for the Gauss law operator considerably, since, when we insert these results into the expression given in App. C2, we obtain

\[
\Gamma^a_B = \Gamma^a_{p,mi} P_{mi} + \Gamma^a_{p,m} P_m
\]

\[
= (\Gamma^a_p(x))^{bc}_{ij}(x_1, x_2) P^{bc}_{ij}(x_1, x_2) + (\Gamma^a_p(x))^{b}_{i}(x_1) P^{b}_{i}(x_1)
\]

(102)

with

\[
(\Gamma^a_p(x))^{b_1b_2}_{n_1n_2}(z_1, z_2) = \frac{1}{2} \left( \frac{\delta}{\delta x_{12}}(\hat{D}^{ab}_{x,ix} \Pi^b_{ix}) \right)
\times (U^{-1})^{a_1b_1\cdot l_n_1}(x_1, x_2)
\times (U^{-1})^{a_2b_2\cdot l_n_2}(x_1, x_2)
\]

(103)

\[
(\Gamma^a_p(x))^{b_1}_{n_1}(z_1) = \frac{1}{\sqrt{2}} \left( \frac{\delta}{\delta y_{11}}(\hat{D}^{ab}_{x,ix} \Pi^b_{ix}) \right)
\times (U^{-1})^{a_1b_1\cdot l_n_1}(x_1, x_2)
\]

(104)

where we don’t integrate over \(x\). Further remarks on notation can be found in footnote 22.

One should note that all double indices are summed over except for \(x\) ! Inserting eq. (94) into eq. (93), with the Hamiltonian given in eq. (72), we obtain (apart from \(\Theta_P = \Theta_p = 0\))

\[
\Theta^a_{q,m} F_{mi} = (\mathcal{M}^{-1})^{ab} \Gamma^b_{p,i},
\]

\[
\Theta^a_{Q,m} B^{11}_{minj} = (\mathcal{M}^{-1})^{ab} \Gamma^b_{P,nj},
\]

(105)

where we have used the properties of the stationary point, i.e. the matrices \(A,...,F\) are all real, etc. [cf. sec. V B and eqs. (C8) - (C15)]. We invert the second line in a way that will be discussed below in some detail. Let us only mention at this point that \((B^{11})^{-1}\) is defined by

\[
((B^{11})^{-1})^{i_1m_1j_1n_1}_{i_2m_2} = \frac{1}{2} (\delta_{i_1i_2} \delta_{m_1m_2} + \delta_{i_1m_2} \delta_{i_2m_1}).
\]

(106)

We insert the result into the normalization eq. (92) and obtain

\[
(\mathcal{M}^{-1})^{ac} \left( \Gamma^c_{p,m} (F^{-1})_{mi} \Gamma^b_{p,i} + 2 \Gamma^c_{P,mi} ((B^{11})^{-1})_{minj} \Gamma^b_{P,nj} \right) = \delta^{ab},
\]

(107)

22 In this section we have to give up the super-index notation used until now in some places; instead of one super-index, the operators \(p\) will carry three indices (color, spatial, and position), i.e. \(p_m \rightarrow b_i(x)\) where \(b\) is the color, \(i\) the spatial, and \(x\) the position index, and correspondingly for \(P_{mi}\). In this context, also the index \(a\) carried by \(\Gamma^a_B\) has to be re-examined; in fact it is also a super-index, consisting of a color index \(a\) and a position index \(x\): \(\Gamma^a_B \rightarrow \Gamma^a_B(x)\); the same applies to the super-indices that are carried by the moment-of-inertia tensor: \(M^{ab} \rightarrow M^{ab}(x,y)\). Having clarified this, in the more formal parts of this section, we will still stick to the super-index notation, since otherwise the formulas will become unreadable.
which bears some resemblance to the expression obtained in nuclear physics [16], especially if one notes that $(B^{11})_{\min j} \propto (A + B)_{\min j}$ and $F_{ij} \propto (E + F)_{ij}$ at the stationary point.

### D. Explicit Calculations

Unfortunately, we cannot read off an explicit expression for $(\mathcal{M}^{-1})^{ab}$ from this in general. However, we can do two things: *First*, we can try to evaluate the terms

$$\Gamma^c_{p,m}(F^{-1})_{mi}\Gamma^b_{p,i} \text{ and } 2\Gamma^c_{p,mi}((B^{11})^{-1})_{\min j}\Gamma^b_{p,nj}$$

(108)

further. This will provide us with the moment of inertia - though for the energy correction due to zero modes we will need its inverse and this cannot be given in a general form. *Second*, we can go back to perturbation theory, where we in fact can give the inverse for the leading and next-to leading part in $g^2$ for the moment of inertia.

1. $\Gamma^c_{p,m}(F^{-1})_{mi}\Gamma^b_{p,i}$

   It is a simple exercise to take $\Gamma^b_{p,i}$ and $(F^{-1})_{mi}$ to calculate

   $$(\mathcal{M}^{-1})^{ac}(x,y) \left( (\Gamma^c_p(y))^{b_1}_{n_1}(F^{-1})^{b_2}_{n_1 n_2}(z_1,z_2)(\Gamma^b_p(z))^{b_2}_{n_2}(z_2) \right) = -(\mathbf{D}\mathbf{D})^{bc}(z)(\mathcal{M}^{-1})^{ca}(z,x).$$

   (109)

2. $2\Gamma^c_{p,mi}((B^{11})^{-1})_{\min j}\Gamma^b_{p,nj}$

   The evaluation of this expression needs a bit more thought. The first step is to give a practical expression for $((B^{11})^{-1})_{\min j}$ beyond its implicit definition in eq. (106). For this purpose, it is useful to rewrite $G^{-1}$, cf. eq. (7), as

   $$G^{-1}_{ij} = \sum_A \lambda_A P^A_{ij}$$

   (110)

   where $\lambda_A$ are the eigenvalues of $G^{-1}$ and $P^A_{ij}$ are the projectors onto the corresponding eigenspaces. The projectors are complete in the sense that

   $$\sum_A P^A_{ij} = \delta_{ij},$$

   (111)
Then

\[
B_{i1i2m1m2}^{11} = \frac{1}{2}(G^{-1}_{i1m1} \delta_{i2m2} + G^{-1}_{i1m2} \delta_{i2m1} + G^{-1}_{i2m1} \delta_{i1m2} + G^{-1}_{i2m2} \delta_{i1m1})
\]

(112)

\[
= \frac{1}{2} \sum_{A,B} (\lambda_A + \lambda_B)(P^A_{i1m1} P^B_{i2m2} + P^A_{i1m2} P^B_{i2m1}).
\]

(113)

Using this expression for \(B^{11}\) it is simple to verify that

\[
((B^{11})^{-1})_{i1i2m1m2} = \frac{1}{2} \sum_{A,B} \frac{1}{\lambda_A + \lambda_B}(P^A_{i1m1} P^B_{i2m2} + P^A_{i1m2} P^B_{i2m1}).
\]

(114)

In order to give a compact expression, using super-indices, for \(\Gamma^b_{P,nj}\) we introduce \(\hat{T}^a_x\): with \(n, m\) being super-indices, where \(m\) stands for \((x_1, b, i)\) and \(n\) stands for \((x_2, c, j)\), we define (no integration over \(x\) !)

\[
(\hat{T}^a)_{bc} \delta_{x1} \delta_{x2} \delta_{ij} = (\hat{T}^a_x)_{mn},
\]

(115)

which inherits the property of anti-symmetry from \((\hat{T}^a)_{bc} = f^{bac}\) where \(f^{bac}\) are the \(SU(N)\) structure constants. Then we have

\[
(\Gamma^a_P(x))_{n1n2} = \frac{g}{2}((UT^a_x U^{-1})_{n2n1} + (U\hat{T}^a_x U^{-1})_{n1n2}).
\]

(116)

This expression can now be cast in the same form as \((B^{11})^{-1}\) by using the eigenvalue decomposition:

\[
U^{-1}_{ij} = \sum_A \sqrt{\lambda_A} P^A_{ij} \quad \text{and} \quad U_{ij} = \sum_A \frac{1}{\sqrt{\lambda_A}} P^A_{ij},
\]

(117)

namely one obtains

\[
(\Gamma^a_P(x))_{n1n2} = \frac{g}{2} \sum_{A,B} \frac{\lambda_B - \lambda_A}{\sqrt{\lambda_A \lambda_B}} (P^A \hat{T}^a_x P^B)_{n2n1}.
\]

(118)

Thus, we obtain overall

\[
2(\Gamma^c_P(x))_{i1i2}((B^{11})^{-1})_{i1i2m1m2}(\Gamma^\phi_P(y))_{m1m2} = -\frac{g^2}{2} \sum_{A,B} \left[ \frac{(\lambda_B - \lambda_A)^2}{\lambda_A \lambda_B (\lambda_A + \lambda_B)} \text{Tr}(P^A \hat{T}^c_x P^B \hat{T}^\phi_y) \right].
\]

(119)

Of course, the moment-of-inertia calculations ultimately have the goal to be connected to the static quark potential. It turns out to be interesting to compare the structure presented in eq. (119) to the structures one obtains in a perturbative Coulomb gauge investigation into the static quark potential [37].
In Coulomb gauge, the charge density consists of two parts, namely the external part $\rho_{\text{ext}}^a$ and the gauge part $\rho_{\text{gauge}}^a = f^{abc} A_i^b (\Pi_{\text{tr}}^c)^i$, where $\Pi_{\text{tr}}$ denotes the transversal components of the momentum (the longitudinal component is eliminated using the Gauss law constraint). We then decompose the Coulomb gauge Hamiltonian into a free part, independent of the coupling, and a rest. In Coulomb gauge, the Hamiltonian contains arbitrary powers of $g$. The free part (i.e. that part of the Hamiltonian which is left if we set $g = 0$) defines what in the following is to be called gluons. Since we are in Coulomb gauge these are given in terms of the transversal fields and the corresponding momenta.

