EMITTANCE GROWTH FROM MULTIPLE SCATTERING IN THE PLASMA BEAT-WAVE ACCELERATOR

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ABSTRACT

Linear colliders for very high energies will require beams of extremely small emittance. In the plasma Beat-Wave Accelerator, multiple scattering from the plasma ions can cause emittance growth. A calculation of this growth is given for an example of a 1 TeV BWA.

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1. INTRODUCTION

The plasma Beat-Wave Accelerator (BWA) proposed by Tajima and Dawson\(^1\) offers a possibility of obtaining the extremely high accelerating fields that will be necessary in order to build, within an acceptable size and cost, linear colliders in the energy range of 1 to 10 TeV. Such intense fields can in principle be obtained in a highly-ionised plasma from charge separation induced by two laser beams, whose difference frequency resonates with the plasma frequency. The details of this process are described in Refs. 2, 3 and 4, which contain more extensive bibliographies.

So far no fundamental objection to the principle has appeared, and at the recent Frascati Workshop\(^5\) the BWA remained a serious contender for high-field generation. However, much theoretical and experimental work will be required before the feasibility can be properly assessed, and many questions remain to be resolved.

One of these questions, which is the subject of this report, concerns multiple scattering of the accelerated beam by the particles in the plasma. To obtain high luminosity within reasonable limits of beam power requires very small beam dimensions at the collision point and hence low beam emittance. Appreciable enhancement of the emittance by collisions with plasma particles would degrade the performance of the machine. This consideration was evoked by the RAL Study Group\(^4\) and further discussed at the Frascati Workshop\(^5\), from which it was apparent that a more precise estimate is needed.

Scattering of accelerated particles by the residual gas in a synchrotron is a familiar problem in accelerator physics which was already treated in 1948 by Blachman and Courant\(^6\), and subsequently by many others. The situation in a BWA, though similar in principle, differs in detail for two main reasons. Firstly, we are dealing with an ionised plasma rather than neutral atoms or molecules, which has an influence on the effective range of the Coulomb force. Secondly, the transverse focusing of the accelerated beam in the BWA will in general be a function of energy which will influence the rate of emittance growth along the machine. It is therefore appropriate to derive scattering formulae which take into account these features.

As a basis for the calculation we take the conceptual design of a BWA elaborated at the Frascati Workshop\(^5\). It consists of an e\(^+\)-e\(^-\) linear collider of 1 TeV per beam, with injection energy assumed to be about 10 GeV. The
accelerated electrons are therefore highly relativistic over the whole energy range. A plasma of density \( n = 10^{17}\text{cm}^{-3} \) yields an accelerating field of 6 GeV m\(^{-1}\) with a stage length of 5.4 m, giving a total length per linac of about 170 m. We assume a fully-ionised hydrogen plasma. In line with most of the references cited, c.g.s. units will mainly be used, though with energies in electron-volts.

2. SMALL-ANGLE SCATTERING

In the approximation of small angles \( \theta \ll 1 \), elastic scattering by a Coulomb central force is well described by the simplified Rutherford formula:

\[
\theta = \frac{\Delta p}{p} = \frac{2Ze^2}{pvd}
\]

where \( p, v, Ze \) are respectively the momentum, velocity and charge of the incident particle, \( Ze \) is the charge of a stationary nucleus, \( \Delta p \) is the momentum transfer in the collision and \( b \) is the impact parameter, i.e. the perpendicular distance from the nucleus to the initial trajectory of the incident particle. For a point Coulomb field in the absence of screening, the Rutherford formula is exact quantum mechanically for the statistical distribution of scattering angles, even under conditions where classical trajectories cannot strictly be defined. Since we are considering highly relativistic incident particles we put \( v = c \) in Eq.(1) which, for unit charge of both beam and plasma particles, becomes:

\[
\theta = \frac{\Delta p}{p} = \frac{2e^2}{pcb}
\]