In order to compute the energy corrections to the energy of the perturbative ground state by the presence of external charges to $\mathcal{O}(g^4)$ in Rayleigh-Schrödinger perturbation theory, we need on the one hand the $\mathcal{O}(g^4)$ contribution to the interaction Hamiltonian in order to compute

\[ \langle 0 | H_{\text{int}} | 0 \rangle \] (120)

and on the other hand the $\mathcal{O}(g^2)$ contribution to the interaction Hamiltonian in order to compute

\[ \sum_{N \neq 0} \frac{|\langle 0 | H_{\text{int}} | N \rangle|^2}{-E_N^0} \] (121)

where the sum runs over all states except for the (perturbative) vacuum, and $-E_N^0$ is the energy of this state (the energy of the perturbative vacuum has been set to zero). Since we are interested only in that part of the interaction energy that is proportional to $\rho_{\text{ext}}(x_1) \times \rho_{\text{ext}}(x_2)$, we drop everything from $H_{\text{int}}$ which is not quadratic in $\rho_{\text{ext}}$ for the first term, and not linear in $\rho_{\text{ext}}$ for the second. Since the term of the Hamiltonian which contains $\rho$ at all is

\[ \rho^a \left( \frac{1}{D \cdot \nabla} \Delta \frac{1}{D \cdot \nabla} \right)^{ab} \rho^b = \rho^a \mathcal{O}^{ab} \rho^b \] (122)

(which consequently has to be expanded in powers of $g$), for the second term we need $\rho_{\text{ext}}^a (\mathcal{O}^{ab} + \mathcal{O}^{ba}) \rho_{\text{gauge}}^b$. Whereas the terms stemming from the first order contribution give the anti-screening part of the one-loop $\beta$-function, the second order contribution gives the screening part of the one-loop $\beta$-function (which is given by $-\frac{1}{12}$ times the anti-screening part). We will go into a little bit more detail of this latter contribution. What goes really
into the computation of eq. (121) is

\[ \langle 0 | \rho_{gauge}^a | N \rangle = f^{abc} \langle 0 | \mathbf{A}_i^b (\Pi_i^c)^a | N \rangle. \]  

(123)

Using the representation in terms of creation and annihilation operators it becomes clear that the matrix element \( \langle 0 | \rho_{gauge}^a | N \rangle \) is non-zero only if \( | N \rangle \) contains two gluons. If the gluons carry energy \( \omega_{k_1}, \omega_{k_2} \) (and color index \( m_1, m_2 \)) one obtains

\[ \langle N | \rho_{gauge}^a | 0 \rangle \propto f^{am_1m_2} \omega_{k_2} - \omega_{k_1} \frac{\sqrt{\omega_{k_1}\omega_{k_2}}}{\omega_{k_1}\omega_{k_2}}. \]  

(124)

The comparison to eq. (118) is striking, especially if one takes into account that in the case of absence of condensates the eigenvalues of \( G^{-1} \) can be interpreted as single-particle energies, cf. app. A. The second quantity needed is \( 1/E_N^0 \); \( E_N^0 \) is obviously the sum of the energies of the two gluons, i.e. \( E_N^0 = \omega_{k_1} + \omega_{k_2} \), and therefore the similarity of \( 1/E_N^0 \) to \( (B_{11})^{-1} \) is obvious, cf. eq. (119). Therefore, it appears that the “quantum part” of the moment of inertia contains only the (albeit generalized) screening component of the static interaction potential, and the (obviously dominant) anti-screening component has to be found elsewhere. This has to be subject to further investigations.

4. Evaluation to Leading Order in Perturbation Theory

After this excursion, we go back to perturbation theory to make an evaluation of our expressions for the two components needed for the moment of inertia. We note\(^{23}\) that \( \Gamma_p^a \) is, in the perturbative scaling scheme given in app. B\(^{22}\) of higher order in \( g \) than\(^{24}\) \( \Gamma_p^a \). Thus, we just compute the leading order piece of the moment of inertia:

\[ (\mathcal{M}^{-1})^{ac}(x, y) \left( (\Gamma_p^a(y))_{b_1}^{b_1} (z_1) (F^{-1})^{b_1b_2}_{n_1n_2} (z_1, z_2) (\Gamma_p^b(z))_{b_2}^{b_2} (z_2) \right) = -(\hat{D} \hat{D})^{bc}(z) (\mathcal{M}^{-1})^{ca}(z, x) \overset{!}{=} \delta^{ba} \delta_{xx}. \]  

(125)

From this we conclude that the leading order piece of the moment of inertia in a perturbative expansion is just the Green’s function of the covariant Laplacian in the background field \( \bar{A} \):

\[ (\mathcal{M}^{-1})^{ab}(x, y) = -G^{ab}(x, y) + \mathcal{O}(g^2) \]  

(126)

---

\(^{23}\) This can be read off the expressions eqs. (103), (104) together with the mean-field expectation value of \( \Gamma_p^a(x) \) given in eq. (C16).

\(^{24}\) For the purpose of counting powers of \( g \), we take \( \bar{A} \) to be of \( \mathcal{O}(g^{-1}) \), s.t. \( \hat{D} \) is completely of \( \mathcal{O}(g^0) \).
with
\[ \hat{D}_{x_l}^{ad} \hat{D}_{x_l}^{db} G_{\Delta}(x, y) = \delta^{ac} \delta_{xy}. \] (127)

Obviously, in the case of a vanishing background field, we obtain Coulomb’s law:
\[ (M^{-1})^{ab}(x, y) = \frac{\delta^{ab}}{4\pi} \frac{1}{|x - y|}. \] (128)

Let us now consider the perturbative evaluation of eq. (119). If we solve the mean-field equation in a perturbative approximation (cf. e.g. [1, 5, 38]) for vanishing \( \tilde{A} \), we obtain for \( G^{-1} \)
\[ (G^{-1})_{ij}^{ab}(x, y) = 2\delta^{ab} \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (x - y)} |k| (\delta_{ij} - \frac{k_i k_j}{k^2}). \] (129)

This shows immediately that we have a problem of building a gRPA treatment on top of the perturbative ground state, since \( G^{-1} \) has zero modes which is (as we have argued before) unacceptable for the gRPA formalism. Since this section has mainly illustrative purposes, we proceed by simply ignoring the zero modes in what follows. The non-zero eigenvalues of \( G^{-1} \) are given by
\[ \lambda_{2|q|}^2 |q| = 2 |q|, \]
and the corresponding projector onto this eigenspace is given by
\[ (P_{2|k|})_{ij}^{ab}(x, y) = \int \frac{|k|^2 d\Omega_k}{(2\pi)^3} \delta^{ab} (\delta_{ij} - \frac{k_i k_j}{k^2}) e^{ik \cdot (x - y)}, \] (130)
where \( d\Omega_k \) denotes the solid angle in \( k \)-space. Inserting these expressions for \( \lambda_{2|k|} \) and \( (P_{2|k|})_{ij}^{ab} \) into eq. (119), we obtain
\[ 2(\Gamma_F^c(x))_{i_1i_2} ((B^{11})^{-1})_{i_1j_2m_1m_2} (\Gamma_p^b(y))_{m_1m_2} = \frac{Ng^2}{48\pi^2} \ln (\Lambda_{UV}^2/\Lambda_{IR}^2) \int \frac{d^3p}{(2\pi)^3} P^2 e^{ip \cdot (x - y)} \delta^{bc} + \text{finite terms} \] (131)
where finite terms are not determined explicitly. \( \Lambda_{IR} \) is an IR-cutoff needed in the calculation due to the primitive way used to evaluate the loop integral. The main point is here, however, that the “quantum part” of the moment of inertia is just the classical part, multiplied by a divergent constant. Therefore, abbreviating \( \frac{Ng^2}{48\pi^2} \ln (\Lambda_{UV}^2/\Lambda_{IR}^2) \) by \( \alpha \), we can now rewrite eq. (107) as
\[ (-\Delta_z (M^{-1})^{ab}(x, z))(1 + \alpha) = \delta^{ab} \delta_{xz}, \] (132)
where \( \Delta_z \) is the Laplacian w.r.t. the coordinates \( z \). Thus,
\[ (M^{-1})^{ab}(x, z) = \frac{\delta^{ab}}{1 + \alpha} \frac{1}{4\pi} \frac{1}{|x - z|}. \] (133)
If we introduce a renormalized coupling in the standard fashion via

\[ \frac{1}{g_{R}^2(\mu)} = \frac{1}{g^2} + \beta_0 \ln \left( \frac{\Lambda_{U,V}^2}{\mu^2} \right) \]  

(134)

the expression for \( \mathcal{M} \) is made finite only for \( \beta_0 = \alpha \), thereby corroborating our earlier claim that the moment of inertia can only account for the screening contribution of the static quark potential.

It is interesting to note that the moment of inertia obtained here is the same as in a cranking type calculation. This also could have been anticipated, as it is well-known in nuclear physics that the moments of inertia determined from RPA and those from cranking are in fact identical in nuclear physics, cf. [22].

**E. Interpretation of Moment of Inertia as Static Quark Potential**

We have to conclude that we *cannot interpret* the moment of inertia as the static quark potential directly. However, this is not necessarily fatal for a gRPA treatment of Yang-Mills theory. Due to the approximations made to the Gauss law operator (e.g. making it Abelian) it is not necessarily clear that \( |\gamma\rangle = e^{i\gamma^{a}\Theta^{a}}|\text{RPA}\rangle \) is a correctly charged state, and thus \( \langle \gamma| (\mathcal{M}^{-1})^{ab} \Gamma^{a}_{B} \Gamma^{b}_{B} |\gamma\rangle \) is also not necessarily the energy of a state with a certain prescribed charge. This demonstrates, however, that further developments are necessary in two areas: First, one needs a systematic expansion where the quasi-boson approximation is the leading order. Second, a clear interpretation of gRPA states in terms of real physical states appears necessary.

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25 The gRPA vacuum \( |\text{RPA}\rangle \) is annihilated by all gRPA annihilation operators \( Q^{B}_{\nu} \) and by the gRPA approximated symmetry generators, \( \Gamma^{a}_{B}|\text{RPA}\rangle = 0 \). Whereas in the case of non-zero modes the excited states are generated by the creation operators, in the case of zero modes the excited states \( |\gamma\rangle \) are given as plane waves and employ the canonically conjugate coordinate of the symmetry generator, \( |\gamma\rangle = e^{i\gamma^{a}\Theta^{a}}|\text{RPA}\rangle \). These excited states are obviously eigenstates of the symmetry generators: \( \Gamma^{a}|\gamma\rangle = \gamma^{a}|\gamma\rangle \). Their energy can be determined easily, too: \( H^{B}_{B}|\gamma\rangle = \frac{1}{2} (\mathcal{M}^{-1})^{ab} \gamma^{a} \gamma^{b} |\gamma\rangle \).
We want to close this section with a few remarks and observations on electrodynamics. In electrodynamics the (full) Gauss law operator is given by

\[ \Gamma(x) = \nabla_i \Pi_i(x). \]  

We guess immediately the form of the corresponding “angle” coordinate \( \Theta \) as was discussed before in the context of gRPA; if we take

\[ \Theta(x) = -G_\Delta(x,y) \nabla_i^y A_i(y) \text{ with } \Delta^x G_\Delta(x,y) = \delta_{xy} \]  

then on the one hand

\[ [\Theta(x), \Gamma(y)] = i \delta_{xy} \]  

and on the other hand

\[ [\Theta(x), H] = -G_\Delta(x,y) \Gamma(y), \]  

thereby identifying the moment of inertia as

\[ (M^{-1})(x,y) = -G_\Delta(x,y). \]  

It seems interesting that the angle coordinate eq. (136) shown here appears to be closely related to the dressing function used in [39, 40], cf. also [41], to construct gauge invariant electron fields,

\[ \psi_c(x) = e^{i e G_\Delta(x,y) \nabla_i^y A_i(y)} \psi(x). \]