Deviations from Eq.(2) occur both for very close collisions (small \( b \)) and for distant collisions (large \( b \)). Outside these limits the momentum transfer \( \Delta p \) is less than that given by a point Coulomb field, due to saturation or screening. This is taken into account to a good approximation by imposing limits \( b_{\text{min}} \) and \( b_{\text{max}} \), corresponding to \( \theta_{\text{max}} \) and \( \theta_{\text{min}} \) respectively, on the range of integration of the scattering cross-section. The precise evaluation of these limits depends on the system in question and may be governed by either quasi-classical or strictly quantum-mechanical constraints. A very lucid and detailed discussion of these questions appears in a famous review by Bohr\(^7\).
Scattering by neutral atoms or molecules has been extensively studied and the results are widely applied as approximate formulae based on theoretical models of atoms adjusted by experimental observations. The minimum impact parameter $b_{\text{min}}$ is governed by the effective radius of the nucleus, whilst $b_{\text{max}}$ is determined by the shielding effect of the atomic electron shells. In a fully-ionised plasma, however, the electrons are not bound by individual nuclear charges but rather by the aggregate of Coulomb fields from many nucleons. The cut-off of the Coulomb field will then be determined by the Debye shielding distance, which is generally much greater than atomic radii. In the following we therefore examine multiple scattering in the BWA from basic principles, using a number of results from Jackson and applying cut-offs appropriate to a hydrogen plasma. For highly energetic incident particles we need only consider scattering from the protons, since the plasma electrons contribute very little to the momentum transfer. Also, spin effects can be neglected.

2.1 Minimum impact parameter $b_{\text{min}}$

For highly-relativistic incident electrons, $b_{\text{min}}$ and therefore $\theta_{\text{max}}$ are governed by one of two quantum-mechanical limits, both related to the de Broglie wavelength $\lambda = \hbar / p$ of the electrons. The first of these requires that, for the impact parameter $b$ to be definable,

$$b > \lambda$$

which leads to the approximate cut-off

$$b_{\text{min}} \approx \frac{\hbar}{p} \quad (3)$$

It is shown in ref. 9), by considering the situation in the reference frames of both the lighter and the heavier particle, that it is the de Broglie wavelength of the lighter particle which applies. From Eq.(2) the maximum scattering angle is then

$$\theta_{\text{max}} = 2e^2 / \hbar c = 2\alpha \quad (4)$$

where $\alpha = 1/137$ is the fine structure constant.
The second quantum condition relates to diffraction of the de Broglie wavefront by an object of non-vanishing radius $R$. This sets an upper limit to the scattering angle such that

$$\theta < \frac{\lambda}{R}$$

and hence

$$\theta_{\text{max}} = \frac{\hbar}{pR} \tag{5}$$

with the corresponding impact parameter

$$b_{\text{min}} = 2\alpha R \tag{6}$$

The stronger condition corresponds to the larger of the two values of $b_{\text{min}}$ given by Eqs.(3) and (6). Since $p = mc\gamma$, the two values are equal for

$$\gamma = \frac{\hbar}{2\alpha Rmc} = \frac{\lambda_c}{2\alpha R} \tag{7}$$

where $\lambda_c = \hbar/(mc) = 3.862 \times 10^{-11}$cm is the electron Compton wavelength.

The quantity $R$ is the effective radius of the proton for electromagnetic interactions, and is related to the flattening of the Coulomb potential in the vicinity of the proton. It has a value of about $0.7 \times 10^{-13}$cm (0.7 fermi) and is thus close to the value of the classical electron radius. Putting the numbers into Eq.(7) we find

$$\gamma = 3.8 \times 10^4$$

for equal $b_{\text{min}}$ in Eqs.(3) and (6). This corresponds to an electron energy of about 19 GeV, which is not much above that assumed for injection into the BWA.

At higher energies the cut-off is determined by Eqs.(5) and (6), and since this appears only in the Coulomb logarithm we can use these equations for the whole energy range as a good approximation.
2.2 Maximum impact parameter $b_{\text{max}}$