The Hamiltonian of electrodynamics furthermore allows for a decomposition very similar to the one made possible by the gRPA treatment for general bosonic theories, cf. eq. (90). If we introduce the longitudinal and transversal projectors \( P_{ij}^L(x,y), P_{ij}^T(x,y) \) as

\[ P_{ij}^L(x,y) = \nabla_i^x G_\Delta(x,y) \nabla_j^y \text{ and } P_{ij}^T(x,y) = \delta_{ij} \delta_{xy} - P_{ij}^L(x,y), \]

we can decompose the Hamiltonian into one depending solely on transversal degrees of freedom, \( \Pi_i^T(x) = P_{ij}^T(x,y) \Pi_j(y), \text{ } A_i^T(x) = P_{ij}^T(x,y) A_j(y) \), and one depending solely on the Gauss law operator (symmetry generator):

\[ H = \frac{1}{2} \int d^3 x \left( \Pi_i^T(x) \Pi_i^T(x) + B^2[A^T] \right) + \frac{1}{2} \int d^3 x_1 d^3 x_2 G_\Delta(x_2, x_1) \Gamma(x_1) \Gamma(x_2). \]
The gRPA treatment, however, differs from the general treatment discussed above, though not in an unexpected manner. Since electrodynamics is eventually a free theory, the solution of the full Schrödinger equation gives the same result as the mean-field treatment, thus we can use as reference state the well-known ground state of electrodynamics \([42]\):

\[
\psi[A] \propto \exp\left\{-\frac{1}{2} \int d^3x \int d^3y A_i(x) \left[ \int \frac{d^3p}{(2\pi)^3} |p| (\delta_{ij} - \frac{p_i p_j}{p^2}) e^{ip(x-y)} \right] A_j(y) \right\}. \tag{143}
\]

This state, however, is annihilated by the Gauss law operator and consequently we expect \(\Gamma_B\) to vanish, and indeed this is verified by explicit computation. It should be noted in passing, however, that an application of eq. (125) for the moment of inertia gives the correct result (since the zeros in \(\Gamma_p\), which come about since \(\nabla\) acts upon the transversal \(U^{-1}\), are canceled by \(F^{-1} = G\) which is infinite in the longitudinal subspace, \(G^{-1}\) being proportional to the transversal projector).

**VII. SUMMARY AND CONCLUSION**

Let us shortly summarize what has been achieved in this paper, and which problems remain to be solved. We started by introducing shortly the canonical quantization and Schrödinger picture treatment of general bosonic theories with a standard kinetic term. We then considered the operator formulation of the (generalized) Random Phase Approximation that is the one common in nuclear physics, and which has recently also been investigated in the context of pion physics. We demonstrated that this approach can also be implemented for a general class of bosonic field theories, but that the class of excitation operators to be considered has to be larger than in nuclear physics; namely, one has to also allow for terms linear in creation/annihilation operators of the fundamental boson fields. It turned out to be possible (at least for a certain class of Hamiltonians with standard kinetic term) to prove the equivalence of the operator formulation to the formulation starting from the time-dependent variational principle. Then we demonstrated that, in the absence of zero modes, the gRPA Hamiltonian is just a collection of harmonic oscillators. The zero modes required special attention, but the problems could be solved along lines paralleling nuclear physics. We then considered the question of how conservation laws of the full theory translate into conservation laws of the theory in generalized Random Phase Approximation, and saw that, at least in the case of symmetries generated by one-body operators, existence of symmetries in
the full theory implies existence of symmetries in the gRPA-approximated theory, although these symmetries need not be the same; in Yang-Mills theory, the generalized RPA only carries an Abelian symmetry. This also opened up a possible way of improving the gRPA by requiring a certain fulfillment of the commutation relations by the approximated operators. Then we investigated the difference between the mean-field energy and the energy of the gRPA ground state with special emphasis on the energy contribution of the zero modes. For this purpose we had to calculate the moment-of-inertia tensor which (at least to lowest order in perturbation theory) turned out to be the Green’s function of the covariant Laplacian in the background field at the stationary point.

Furthermore, we compared the quantum corrections to the moment of inertia, and found that their structure is very similar to the screening contribution in a Coulomb gauge perturbative calculation of the static quark potential.

As a last point, we made some observations of electrodynamics where the concepts of moment of inertia and zero mode operators can be applied to the system without any form of approximation. In the gRPA treatment of electrodynamics some of the general observations can be verified explicitly; the most interesting point seems to be that - even though the Gauss law operator vanishes in the gRPA treatment, due to the fact that the reference state is gauge invariant, - it is still possible to determine correctly the moment of inertia.

To put the method into perspective, let us summarize the main positive and negative aspects:

On the positive side, we first have to mention that in the generalized RPA only energy differences w.r.t. the ground state are computed. This simplifies matters, since that part of renormalization that is usually done by normal-ordering is automatically taken care of. This brings us directly to the second point: energies of excited states can be computed. This is usually very difficult if one relies upon e.g. the Rayleigh-Ritz principle. The most important point, however, is the effective implementation of the Gauss law constraint. Even though the gauge symmetry is broken in the mean-field treatment, the unphysical excitations generated by the (gRPA approximated) Gauss law operator (more precisely given by plane waves given in the coordinate conjugate to the Gauss law operator) are orthogonal to all the other physical excited states, which is almost as good as if they didn’t even exist.

But there are also a number of drawbacks of the method presented. The first has to do with the ground state energy: Whereas one does obtain a lowering of the ground state energy
w.r.t. the mean-field energy due to deformation, in a manner which is even formally quite similar to the lowering obtained e.g. in the second order Kamlah expansion (at least to the order in perturbation theory considered) and also in other methods, the point of the calculation at which the energy correction due to the deformation is considered is fundamentally different. In the Kamlah expansion and also in the Thouless-Valatin method, the corrections are subtracted before the variation is carried out, whereas in the generalized RPA they are subtracted only after the variation. Therefore the energy correction does not have any influence on the parameters of the mean-field. This brings us directly to the next problematic point: of course the mean-field vacuum is not the gRPA ground state. In nuclear physics, one was able, however, to construct the gRPA ground state from the mean-field ground state; but, in the presence of zero modes, this state has a divergent norm. This is due to the fact that the gRPA is a 'small angle' approximation [22], or in other words, the compact nature of the non-Abelian symmetry is lost. We have seen explicitly that in the case of Yang-Mills theory, the compact $SU(N)$ symmetry is replaced by the non-compact $U(1)^{N^2-1}$ symmetry. In nuclear physics, at least in the case of two-dimensional rotations, the fact that RPA is only a small-angle approximation was not so much of a problem, since the global dependence on the angle is generally known and therefore one can extract enough information from the small-angle approximation to determine the whole wave function [22]. In Yang-Mills theory, we have no comparable knowledge that could be put to use practically and therefore, we cannot compute the ground-state wave functional. Lacking this knowledge, however, one needs other methods to evaluate matrix elements of operators, which up to now we haven’t developed. Therefore, the only quantities we currently can calculate are the energies of the excited states. As a last though very important point, we have to mention that we have not dealt with the problems of renormalization. However, it should be possible to deal with them, since (at least in the case of $\phi^4$ theory) this problem has been faced already by Kerman et al. [14, 15] in the context of the generalized RPA derived from the time-dependent variational principle.

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**APPENDIX A: GENERALIZED RPA FROM THE TIME-DEPENDENT VARIATIONAL PRINCIPLE**

In this appendix we present the approach to the generalized Random Phase Approximation via the time-dependent variational principle as pioneered by Kerman *et al.* \cite{13, 14, 15}. We show that it leads to equations of motion of coupled harmonic oscillators. The purpose of this appendix is to make this paper reasonably self-contained, so that the reader can follow our claim that (under the circumstances outlined in the main text) the operator approach and the time-dependent approach to the generalized RPA yield identical equations of motion.

We don’t want to deal with a specific theory at the moment, we will only require the Hamiltonian to be of the form

$$H = \frac{1}{2} \pi_i^2 + V[\phi_i],$$  \hspace{1cm} (A1)

where $\phi_i$ are a set of fields, and $\pi_i$ are the canonical momenta conjugate to $\phi_i$, in the field (coordinate) representation under consideration

$$\pi_i = \frac{1}{i} \delta \phi_i$$  \hspace{1cm} (A2)

and $i$ is a super-index, containing a position variable $x$, and all other indices required (like color, spatial etc). The Einstein summation convention is adopted, implying sums over all discrete and integrals over all continuous variables. $V[\phi]$ is a functional of the field operators, in the following referred to as 'potential'.

The states that we consider as trial states for the time-dependent variational principle are the most general time-dependent Gaussian states (we only indicate the time-variable explicitly, $i, j$ are super-indices and $\mathcal{N}$ is a normalization constant):

$$\psi[\phi, t] = \mathcal{N} \exp \left( - (\phi - \bar{\phi}(t))_i \left( \frac{1}{4} G^{-1}(t) - i \Sigma(t) \right)_{ij} (\phi - \bar{\phi}(t))_j + i \pi_i(t)(\phi - \bar{\phi}(t))_i \right).$$  \hspace{1cm} (A3)
The meaning of the parameters becomes clear by considering expectation values:

\begin{align}
\langle \psi(t)|\phi_i|\psi(t) \rangle & = \bar{\phi}_i(t) \tag{A4} \\
\langle \psi(t)|\phi_i\phi_j|\psi(t) \rangle & = \bar{\phi}_i(t)\bar{\phi}_j(t) + G_{ij}(t) \tag{A5} \\
\langle \psi(t)|\pi_i|\psi(t) \rangle & = \bar{\pi}_i(t) \tag{A6} \\
\langle \psi(t)|\pi_i\pi_j|\psi(t) \rangle & = \bar{\pi}_i\bar{\pi}_j + \frac{1}{4}G^{-1}_{ij}(t) + 4(\Sigma(t)G(t)\Sigma(t))_{ij}. \tag{A7}
\end{align}

We can now compute the action of the time-dependent variational principle \cite{13}

\[ S = \int dt \langle \psi(t)|i\partial_t - H|\psi(t) \rangle \tag{A8} \]

and obtain

\[ S = \int dt \left\{ \bar{\pi}_i(t)\dot{\phi}_i(t) - \text{tr}(\dot{\Sigma}G) + \frac{i}{4}\text{tr}(\dot{G}^{-1}G) \right\} - H(t)[\bar{\phi}, \bar{\pi}, G, \Sigma] \tag{A9} \]

with

\[ H(t) = \langle \psi(t)|H|\psi(t) \rangle. \tag{A10} \]

We can now add a total time derivative\textsuperscript{26} that does not change the equations of motion, and obtain for the action

\[ S = \int dt (\Sigma_{ij}(t)\dot{G}_{ij}(t) + \bar{\pi}_i(t)\dot{\phi}_i(t) - H(t)), \tag{A11} \]

which shows that \( \Sigma \) is to be considered as the canonical momentum conjugate to \( G \), and \( \bar{\pi} \) that of \( \bar{\phi} \). The parameters of the wave functional are now determined via Hamilton’s classical equations of motion:

\begin{align}
\dot{\phi}_i(t) & = \frac{\delta H}{\delta \bar{\pi}_i} \ ; \ \dot{\bar{\pi}}_i(t) = -\frac{\delta H}{\delta \phi_i} \tag{A12} \\
\dot{G}_{ij}(t) & = \frac{\delta H}{\delta \Sigma_{ij}} \ ; \ \dot{\Sigma}_{ij}(t) = -\frac{\delta H}{\delta G_{ij}}. \tag{A13}
\end{align}

However, in general it will be much too complicated to solve these equations, therefore we now consider a two-step procedure

1. look for static solutions to the equations of motion

\textsuperscript{26} Its form can e.g. be found in \cite{14} as (here only symbolically) \( \frac{d}{dt}(-\frac{i}{4}\log(G^{-1}) - \Sigma G) \).