For relativistic particles the minumum scattering angle also is determined by quantum-mechanical conditions. To some given impact parameter $b$ corresponds an uncertainty $\delta p$ in the momentum transfer given by

$$ b \delta p > \hbar . \quad (8) $$

If we choose the minimum momentum transfer to be $\Delta p = \delta p$ we then have

$$ \theta_{\text{min}} = \frac{\Delta p}{\rho} = \frac{\hbar}{\rho b} $$

and putting $b = \lambda_D$, the Debye distance in the plasma, yields

$$ \theta_{\text{min}} = \frac{\hbar}{\rho \lambda_D} . \quad (9) $$

Assuming that Eq.(2) is valid down to this cut-off angle we have, with the previous assumptions

$$ b_{\text{max}} = \frac{2e^2}{pc\theta_{\text{min}}} = 2\pi \lambda_D . \quad (10) $$

2.3 Mean-square scattering angle

Since Eq.(5) can be expressed as:

$$ \theta_{\text{max}} = \frac{\hbar}{mcYR} = \frac{\chi_C}{\gamma R} , $$

it is evident that, for $\gamma > 10^4$, $\theta_{\text{max}} \ll 1$ and we can use small-angle approximations with good accuracy. The differential cross-section for a cut-off Coulomb potential can then be written in the form$^9$

$$ \frac{d\sigma}{d\Omega} = \left( \frac{2e^2}{pc} \right)^2 \cdot \frac{1}{(\theta^2 + \theta_{\text{min}}^2)^2} . \quad (11) $$
The total cross section $\sigma$ is obtained by integrating Eq.(11) from 0 to some suitably large angle $\theta_0$. Since the small angles will dominate we can write
\[
d\Omega = 2\pi \sin \theta d\theta = 2\pi d\theta
\]
and hence
\[
\sigma = \int_0^{\theta_0} \frac{d\sigma}{d\Omega} d\Omega = 2\pi \left( \frac{2e^2}{pc} \right)^2 \int_0^{\theta_0} \frac{d\theta}{(\theta^2 + \theta_{\text{min}}^2)^2}
\]
\[
= \pi \left( \frac{2e^2}{pc} \right)^2 \left[ \frac{1}{\theta_{\text{min}}^2} - \frac{1}{\theta_0^2} \right]
\]
and for $\theta_0 \gg \theta_{\text{min}}$ we have
\[
\sigma = \pi \left( \frac{2e^2}{pc\theta_{\text{min}}} \right)^2 \quad .
\] (12)

The mean-square scattering angle $\langle \theta^2 \rangle$ is then
\[
\langle \theta^2 \rangle = \frac{1}{\sigma} \int_0^{\theta_0} \theta^2 \frac{d\sigma}{d\Omega} d\Omega
\]
whence, using Eqs.(11) and (12), with $\theta_{\text{max}}^2 \gg \theta_{\text{min}}^2$,
\[
\langle \theta^2 \rangle = 2\theta_{\text{min}}^2 \ln \left( \frac{\theta_{\text{max}}}{\theta_{\text{min}}} \right) - \frac{1}{2}
\] . (13)

Now from Eqs.(5) and (9),
\[
\frac{\theta_{\text{max}}}{\theta_{\text{min}}} = \frac{\lambda_0}{R}
\]
and, since $\lambda_0$ is typically of order $10^{-6}$cm or more and $R \approx 10^{-3}$cm, we can use the stronger assumption $\ln(\theta_{\text{max}}/\theta_{\text{min}}) \gg 1$, approximating Eq.(13) to
\[
\langle \theta^2 \rangle \approx 2\theta_{\text{min}}^2 \ln \left( \frac{\theta_{\text{max}}}{\theta_{\text{min}}} \right) .
\] (14)

2.4 Multiple scattering

Eq.(14) gives the mean square angle for individual scatterings. For incident particles undergoing a large number $N$ of collisions in a distance $l$,
the angular distribution will be approximately Gaussian with a mean square angle \( \langle \sigma^2 \rangle = N\langle \theta^2 \rangle \). The number of collisions occurring over a distance \( l \) in a medium with \( n \) scattering centres per unit volume is

\[
N = n \sigma l
\]

and, with \( \sigma \) given by Eq.(12), the increase in angular divergence from the multiple scattering is

\[
\langle \sigma^2 \rangle = 2n \sigma l \left( \frac{2e^2}{pc} \right)^2 \ln\left( \frac{\theta_{\text{max}}}{\theta_{\text{min}}} \right).
\]

Using Eqs.(5) and (9) respectively for \( \theta_{\text{max}} \) and \( \theta_{\text{min}} \) yields

\[
\langle \sigma^2 \rangle = 2n \sigma l \left( \frac{2e^2}{pc} \right)^2 \ln\left( \frac{\lambda D}{R} \right).
\]

In Eq.(16), \( \langle \sigma^2 \rangle \) is the spatial angular divergence; we shall in fact use the projection on to one plane \( \langle \sigma^2 \rangle_p = 1/2 \langle \sigma^2 \rangle \), expressed in differential form instead of for a finite length \( l \), which results in

\[
\frac{d}{dz} \langle \sigma^2 \rangle_p = \pi n \left( \frac{2e^2}{pc} \right)^2 \ln\left( \frac{\lambda D}{R} \right).
\]

3. **EMITTANCE GROWTH IN A FOCUSED SYSTEM**

When a beam of particles undergoes multiple scattering in the presence of transverse focusing the effect of the random collisions is to excite betatron oscillations, leading to a growth in the beam emittance. Scattering by the residual gas in a synchrotron is a well-understood phenomenon and the established theory can readily be adapted to the situation in a plasma.

We adopt here the emittance definition normally used for electron storage rings, that is, measured at one standard deviation of an approximately Gaussian distribution. In an electron linac, in contrast to an electron storage ring, synchrotron radiation is weak and is assumed here to have a negligible effect on the beam emittance. Consequently, we distinguish between the absolute emittance \( E \) given by the product of the beam radius and angular divergence, and the invariant emittance \( \epsilon = \gamma E \) (for \( \gamma = c \)) which is conserved if there is no multiple scattering or other random process.
In the absence of collision phenomena, the conservation of invariant emittance under acceleration implies

\[ \frac{d\varepsilon}{dz} = \frac{d}{dz} (\gamma \varepsilon) = 0 \]

and hence

\[ \left( \frac{d\varepsilon}{dz} \right)_{ad} = -\frac{E}{\gamma} \frac{dv}{dz} \] \hspace{1cm} (18)

This so-called adiabatic damping arises from the use of co-ordinates \( x, x' \) which are not canonically conjugate, resulting in the absolute emittance \( E \) decreasing with increasing energy. This is an important consideration in obtaining high luminosity in a high-energy linear collider.

Emittance growth due to multiple scattering is in competition with adiabatic damping and results in an increase in the (otherwise) invariant emittance \( \varepsilon \). A convenient formulation of such growth produced by gas scattering in a synchrotron has been given by Hardt\(^{10}\). From the Fokker-Planck equation for a Gaussian distribution in a 4-dimensional phase space he obtains a diffusion equation in each of the two transverse space projections. Using the conventional definition of electron-beam emittance at one standard deviation \( \sigma_x \)

\[ E = \frac{\sigma_x^2}{\beta_x} \] \hspace{1cm} (19)

where \( \beta_x \) is the betatron focusing function, the diffusion equation for one plane can be written in terms of the emittance growth rate

\[ \left( \frac{d\varepsilon}{dz} \right)_{diff} = \frac{\beta_x}{2} \frac{d}{dz} \langle \varepsilon^2 \rangle_p \]

and adding the adiabatic damping term, Eq.(18), yields

\[ \frac{d\varepsilon}{dz} = \left( \frac{d\varepsilon}{dz} \right)_{diff} + \left( \frac{d\varepsilon}{dz} \right)_{ad} \]

\[ = \frac{\beta_x}{2} \frac{d}{dz} \langle \varepsilon^2 \rangle_p - \frac{E}{\gamma} \frac{dv}{dz} \]
It follows immediately that

\[
\frac{d\epsilon}{dz} = \frac{d}{dz} (\gamma E) = \frac{\gamma B_x}{2} \frac{d}{dz} \langle \sigma^2 \rangle p ,
\]

which, from Eq.(17) with \( p = mc\gamma \) results in

\[
\frac{d\epsilon}{dz} = \frac{\pi B_x \eta}{2\gamma} \left( \frac{2e^2}{mc^2} \right)^2 \ln \left( \frac{\lambda p}{R} \right).
\]

(20)

Transverse focusing of the accelerated beam can occur due to radial fields in the plasma, which arise because of the finite lateral extent of the laser beams, and therefore of the plasma beat wave. The consequent radial dependence of the accelerating field \( E_z \) is associated, through Maxwell's equations, with a radial field which is focusing over a certain range of phase.