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2. Consider small fluctuations around these static solutions.

The static solutions are (obviously) determined via

\[ \dot{\bar{\phi}}_i(t) = 0 ; \quad \dot{\bar{\pi}}_i(t) = 0 ; \quad \dot{G}_{ij}(t) = 0 ; \quad \dot{\Sigma}_{ij}(t) = 0. \]  
(A14)

But these are nothing but the equations resulting from the Rayleigh-Ritz principle:

\[ \frac{\delta H}{\delta \bar{\pi}_i} = 0 ; \quad \frac{\delta H}{\delta \bar{\phi}_i} = 0 ; \quad \frac{\delta H}{\delta \Sigma_{ij}} = 0 ; \quad \frac{\delta H}{\delta G_{ij}} = 0. \]  
(A15)

Thus, for a static solution of the time-dependent variational principle, the parameters are those which minimize (or at least extremize) the energy. This should not really come as a surprise, since for a static state \( \psi[\bar{\phi}] \), the action reduces just to minus the energy times the respective time interval under consideration.

For the next step, we decompose the general, time-dependent parameters into the static solution plus a time-dependent contribution that later on is considered to be small, e.g. for \( \bar{\phi} \):

\[ \bar{\phi}_i(t) = \bar{\phi}_{i,s} + \delta \bar{\phi}_i(t), \]  
(A16)

where \( \bar{\phi}_{i,s} \) denotes the static solution\(^{27} \), and \( \delta \bar{\phi}_i(t) \) the 'small' time-dependent part. We insert this decomposition into the equations of motion eqs. (A12), (A13), and obtain

\[ \dot{\delta \bar{\phi}}_i(t) = \frac{\delta^2 H}{\delta \bar{\pi}_i^2} \bar{\phi}_{s} + \delta \bar{\phi}(t), \ldots \]

\[ = \frac{\delta H}{\delta \bar{\pi}_i}[\bar{\phi}_s, \bar{\pi}_s, G_s, \Sigma_s] \]

\[ + \frac{\delta^2 H}{\delta \bar{\pi}_j \delta \bar{\pi}_i}[\bar{\phi}_s, \bar{\pi}_s, G_s, \Sigma_s] \delta \bar{\pi}_j + \ldots + \mathcal{O}(\delta^3). \]  
(A17)

Now the meaning of \( \delta \bar{\phi} \), etc. being small is clarified: in the equations of motion terms of higher than linear order are neglected (in the action, it would be terms of higher than quadratic order). The first contribution \( \frac{\delta H}{\delta \bar{\pi}_i}[\bar{\phi}_s, \bar{\pi}_s, G_s, \Sigma_s] \) vanishes by virtue of the static equations of motion eqs. (A12), (A13). The same construction can be carried out for all four parameter types \( (\bar{\phi}, \bar{\pi}, \Sigma, G) \), and the resulting equations of motion can be nicely summarized

\(^{27} \) In other words, if we evaluate the first derivative \( \frac{\delta H}{\delta \bar{\pi}_i} \) for \( \bar{\pi} = \bar{\pi}_s \), it is zero, and correspondingly for \( \bar{\phi}, G, \Sigma. \)
as follows:

\[
\begin{pmatrix}
\delta \dot{\phi} \\
\delta \dot{\pi} \\
\delta \dot{G} \\
\delta \dot{\Sigma}
\end{pmatrix}
= \Omega
\begin{pmatrix}
\frac{\partial^2 H}{\partial \pi \delta \phi} & \frac{\partial^2 H}{\partial \delta \phi \delta \pi} & \frac{\partial^2 H}{\partial \delta \phi \delta G} & \frac{\partial^2 H}{\partial \delta \phi \delta \Sigma} \\
\frac{\partial^2 H}{\partial \delta \phi \delta \pi} & \frac{\partial^2 H}{\partial \delta \pi \delta \phi} & \frac{\partial^2 H}{\partial \delta \pi \delta G} & \frac{\partial^2 H}{\partial \delta \pi \delta \Sigma} \\
\frac{\partial^2 H}{\partial \delta \phi \delta G} & \frac{\partial^2 H}{\partial \delta G \delta \phi} & \frac{\partial^2 H}{\partial \delta G \delta \pi} & \frac{\partial^2 H}{\partial \delta G \delta \Sigma} \\
\frac{\partial^2 H}{\partial \delta \phi \delta \Sigma} & \frac{\partial^2 H}{\partial \delta \Sigma \delta \phi} & \frac{\partial^2 H}{\partial \delta \Sigma \delta \pi} & \frac{\partial^2 H}{\partial \delta \Sigma \delta \Sigma}
\end{pmatrix}
\begin{pmatrix}
\delta \phi \\
\delta \pi \\
\delta G \\
\delta \Sigma
\end{pmatrix},
\]  

(A18)

where \( \frac{\delta^2 H}{\delta A \delta B} = \frac{\delta^2 H}{\delta A \delta B}[\phi, \pi, G, \Sigma] \) and can be made even more transparent if one introduces an auxiliary matrix \( \Omega \):

\[
\Omega = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]  

(A19)

Then one obtains

\[
\begin{pmatrix}
\delta \dot{\phi} \\
\delta \dot{\pi} \\
\delta \dot{G} \\
\delta \dot{\Sigma}
\end{pmatrix}
= \Omega
\begin{pmatrix}
\frac{\partial^2 H}{\partial \pi \delta \phi} & \frac{\partial^2 H}{\partial \delta \phi \delta \pi} & \frac{\partial^2 H}{\partial \delta \phi \delta G} & \frac{\partial^2 H}{\partial \delta \phi \delta \Sigma} \\
\frac{\partial^2 H}{\partial \delta \phi \delta \pi} & \frac{\partial^2 H}{\partial \delta \pi \delta \phi} & \frac{\partial^2 H}{\partial \delta \pi \delta G} & \frac{\partial^2 H}{\partial \delta \pi \delta \Sigma} \\
\frac{\partial^2 H}{\partial \delta \phi \delta G} & \frac{\partial^2 H}{\partial \delta G \delta \phi} & \frac{\partial^2 H}{\partial \delta G \delta \pi} & \frac{\partial^2 H}{\partial \delta G \delta \Sigma} \\
\frac{\partial^2 H}{\partial \delta \phi \delta \Sigma} & \frac{\partial^2 H}{\partial \delta \Sigma \delta \phi} & \frac{\partial^2 H}{\partial \delta \Sigma \delta \pi} & \frac{\partial^2 H}{\partial \delta \Sigma \delta \Sigma}
\end{pmatrix}
\begin{pmatrix}
\delta \phi \\
\delta \pi \\
\delta G \\
\delta \Sigma
\end{pmatrix}.
\]  

(A20)

At this point it becomes clear what determines the spectrum of small fluctuations around a static mean-field solution: it’s the stability matrix of this static mean-field solution. Eqs. (A20) are usually called the generalized RPA equations. Up to now, the only assumption that has been used was the assumption of \( \psi \) being a Gaussian state. But we have also restricted the choice of Hamiltonians that we want to consider by eq. (A1). This restriction will allow to carry the calculation a bit further. \( H \) depends only via \( \langle \pi^2 \rangle \) on both \( \pi \) and \( \Sigma \), \( V[\phi] \) depends only on \( \phi \) and \( G \). Thus for all Hamiltonians that we are considering, we have the kinetic energy written as

\[
\langle \psi | \frac{1}{2} \pi_i^2 | \psi \rangle = \frac{1}{2} \bar{\pi}_i \bar{\pi}_i + \frac{1}{8} \text{Tr}(G^{-1}) + 2 \text{Tr}(\Sigma G \Sigma),
\]  

(A21)

where \( \text{Tr} \) denotes the trace over the super-indices. The static solutions \( \bar{\pi}_{i,s}, \Sigma_{ij,s} \) are determined via

\[
\frac{\delta H}{\delta \bar{\pi}_i} = 0, \quad \frac{\delta H}{\delta \Sigma_{ij}} = 0
\]  

(A22)

and result in

\[
\bar{\pi}_{i,s} = 0, \quad \Sigma_{ij,s} = 0.
\]  

(A23)
This information, together with the knowledge that \( V[\phi] \) does neither depend on \( \bar{\pi} \) nor on \( \Sigma \), determines a number of second derivatives:

\[
\begin{align*}
\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \phi} &= 0, \\
\frac{\delta^2 \mathcal{H}}{\delta \Sigma \delta \phi} &= 0, \\
\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \Sigma} &= 0, \\
\frac{\delta^2 \mathcal{H}}{\delta \Sigma \delta \Sigma} &= \delta_{ij}, \\
\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \pi} &= 0, \\
\frac{\delta^2 \mathcal{H}}{\delta \pi \delta \phi} &= 0, \\
\frac{\delta^2 \mathcal{H}}{\delta \pi \delta \Sigma} &= 0, \\
\frac{\delta^2 \mathcal{H}}{\delta \Sigma \delta \pi} &= (G\|),
\end{align*}
\]

where we have used the abbreviation \(^{28}\)

\[
(G\|)_{ij;kl} = 2 \frac{\delta}{\delta \Sigma_{ij}}\frac{\delta}{\delta \Sigma_{kl}} \text{tr}(\Sigma G \Sigma) = G_{ki}\delta_{ij} + G_{kj}\delta_{li} + G_{jl}\delta_{ki} + G_{il}\delta_{kj}.
\]  

These results can be used to simplify eq. (A18):

\[
\begin{pmatrix}
\delta \dot{\phi} \\
\delta \dot{\pi} \\
\delta \dot{G} \\
\delta \dot{\Sigma}
\end{pmatrix} = 
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \phi} & 0 & -\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \Sigma} & 0 \\
0 & 0 & 0 & (G\|) \\
-\frac{\delta^2 \mathcal{H}}{\delta \Sigma \delta \phi} & 0 & -\frac{\delta^2 \mathcal{H}}{\delta \Sigma \delta \Sigma} & 0
\end{pmatrix}
\begin{pmatrix}
\delta \dot{\phi} \\
\delta \dot{\pi} \\
\delta \dot{G} \\
\delta \dot{\Sigma}
\end{pmatrix},
\]

with \((\|)_{ij} = \delta_{ij}\). By taking the derivative of eq. [A26] with respect to time, and reinserting eq. [A26], one obtains two sets of partially decoupled equations

\[
\begin{align*}
\text{(I)} & \quad \begin{pmatrix}
\delta \ddot{\phi} \\
\delta \ddot{G}
\end{pmatrix} = - \begin{pmatrix}
\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \phi} & \frac{\delta^2 \mathcal{H}}{\delta \phi \delta \Sigma} \\
\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \phi} & (G\|)
\end{pmatrix}
\begin{pmatrix}
\delta \ddot{\phi} \\
\delta \ddot{G}
\end{pmatrix}, \\
\text{(II)} & \quad \begin{pmatrix}
\delta \ddot{\pi} \\
\delta \ddot{\Sigma}
\end{pmatrix} = - \begin{pmatrix}
\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \phi} & \frac{\delta^2 \mathcal{H}}{\delta \phi \delta \Sigma} \\
\frac{\delta^2 \mathcal{H}}{\delta \phi \delta \phi} & (G\|)
\end{pmatrix}
\begin{pmatrix}
\delta \ddot{\pi} \\
\delta \ddot{\Sigma}
\end{pmatrix}.
\end{align*}
\]

If we now make the \textit{ansatz} of a harmonic time-dependence

\[
\begin{align*}
\text{(I)} & \quad \begin{pmatrix}
\delta \phi_i(t) \\
\delta G_{ij}(t)
\end{pmatrix} = \begin{pmatrix}
\delta \phi_i^{(0)} \\
\delta G_{ij}^{(0)}
\end{pmatrix} \cos(\omega t + \delta_1), \\
\text{(II)} & \quad \begin{pmatrix}
\delta \pi_i(t) \\
\delta \Sigma_{ij}(t)
\end{pmatrix} = \begin{pmatrix}
\delta \pi_i^{(0)} \\
\delta \Sigma_{ij}^{(0)}
\end{pmatrix} \cos(\omega t + \delta_2)
\end{align*}
\]

\(^{28}\) In this context, one should note that \(\frac{\delta \Sigma_{ij}}{\delta \Sigma_{kl}} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\).
we obtain the eigenvalue equations

\[
\begin{align*}
(\text{I}) \quad \omega^2 \begin{pmatrix} \delta \bar{\phi} \\ \delta G \end{pmatrix} &= \begin{pmatrix} \frac{\delta^2 H}{\delta \phi \delta \phi} & \frac{\delta H}{\delta G \delta \phi} \\ \frac{\delta H}{\delta \phi \delta G} & \frac{\delta^2 H}{\delta G \delta G} \end{pmatrix} \begin{pmatrix} \delta \bar{\phi} \\ \delta G \end{pmatrix}, \\
(\text{II}) \quad \omega^2 \begin{pmatrix} \delta \bar{\pi} \\ \delta \Sigma \end{pmatrix} &= \begin{pmatrix} \frac{\delta^2 H}{\delta \phi \delta \phi} & \frac{\delta^2 H}{\delta G \delta \phi} \\ \frac{\delta^2 H}{\delta \phi \delta G} & \frac{\delta^2 H}{\delta G \delta G} \end{pmatrix} \begin{pmatrix} \delta \bar{\phi} \\ \delta G \end{pmatrix}.
\end{align*}
\]

(A31) (A32)

In analyzing properties of the generalized RPA equations eqs. (A31), (A32) it is often simpler to study the Hamiltonian they are derived from than to study the equations themselves. In this context one can imagine eq. (A26) to originate from a Hamiltonian:

\[
H = \frac{1}{2} \begin{pmatrix} \delta \bar{\pi} & \delta \Sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (G\|) \end{pmatrix} \begin{pmatrix} \delta \bar{\pi} \\ \delta \Sigma \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \delta \bar{\phi} \\ \delta G \end{pmatrix} \begin{pmatrix} \frac{\delta^2 H}{\delta \phi \delta \phi} & \frac{\delta H}{\delta G \delta \phi} \\ \frac{\delta H}{\delta \phi \delta G} & \frac{\delta^2 H}{\delta G \delta G} \end{pmatrix} \begin{pmatrix} \delta \bar{\phi} \\ \delta G \end{pmatrix}.
\]

(A33)

This evidently is the Hamiltonian of a set of coupled oscillators. The important point is that the signs of the eigenvalues of the decoupled oscillators are determined by the reduced stability matrix\(^{30}\) containing only second derivatives w.r.t. to \(\bar{\phi}\) and \(G\) (We assume here that \((G\|)\) is positive definite and since \((G\|)\) is only multiplied by objects symmetric in their two indices, e.g. \(\delta \Sigma_{ij} = \delta \Sigma_{ji}\), this boils down to assuming that \(G\) is positive definite. This is a sensible assumption connected to the normalizability of \(\psi[\phi]\) and is discussed extensively in sec. IIIA). An interesting observation can be made immediately: usually \(\frac{\delta^2 H}{\delta \phi \delta \phi}\) will be only non-zero if there is a condensate (i.e. \(\bar{\phi} \neq 0\)) in the system. Thus if we don’t have a condensate, the equations for the one- and two-particle content\(^{31}\) decouple, and, since in a system without condensate \(G\) will usually be translation invariant, we can see by considering the Fourier transformed quantities that \(1/\tilde{g}(p)\) just describes the energy spectrum of single-particle excitations with momentum \(p\), where \(\tilde{g}(p)\) is defined by the following procedure.

\(^{29}\) They originate by the canonical equations of motion, e.g. \(\delta \dot{\bar{\pi}} = -\delta H/\delta (\bar{\phi})\).

\(^{30}\) The problem of coupled oscillators is well-known, cf. [43, 44]. If the reduced stability matrix is a positive matrix (and thus our mean field vacuum is indeed a minimum), all the eigenvalues will be positive, and thus all oscillator frequencies will be real.

\(^{31}\) We see in sec. VIB that in a creation/annihilation operator formalism \(\delta \bar{\phi}, \delta \bar{\pi}\) are connected to the amplitudes of operators containing one creation/annihilation operator, whereas \(\delta G, \delta \Sigma\) are connected to the amplitudes containing two creation/annihilation operators.
We introduce the Fourier transform $\tilde{G}(p, q)$ by

$$G(x, y) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} e^{ip \cdot x} \tilde{G}(p, q) e^{-iq \cdot y}. \quad (A34)$$

If we now require that $G$ shall be translation invariant, i.e. $G(x, y) = G(x - y)$, we obtain for $\tilde{G}(p, q)$

$$\tilde{G}(p, q) = (2\pi)^3 \delta(p - q) \tilde{g}(p). \quad (A35)$$

This defines $\tilde{g}(p)$.

**APPENDIX B: HAMILTONIAN TREATMENT OF YANG-MILLS THEORY**

1. Yang-Mills Theory

There is not much to be said about the Hamiltonian treatment of Yang-Mills theory in particular, since there is an excellent treatment in [45]. The main points to be kept in mind are: First, in order to work in the Hamiltonian formalism at all one chooses the Weyl gauge, i.e. $A_0 = 0$. The price to be paid in this gauge is that the classical constraint equation

$$\hat{D}^{ab} E^b_i = 0 \quad (E \text{ is the color-electric field, for the definition of } \hat{D} \text{ cf. eq. (B4) below}),$$

i.e. the Gauss law equation, has to be implemented as a constraint on states: $|\psi\rangle$ is a physical state if it is annihilated by the Gauss law operator (which is constructed from $\hat{D}^{ab} E^b_i$ upon quantization). Even in the presence of this constraint the canonical pair of $A, \Pi$ can be quantized straightforwardly without the appearance of non-trivial metric tensors [46].

To put our treatment of Yang-Mills theory in a nutshell: We consider it to be described as a canonical system, defined in terms of coordinates $A_a^i(x)$ and conjugate momenta $\Pi^i_a(x)$ which satisfy ordinary commutation relations:

$$[A^i_a(x), \Pi^b_j(y)] = i \delta^{ab} \delta_{ij} \delta_{xy}. \quad (B1)$$

The states in the physical subspace have to satisfy Gauss’ law, i.e.

$$\Gamma^a(x)|\psi\rangle = 0, \quad (B2)$$

where we have introduced both the Gauss law operator

$$\Gamma^a(x) = \hat{D}^{ab}_i(x) \Pi^b_i(x) \quad (B3)$$
and the covariant derivative in the adjoint representation

\[ \hat{D}^{ab}_i(x) = \delta^{ab} \nabla_i - (g) f^{abc} A^c_i(x). \]  

(B4)

We have put the coupling constant in brackets since its appearance depends on whether we have chosen “perturbative” or “non-perturbative” scaling, cf. app. B2. The SU(N) structure constants are denoted as \( f^{abc} \). One should note that the Gauss law operator used here may differ from those in other publications by some proportionality factors, as it is not directly the generator of (small) gauge transformations. The reason for choosing this form of the Gauss law operator is that the only factors of \( g \) appearing during a change from perturbative to non-perturbative scaling appear inside the covariant derivative.

The wave functional of the reference state taken for Yang-Mills theory is of Gaussian form

\[
\psi[A] = \mathcal{N} \exp \left\{ - \left( A^a_i(x) - \bar{A}^a_i(x) \right) \left[ \frac{1}{4} (G^{-1})^{ab}_{ij}(x,y) - i \Sigma^{ab}_{ij}(x,y) \right] \left( A^b_j(y) - \bar{A}^b_j(y) \right) \right\} 
\times \exp \left\{ i \bar{\pi}^a_{ij}(x) \left( A^a_i(x) - \bar{A}^a_i(x) \right) \right\},
\]  

(B5)

falling into the class of reference states taken in the main text for generic bosonic theories. The remaining definitions, like the Hamiltonian, can be found in the next section, app. B2.

2. Factors of \( g \)

In Yang-Mills theory one has basically two options concerning where one wants to put the coupling constant, either in front of the action (here called 'non-perturbative scaling'), or in front of the commutator term in the field strength (here called 'perturbative scaling'). In table I we give a short list concerning which convention leads to which placing of factors of \( g \) in other quantities of interest.

<table>
<thead>
<tr>
<th></th>
<th>non-perturbative scaling</th>
<th>perturbative scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>covariant derivative</td>
<td>( D_\mu = \partial_\mu - iA_\mu )</td>
<td>( D_\mu = \partial_\mu - i g A_\mu )</td>
</tr>
<tr>
<td>field strength</td>
<td>( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] )</td>
<td>( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu] )</td>
</tr>
<tr>
<td></td>
<td>( = i[D_\mu, D_\nu] )</td>
<td>( = i g [D_\mu, D_\nu] )</td>
</tr>
<tr>
<td>action</td>
<td>( S = -\frac{1}{4g} F^{a}_{\mu\nu}(x) F^{a \mu\nu}(x) )</td>
<td>( S = -\frac{1}{4} F^{a}_{\mu\nu}(x) F^{a \mu\nu}(x) )</td>
</tr>
<tr>
<td>electrical field</td>
<td>( E^a_i = F^a_{0i} )</td>
<td>( E^a_i = F^a_{0i} )</td>
</tr>
</tbody>
</table>
non-perturbative scaling & perturbative scaling

<table>
<thead>
<tr>
<th>term</th>
<th>non-perturbative scaling</th>
<th>perturbative scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>magnetic field</td>
<td>$B^a_i = -\frac{1}{2} \varepsilon_{ijk} F^{ajk}$</td>
<td>$B^a_i = -\frac{1}{2} \varepsilon_{ijk} F^{ajk}$</td>
</tr>
<tr>
<td></td>
<td>$(\nabla \times A)^a_i \equiv -\frac{1}{2} f^{abc} (A^b \times A^c)_i$</td>
<td>$(\nabla \times A)^a_i \equiv -\frac{1}{2} f^{abc} (A^b \times A^c)_i$</td>
</tr>
<tr>
<td>momenta $\Pi^a_i = \frac{\partial L}{\partial \dot{A}^a_i}$</td>
<td>$\Pi^a_i = \frac{1}{g} F^a_{i0} = -\frac{1}{g} E^a_i$</td>
<td>$\Pi^a_i = F^a_{i0} = -E^a_i$</td>
</tr>
<tr>
<td>Hamiltonian</td>
<td>$H = \frac{g^2}{2} \Pi^a_i(x) \Pi^a_i(x) + \frac{1}{2g^2} B^a_i(x) B^a_i(x)$</td>
<td>$H = \frac{1}{2} \Pi^a_i(x) \Pi^a_i(x) + \frac{1}{2} B^a_i(x) B^a_i(x)$</td>
</tr>
<tr>
<td>wave functional of 'free' theory</td>
<td>$\psi[A] \sim e^{-\frac{1}{g^2} AG^{-1}A}$</td>
<td>$\psi[A] \sim e^{- AG^{-1}A}$</td>
</tr>
<tr>
<td>generators of time-independent gauge trasfos</td>
<td>$\left[(-\Gamma^a_x), (-\Gamma^a_y)\right] = i \delta_{xy} f^{abc} (-\Gamma^c_x)$</td>
<td>$\left[\frac{1}{g} (-\Gamma^a_x), \frac{1}{g} (-\Gamma^a_y)\right] = \frac{g}{g} i \delta_{xy} f^{abc} \frac{1}{g} (-\Gamma^c_x)$</td>
</tr>
<tr>
<td>finite gauge trasfos (gluonic part)</td>
<td>$e^{i \int \phi^a \Gamma^a}$</td>
<td>$e^{i \int \phi^a \frac{1}{g} \Gamma^a}$</td>
</tr>
</tbody>
</table>

TABLE I: Placement of the coupling constant $g$ in the 'non-perturbative' and the 'perturbative' scaling scheme. We have used the shorter form $\Gamma^a_x$ for $\Gamma^a(x)$.

APPENDIX C: EXPLICIT EXPRESSIONS

1. Matrices A-F

In this appendix, we want to give explicit expressions that are valid for the theories that have a Hamiltonian of the form eq. (6). As we have already mentioned, terms in the Hamiltonian that contain more than four $c/a$ operators do not contribute to any of the matrix elements due to the second gRPA approximation\(^{32}\). Then the computation is straightforward: we simply insert eq. (16) into eq. (26), and compute the double commutators. The

---

\(^{32}\) One can use Wick’s theorem to compute the double commutators that appear in the definition of the matrices $A, \ldots, F$. Then one realizes that for those terms of the Hamiltonian that contain more than four $c/a$ operators the evaluated double commutators still contain at least one $c/a$ operator. Taking the vacuum expectation values - where according to the second gRPA approximation we use the mean-field vacuum state - of these expressions sets these terms then to zero.
computation is simplified by the observation that only those terms of $H$ contribute to the matrices where the number of creation operators together with the number of creation operators in the definition of the matrix elements match the respective number of annihilation operators. This is the reason for the matrices $A, C, D, E$ having contributions from one term of $H$ only. $B$ obtains contributions from two terms of $H$ as is to be expected, whereas $F$ obtains only one. Individually, the matrices read:

- **Matrix A**
  \[ A_{\minj} = -24H_{04}^{\{\minj\}} \]  
  \hspace{1cm} (C1)

- **Matrix B**
  \[ B_{\minj} = B_{\minj}^{11} + B_{\minj}^{22} \]  
  with
  \[ B_{\minj}^{11} = (H_{i1}^{ji}\delta_{nm} + H_{11}^{ni}\delta_{jm} + H_{11}^{jm}\delta_{ni} + H_{11}^{nm}\delta_{ji}) \hspace{0.5cm} ; \hspace{0.5cm} B_{\minj}^{22} = 4H_{22}^{\{nj\}\{mi\}} \]  
  \hspace{1cm} (C3)

- **Matrix C**
  \[ C_{jmi} = -6H_{03}^{\{jmi\}} \]  
  \hspace{1cm} (C4)

- **Matrix D**
  \[ D_{jmi} = 2H_{12}^{i(jm)} \]  
  \hspace{1cm} (C5)

- **Matrix E**
  \[ E_{mi} = -2H_{02}^{\{mi\}} \]  
  \hspace{1cm} (C6)

- **Matrix F**
  \[ F_{mi} = H_{11}^{im} \]  
  \hspace{1cm} (C7)

where $^{\{ij\}}$ means: symmetrize in the indices $i, j$ (i.e. add all permutations and divide by the number of permutations) and $H_{ab}^{ij}$ means ‘that factor in the Hamiltonian that multiplies $a$ creation and $b$ annihilation operators’. We see that at the stationary point they simplify...
considerably:

\[
A_{mj} = -4U_{mm}U_{ii}U_{nm}U_{jj} \frac{\delta^2}{\delta G_{mii} \delta G_{njj}} \langle V \rangle, \tag{C8}
\]

\[
B_{mj} = B_{mj}^{11} + B_{mj}^{22}, \tag{C9}
\]

\[
B_{mj}^{11} = (F_{m} \delta_{ij} + F_{m} \delta_{im} + F_{m} \delta_{mj} + F_{i} \delta_{mn}), \tag{C10}
\]

\[
B_{mj}^{22} = 4U_{mm}U_{ii}U_{nm}U_{jj} \frac{\delta^2}{\delta G_{mii} \delta G_{njj}} \langle V \rangle, \tag{C11}
\]

\[
C_{jmi} = -2U_{jj}U_{mm}U_{ii} \frac{\delta^2}{\delta \phi_{ji} \delta G_{mii}} \langle H \rangle, \tag{C12}
\]

\[
D_{jmi} = C_{jmi}, \tag{C13}
\]

\[
E_{mi} = 0, \tag{C14}
\]

\[
F_{mi} = \frac{1}{2} G_{mi}, \tag{C15}
\]

2. Gauss Law Operator

The Gauss law operator was defined in eq. (B3). Here, we give its decomposition into creation/annihilation operators. After normal ordering, we obtain the result

\[
\hat{D}_{x,i}^{ab} \Pi_{ix}^{b} = \hat{D}_{x,i}^{ab} \Pi_{ix}^{b} + 2 \text{tr} (\hat{T}^{a} \sum_{x} G_{xix})
\]

\[
+ \left( \frac{\delta}{\delta A_{1i}^{a_{1}}} (\hat{D}_{x,i}^{ab} \Pi_{ix}^{b}) \right) U_{a_{1}a_{1}} (a_{1}^{b_{1}a_{1}} + a_{1}^{b_{1}a_{1}})
\]

\[
+ 2 \left( \frac{\delta}{\delta \Pi_{1i}^{a_{1}}} (\hat{D}_{x,i}^{ab} \Pi_{ix}^{b}) \right) \left( \frac{i}{4} (U-1)_{a_{1}a_{1}} (a_{1}^{b_{1}a_{1}} - a_{1}^{b_{1}a_{1}}) + \sum_{x} U_{a_{1}a_{1}} (a_{1}^{b_{1}a_{1}} + a_{1}^{b_{1}a_{1}}) \right)
\]

\[
+ \left( \frac{\delta}{\delta G_{x_{1}x_{2}} (\hat{D}_{x,i}^{ab} \Pi_{ix}^{b})} \right) U_{a_{1}a_{1}} U_{a_{2}a_{2}} x (a_{2}^{b_{1}a_{1}} + a_{2}^{b_{1}a_{1}})
\]

\[
+ \left( \frac{\delta}{\delta \sum_{x_{1}x_{2}} (\hat{D}_{x,i}^{ab} \Pi_{ix}^{b})} \right) \frac{i}{4} (U-1)_{a_{1}a_{1}} (U-1)_{a_{2}a_{2}} x (a_{2}^{b_{1}a_{1}} - a_{2}^{b_{1}a_{1}}) + g \frac{i}{2} (\hat{T}^{a})_{a_{1}a_{2}} U_{a_{1}a_{1}} U_{a_{2}a_{2}} x (a_{2}^{b_{1}a_{1}} + a_{2}^{b_{1}a_{1}}). \tag{C16}
\]

Here we have employed the notation

\[
(\hat{T}^{a})_{a_{1}a_{2}} = f^{a_{1}a_{2}} \tag{C17}
\]

where \(f^{a_{1}a_{2}} \) denote the \(SU(N)\) structure constants.
APPENDIX D: COMMUTATION RELATIONS OF NORMAL MODES

In this appendix we want to consider the question 'what are the conditions under which the eigenmodes $Q_B^\dagger \nu$ can be treated as harmonic oscillators?' We start with the observation that eq. (40) implies by hermitian conjugation

$$[H_B, Q_B^\nu] = -\Omega_\nu Q_B^\nu.$$  \hspace{1cm} (D1)

Thus, to every eigenfrequency, the negative eigenfrequency also belongs to the spectrum. Thus, without loss of generality\textsuperscript{33} in the following we will assume that

$$\Omega_\nu \geq 0,$$  \hspace{1cm} (D2)

otherwise we just exchange the respective $Q_B^\nu, Q_B^\dagger \nu$. The next point is that all the commutators

$$[Q_B^\dagger \nu, Q_B^\dagger \mu] \quad [Q_B^\nu, Q_B^\dagger \mu] \quad [Q_B^\nu, Q_B^\mu]$$  \hspace{1cm} (D3)

are pure numbers. This is due to the fact that all operators considered in this context are by construction linear in $B, B^\dagger, a, a^\dagger$. We denote these numbers as

$$[Q_B^\dagger \mu, Q_B^\dagger \nu] = M_{\mu\nu}; \quad [Q_B^\mu, Q_B^\dagger \nu] = N_{\mu\nu}; \quad [Q_B^\mu, Q_B^\nu] = O_{\mu\nu}.$$  \hspace{1cm} (D4)

Up to now these numbers are arbitrary; one can, however, put the equations of motion to some good use. Consider

$$\Omega_\nu [Q_B^\mu, Q_B^\dagger \nu] \quad \text{eq. (40)} \quad [Q_B^\mu, [H_B, Q_B^\dagger \nu]]$$  \hspace{1cm} (D5)

\begin{align*}
&= \text{Jacobi id.} \quad -[H_B, [Q_B^\dagger \nu, Q_B^\mu]] - [Q_B^\nu, [Q_B^\mu, H_B]] \\
&= [H_B, N_{\mu\nu}] + [Q_B^\dagger \nu, [Q_B^\mu, H_B]] \\
&= 0 \quad = -\Omega_\nu Q_B^\mu \\
&= \Omega_\mu [Q_B^\mu, Q_B^\dagger \nu].
\end{align*}  \hspace{1cm} (D6) (D7) (D8)

In other words:

$$(\Omega_\nu - \Omega_\mu) N_{\mu\nu} = 0 \quad \text{(no sum over } \mu, \nu).$$  \hspace{1cm} (D9)

\textsuperscript{33} We have seen in sec. \textsuperscript{57} that the assumption that the eigenenergies of the modes are real already implies that we are dealing with a stable mean-field solution.
If all eigenvalues are distinct and non-zero then eq. (D9) implies that $N_{\mu\nu}$ is diagonal. Since the gRPA equations are homogeneous equations, we may now normalize the amplitudes in such a way that

$$N_{\mu\nu} = \delta_{\mu\nu}.$$  \hspace{1cm} (D10)

The same procedure can be carried out for $M_{\mu\nu}, O_{\mu\nu}$. We derive in analogy to eq. (D9):

$$(\Omega_{\nu} + \Omega_{\mu})M_{\mu\nu} = 0, \quad (\Omega_{\nu} + \Omega_{\mu})O_{\mu\nu} = 0,$$ \hspace{1cm} (D11)

where again no sum over double indices is performed. From this we conclude, again if there are no zero modes, that

$$M_{\mu\nu} = 0, \quad O_{\mu\nu} = 0.$$ \hspace{1cm} (D12)

Thus, under the aforementioned conditions, the normal modes satisfy ordinary c/a commutation relations. To arrive at this, we have indeed used both gRPA approximations, since the basic ingredient was that $Q^\dagger$ is a one-particle operator (first gRPA approximation) and that all the commutators of $a, a^\dagger, B, B^\dagger$ are pure numbers (quasi-boson approximation).

34 In [16] it is stated that one can show that $N_{\mu\nu}$ is positive if one is considering a positive definite Hessian at the stationary point of the mean-field problem; a possible proof of this statement works as follows: only if the Hessian is a positive matrix it is guaranteed that all eigenvalues $\Omega^2$ are positive. This has been assumed so far (e.g. how we concluded that for every positive frequency there is also a negative one etc.); thus we know that the system is stable. Knowing this, we can argue as follows: we know that $[H, Q_{B\nu}^\dagger] = \Omega_{\nu}Q_{B\nu}^\dagger, [H, Q_{B\nu}] = -\Omega_{\nu}Q_{B\nu}$ and $[Q_{B\nu}, Q_{B\nu}^\dagger] = N_{\nu}\delta_{\nu\nu'}$. From this we conclude that the Hamiltonian has to look like $H = \sum_{\nu}(\Omega_{\nu}/N_{\nu})Q_{B\nu}^\dagger Q_{B\nu}$. Now we can study two different scenarios, namely $N_{\nu}$ can be positive (we call the set of $\nu$ for which this is true $\nu_+$) or negative (correspondingly $\nu_-$). In order to have the usual creation/annihilation commutation relations we have - for $\nu \in \nu_-$ - to interchange ‘creation’ and ‘annihilation’ operators; for clarity, we introduce new letters for them, i.e. for $\nu \in \nu_-, Q_{B\nu} \rightarrow P_{B\nu}^\dagger, Q_{B\nu}^\dagger \rightarrow P_{B\nu}$. Now we normalize $Q_{BS}$ and $P_{BS}$ s.t. for $\nu \in \nu_+, N_{\nu} = 1$ and for $\nu \in \nu_-, N_{\nu} = -1$. The Hamiltonian then reads $H = \sum_{\nu \in \nu_+}(\Omega_{\nu}Q_{B\nu}^\dagger Q_{B\nu} + \sum_{\nu \in \nu_-}(\Omega_{\nu})P_{B\nu}^\dagger P_{B\nu} + \text{const.}$ But here we see that we are dealing with a rather unstable system: the more modes are generated by $P_{B\nu}^\dagger$, the lower the energy becomes. This cannot be true, however, since we know that we started from a stable system (with all frequencies real) ! Thus $\nu_-$ has to be empty.

35 This relation has also as its consequence that the states generated by $Q_{B\nu}^\dagger, Q_{B\nu}^\dagger$ from the gRPA vacuum (defined to be annihilated by all gRPA annihilation operators, $Q_{B\nu}|\text{RPA}\rangle = 0 \forall \nu$) are indeed orthogonal: $\langle \text{RPA}|Q_{B\mu}^\dagger Q_{B\nu}^\dagger|\text{RPA}\rangle = \langle \text{RPA}|Q_{B\mu}^\dagger Q_{B\nu}^\dagger|\text{RPA}\rangle = 0 = \delta_{\mu\nu}$ where we have used the second gRPA approximation in the next to last step [16].
APPENDIX E: GENERAL POTENTIAL

In this appendix we will demonstrate that a potential that is an arbitrary polynomial in the field operators, can - once one decomposes the field operator into creation and annihilation operators - be written in normal ordered form s.t. all the coefficients appearing in front of the normal ordered products of creation/annihilation operators can be written as functional derivatives w.r.t. $G$ and $\bar{\phi}$ of the vacuum expectation value of the potential (the creation/annihilation operators are defined with respect to that vacuum).

For simplicity we take the potential of the form

$$V[\phi] = M_{x_1...x_n} \phi_{x_1} \cdots \phi_{x_n}, \quad (E1)$$

where $x_i$ are super-indices. It is clear that, if the claim holds for this potential, it will hold for an arbitrary polynomial since it will be a sum of terms of type $[E1]$.

Since the field operators commute, it is sufficient to consider an $M_{x_1...x_n}$ that is symmetric in all indices. We have seen in section III A that one can write

$$\phi_x = \bar{\phi}_x + U_{xy}(b^+_y + b_y), \quad (E2)$$

where $x, y$ are super-indices, $U^2 = G$, and $[b_x, b^+_y] = \delta_{xy}$. For our purposes, it will be more useful to define rescaled operators

$$a^+_x = U_{xy}b^+_y \quad ; \quad a_x = U_{xy}b_y \quad \text{with} \quad [a_x, a^+_y] = G_{xy}. \quad (E3)$$

Sometimes we find it also useful to write

$$\phi_x = \bar{\phi}_x + \varphi_x \quad \text{with} \quad \varphi_x = a^+_x + a_x. \quad (E4)$$

After all this notational introduction, let’s come to the proof. We note that, similar to $\phi$, both $\bar{\phi}$ and $\varphi$ commute; thus we can write equally\footnote{It is understood, obviously, that the index of $x$ increases from left to right; if it should ever decrease, as in the case $m = 0$, the $\bar{\phi}$s are to be considered absent.} instead of eq. $[E1]$:

$$V[\phi] = M_{x_1...x_n} \sum_{m=0}^n \binom{n}{m} \bar{\phi}_{x_1} \cdots \bar{\phi}_{x_m} \varphi_{x_{m+1}} \cdots \varphi_{x_n}. \quad (E5)$$
Now we normal order the terms with a fixed $m$:

$$
\varphi_{x_{m+1}} \cdots \varphi_{x_n} \\
= (a_{x_{m+1}} + a_{x_{m+1}}^\dagger) \cdots (a_{x_n} + a_{x_n}^\dagger) \\
= a_{x_{m+1}}^\dagger a_{x_{m+2}}^\dagger \cdots a_{x_n}^\dagger \\
+ a_{x_{m+1}} a_{x_{m+2}} a_{x_{m+3}} a_{x_n}^\dagger + \cdots + a_{x_{m+1}} a_{x_{m+2}} a_{x_{m+3}} a_{x_n} a_{x_{m+4}} a_{x_{m+5}} a_{x_{m+6}} \cdots a_{x_n} \\
+ a_{x_{m+1}} a_{x_{m+2}} a_{x_{m+3}} a_{x_n}^\dagger a_{x_{m+2}} a_{x_{m+3}} a_{x_{m+4}} a_{x_{m+5}} a_{x_{m+6}} \cdots a_{x_n} \\
\vdots \\
+ a_{x_{m+1}} a_{x_{m+2}} \cdots a_{x_n}.
$$

(E6)

We see that every row contains a fixed number of creation/annihilation operators and that it contains all possible permutations of types (creation or annihilation) among the possible indices. Especially for every term

$$
\cdots a_{x_p}^\dagger \cdots a_{x_q} \cdots
$$

(E7)

there also exists a term

$$
\cdots a_{x_p} \cdots a_{x_q}^\dagger \cdots
$$

(E8)

with all undenoted operators identical. We now use Wick’s theorem, cf. e.g. [16], to put every line into normal order. It is practical to deal with the whole line for the following reason: if we use Wick’s theorem naively we obtain the normal ordered expression plus the normal ordered expression of two less operators times their contraction etc. However, a lot of these contractions are zero, since (we denote the contraction of two operators by $C$)

$$
C(a_{i}^\dagger a_j) = 0.
$$

(E9)

However, if we deal with the complete line at once, we can use that, since for every arrangement eq. (E7) there also exists a partner eq. (E8), we will always obtain contractions

$$
C(a_{i}^\dagger a_j + a_i a_j^\dagger) = G_{ij}.
$$

(E10)

It is clear that for $k$ contractions present we need $2^k$ partners to obtain a non-vanishing contribution. The important point is that they exist if we deal with the whole line at once. Remember that the contractions are multiplied by normal ordered terms, and that
\([a,a] = [a^\dagger,a^\dagger] = 0\). Since in addition \(\mathcal{M}_{x_1\ldots x_n}\) is symmetric in all its indices, all terms with a fixed number of contractions, creation, and annihilation operators, will give an identical contribution. Thus the question appears: assume we start out from the line where every term contains \(p\) creation operators, \(q\) annihilation operators and consider now terms with \(k\) contractions; how many terms will we obtain? The answer is simple,

\[
\text{number of terms} = \frac{1}{2^k} \times \frac{(p+q)!}{p!q!} \times \left( \frac{1}{k!} \frac{p!}{(p-k)!} \frac{q!}{(q-k)!} \right),
\]

(E11)

and comes about as follows:\(^{37}\)

1. the line containing \(p\) c’s and \(q\) a’s contains \(\frac{(p+q)!}{p!q!}\) terms;

2. out of the \(p\) c’s and \(q\) a’s we take \(k\) each, and put them together to form contractions: there are obviously \(\frac{p!}{(p-k)!} \frac{q!}{(q-k)!}\) ways to do this.

3. However, it does not matter in which order we perform the contractions since the contractions commute; thus in the step before we have overcounted by a factor of \(k!\).

4. We now have to take into account what was said above: On the one hand, a lot of contractions are zero. On the other hand we can form pairs - this gives as argued above an additional factor of \(2^{-k}\).

With this formula at hand, we can at first answer the following important question: assume that we have started from an expression with \(n\) field operators; upon normal ordering we obtain expressions with \(n,n-2,\ldots,n-2k\) c/a operators each; the question now is: is the relative number of \(P\) c’s and \(Q\) a’s (with \(P + Q\) fixed) always the same, no matter from which \(n\) one starts and how many contractions one needs\(^{38}\) (provided \(n - (P + Q)\) is even) ? This question can be answered in the affirmative in the following way: We start out with an expression that contains \(p\) c’s and \(q\) a’s; after \(k\) contractions we will end up with \(P = p-k\) c’s and \(Q = q-k\) a’s. Since in the beginning \(p+q = n\) was fixed, and we end up with \(P + Q = n'\) fixed we need for every term the same number of contractions, namely \(2k = n - n'\). With this we can rewrite eq. (E11) as

\[
\frac{n!}{2^k k! P!(n' - P)!} = \frac{n!}{2^k k! P!Q!},
\]

(E12)

---

\(^{37}\) In the following, we abbreviate 'creation operators' as c’s and 'annihilation operators' as a’s.

\(^{38}\) As an example: no matter where one starts - if one can get to the expression with, in total, two c’s and a’s, will they always come as \(aa + a^\dagger a^\dagger + 2a^\dagger a\)?
Thus, we have decomposed the number of terms into a factor that depends on $n, n'$ which is a constant for $P + Q = n'$ fixed and $P$ varying, and a factor that depends on the number of creation and annihilation operators. If we go back to eq. (E5) we see that we have different possibilities to end up with $P$ c’s and $Q$ a’s ($P + Q = n'$): either start from $n \varphi$s, and perform $k$ contractions, or start from two $\bar{\varphi}$s, $(n - 2) \varphi$s and perform $k - 1$ contractions etc. Thus the contribution to $\mathcal{M}_{x_1...x_n} \phi_{x_1} \cdots \phi_{x_n}$ containing $n'$ c’s and a’s can be written as

$$\mathcal{M}_{x_1...x_n} \left[ \sum_{p=0}^{n'} \frac{1}{P!(n' - P)!} a^\dagger_{x_1} \cdots a^\dagger_{x_P} a_{x_{P+1}} \cdots a_{x_{n'}} \right]$$

$$\times \sum_{k=0}^{[\frac{1}{2}(n-n')]} \left( \frac{n!}{2^k k!} \frac{1}{(n - n' - 2k)!} G_{x_{n' + 1}x_{n' + 2}} \cdots \times G_{x_{n' + (2k-1)x_{n' + 2k}} \bar{\phi}_{x_{n' + 2k + 1}} \cdots \bar{\phi}_{x_{n}}} \right)$$

(E13)

where the latter sum runs to $\frac{1}{2}(n - n')$ if $n - n'$ is even, and to $\frac{1}{2}(n - n' - 1)$ if it is odd - we will deal with these details below. This expression allows us to write down the vacuum expectation value (VEV) of $V[\phi]$ since it corresponds to the case $n' = 0$. Distinguish

- $n = 2N$: the VEV reads

$$\langle V[\phi] \rangle = \mathcal{M}_{x_1...x_{2N}} \sum_{k=0}^{N} \left( \frac{n!}{2^k k!} \frac{1}{(n - 2k)!} G_{x_1x_2} \cdots G_{x_{(2k-1)x_{2k}} \bar{\phi}_{x_{2k+1}} \cdots \bar{\phi}_{x_{2N}}} \right)$$

(E14)

- $n = 2N + 1$: the VEV reads

$$\langle V[\phi] \rangle = \mathcal{M}_{x_1...x_{2N+1}} \sum_{k=0}^{N} \left( \frac{n!}{2^k k!} \frac{1}{(n - 2k)!} G_{x_1x_2} \cdots G_{x_{(2k-1)x_{2k}} \bar{\phi}_{x_{2k+1}} \cdots \bar{\phi}_{x_{2N}}} \bar{\phi}_{x_{2N+1}} \right)$$

(E15)

For the following treatment, we can treat both cases with the same formula if we realize that $\mathcal{M}_{x_1...x_{2N+1}} \bar{\phi}_{x_{2N+1}}$ has the same properties as $\mathcal{M}_{x_1...x_{2N}}$ and that in the sum in eq. (E15) always at least one $\bar{\phi}$ survives. Thus we only have to treat the case with $n = 2N$. We now

39 Note that the factors of the contractions and of the binomial decomposition of eq. (E5) can be put together in a practical manner: $\frac{(n' + 2k)!}{2^k k! (n + 1 - (n' + 2k + 1))} = \frac{n!}{2^k k! (n - n' - 2k)}$. Note also that we have arranged here $\varphi, \bar{\phi}$ opposite to eq. (155).
consider
\[ \frac{\delta}{\delta G_{y_1 y_2}} \langle V[\phi] \rangle \]
\[ = \frac{\delta}{\delta G_{y_1 y_2}} M_{x_1 \ldots x_{2N}} \sum_{k=0}^{N} \left( \frac{n!}{2^k k!(n - n' - 2k)!} \right)_{n'=0} G_{x_1 x_2} \cdots G_{x_{(2k-1)} x_{2k} \bar{\phi}_{x_{2k+1}} \cdots \bar{\phi}_{x_{2N}}} \]
\[ = M_{y_1 y_2 x_3 \ldots x_{2N}} \sum_{k=1}^{N} \left( \frac{n!}{2^k k!(n - n' - 2k)!} \right)_{n'=0} k G_{x_1 x_4} \cdots G_{x_{(2k-1)} x_{2k} \bar{\phi}_{x_{2k+1}} \cdots \bar{\phi}_{x_{2N}}} \]
\[ = \frac{1}{2N} M_{y_1 y_2 x_{n+1} \ldots x_{2N}} \]
\[ \times \sum_{k=0}^{N-N'} \left( \frac{n!}{2^k k!(n - n' - 2k)!} \right)_{n'=2=2N'} G_{x_{n'+1} x_{n'+2} \cdots G_{x_{(2k-1)} x_{2k} \bar{\phi}_{x_{2k+1}} \cdots \bar{\phi}_{x_{2N}}} \right) . \]

(E16)

In (*) we have used the symmetry of \( M \). Obviously, as has been indicated by the suggestive notation, one is not restricted to one functional derivative but one can also perform \( N' \) of them, and then the restriction \( N' = 1 \) in the last line is rendered unnecessary. We see clearly that, apart from the factor \( 2^{-N'} \) the outcome of \( N' \) derivatives of the vacuum expectation value w.r.t. \( G \) is identical to the prefactor of the addend containing \( P + Q = n' = 2N' \)'s and a's in eq. (E13). Thus we can rewrite eq. (E13) as

\[ 2^{N'} \left[ \sum_{P=0}^{n'} \frac{1}{P!(n' - P)!} a_{y_1}^P \cdots a_{y_{P+1}}^P \cdots a_{y_{n'}}^P \right] \left( \frac{\delta}{\delta G_{y_1 y_2}} \cdots \frac{\delta}{\delta G_{y_{n'-1} y_{n'}}} \right) \langle V[\phi] \rangle . \]

(E17)

The treatment we have presented up to here is valid for \( P + Q \) even. If \( P + Q \) is odd, we have to perform derivatives w.r.t. \( \bar{\phi} \) and thus we have to treat the \( n \) even/odd cases individually.

Let's start with \( n \) even:

\[ \frac{\delta}{\delta \bar{\phi}_{y_1}} \langle V[\phi] \rangle \]
\[ = \frac{\delta}{\delta \bar{\phi}_{y_1}} M_{x_1 \ldots x_{n-1} x_n} \sum_{k=0}^{N} \left( \frac{n!}{2^k k!(n - n' - 2k)!} \right)_{n'=0} G_{x_1 x_2} \cdots G_{x_{(2k-1)} x_{2k} \bar{\phi}_{x_{2k+1}} \cdots \bar{\phi}_{x_{x_n-1}} \bar{\phi}_{x_n}} \]
\[ = M_{x_1 \ldots x_{n-1} y_1} \sum_{k=0}^{N-1} \left( \frac{n!}{2^k k!(n - n' - 2k)!} \right)_{n'=0} G_{x_1 x_2} \cdots G_{x_{(2k-1)} x_{2k} \bar{\phi}_{x_{2k+1}} \cdots \bar{\phi}_{x_{x_n-1}} (n - 2k)} \]
\[ = M_{x_1 \ldots x_{n-1} y_1} \sum_{k=0}^{N-1} \left( \frac{n!}{2^k k!(n - n' - 2k)!} \right)_{n'=1} G_{x_1 x_2} \cdots G_{x_{(2k-1)} x_{2k} \bar{\phi}_{x_{2k+1}} \cdots \bar{\phi}_{x_{x_n-1}}} \]
\[ = M_{y_1 x_2 \ldots x_n} \sum_{k=0}^{N-1} \left( \frac{n!}{2^k k!(n - n' - 2k)!} \right)_{n'=1} G_{x_{n'+1} x_{n'+2} \cdots G_{x_{n'+(2k-1)} x_{n'+2k} \bar{\phi}_{x_{n'+2k+1}} \cdots \bar{\phi}_{x_{x_n}}} \right) . \]

(E18)
The main point happened in line (*) where the fact that we started from even \( n \) played a role. If \( n \) is even, \( k = N \) means that this addend doesn’t contain a single factor of \( \bar{\phi} \), thus its derivative vanishes. This is different in case of \( n \) odd, there the upper boundary is not affected by the first differentiation. It is different if one performs two derivatives w.r.t. \( \bar{\phi} \) since then the upper limit of the sum changes once altogether independent of whether one starts from \( n \) even or odd. Thus one obtains a relation between derivatives w.r.t. \( G \) and to \( \bar{\phi} \)

\[
\frac{1}{2} \frac{\delta^2}{\delta \phi_x \delta \phi_y} \langle V[\phi] \rangle = \frac{\delta}{\delta G_{xy}} \langle V[\phi] \rangle, \quad \text{(E19)}
\]

which will be very useful in proving the equivalence between the generalized RPA using the operator approach and generalized RPA as derived from the time-dependent variational principle.

To put this appendix in a nutshell, we have proved that a potential that is an arbitrary polynomial in the field operators can be decomposed into creation and annihilation operators, s.t. upon normal ordering one obtains a sum of subsums where each subsum contains a fixed number of \( c \)’s and \( a \)’s. The subsum consisting of the terms containing \( n' \) \( c \)’s and \( a \)’s can be written as a standard polynomial in \( c \)’s and \( a \)’s

\[
\left[ \sum_{P=0}^{n'} \frac{1}{P!(n' - P)!} a_{x_1}^\dagger \cdots a_{x_P}^\dagger a_{x_{P+1}} a_{x_{n'}} \right] \quad \text{(E20)}
\]

multiplied by

- if \( n' \) is even, \( n'/2 \) derivatives w.r.t. \( G \) times a factor \( 2^{n'/2} \)

- if \( n' \) is odd, one derivative w.r.t. \( \bar{\phi} \) and \( (n' - 1)/2 \) derivatives w.r.t. \( G \) times a factor \( 2^{(n'-1)/2} \).

We have also shown that each derivative w.r.t. \( G \) may be traded for two derivatives w.r.t. \( \bar{\phi} \).
[7] For an extensive list of references cf. [5].