The corresponding betatron focusing function \( \beta_x \) has been derived in Ref.4 and is given by

\[
\beta_x = \left[ \frac{-\pi \sigma_0^2 mc^2 \gamma}{\lambda_p E_z \cos \phi} \right]^{1/2}
\]

(21)

where \( \sigma_0 \) is the standard deviation of the laser-beam cross-section, \( \lambda_p \) is the wavelength of the plasma beat-wave and \( \phi \) is the accelerating phase angle. It is assumed that both the laser beam and the resultant beat-wave have the same Gaussian distribution transversely. The accelerating field on the axis is given by \( E_z \sin \phi \). As discussed in Ref.4 the phase angle \( \phi \) is constrained to the range

\[
\frac{\pi}{2} < \phi < \frac{13\pi}{16}
\]

because of acceleration, radial stability and phase slip between accelerated bunches and beat wave.

In Eq.(20) we replace \( \beta_x \) from Eq.(21), write \( r_e = e^2/mc^2 \) for the classical electron radius and obtain

\[
\frac{d\epsilon}{dz} = F/\sqrt{\gamma}
\]

(22)

where
\[ F = 2 \pi r_e^2 n \left[ \frac{-\kappa_0^2 m_e^2}{\lambda_p e E_0 \cos \phi} \right]^{1/2} \ln \left( \frac{\lambda_D}{R} \right) \] (23)

has no explicit energy dependence. Assuming a constant acceleration rate \( e E_0 \sin \phi \) we can write

\[ \gamma = \gamma_i + \gamma' z \]

where \( \gamma_i \) corresponds to injection energy and

\[ \gamma' = \frac{dv}{dz} = \frac{e E_0 \sin \phi}{m_e c^2} . \] (24)

Integrating Eq.(22) from \( \gamma_i \) to the final energy \( \gamma_f \) leads to a growth of emittance \( \Delta \varepsilon \) given by

\[ \Delta \varepsilon = \frac{2F}{\gamma_i} \left[ \sqrt{\gamma_f} - \sqrt{\gamma_i} \right] \] (25)

where, by rearranging the factors

\[ \frac{2F}{\gamma_i} = 4 \pi r_e^2 \sigma_0 n \left[ \frac{m_e c^2}{e E_0 \sin \phi} \right]^{3/2} \left[ \frac{\kappa_0^2 m_e^2}{\lambda_p} \right]^{1/2} \ln \left( \frac{\lambda_D}{R} \right) . \] (26)

To evaluate Eq.(26) we take the following parameters adopted at the Frascati Workshop\(^5\):

\[ n = 10^{17} \text{cm}^{-3} \]
\[ e E_0 \sin \phi = 60 \text{ MeV cm}^{-1} \]
\[ \lambda_p = 10^{-2} \text{ cm} \]
\[ \sigma_0 = 2.5 \times 10^{-2} \text{ cm} . \]

The Debye wavelength is given by

\[ \lambda_0 = \left[ \frac{kT}{4 \pi n e^2} \right]^{1/2} = \left[ \frac{kT}{4 \pi n m_e c^2 r_e} \right]^{1/2} \]

where \( k \) is Boltzmann's constant and \( T \) the temperature of the plasma electrons. Assuming a relatively cool plasma of 5 eV (~ \( 5.8 \times 10^4 \) K) and with

\[ r_e = 2.818 \times 10^{-13} \text{ cm} \]
\[ m_e c^2 = 0.511 \times 10^6 \text{ eV} , \]
we have:

\[ \lambda_D = 5.26 \times 10^{-6} \text{cm}. \]

Taking \( R = 0.7 \times 10^{-13} \text{cm} \) for the effective radius of the proton leads to

\[ \ln \left( \frac{\lambda_D}{R} \right) = 18.13. \]

For \( \phi \) we choose the average over the phase slip between \( \pi/2 \) and \( 13\pi/16 \) for a single stage of the BWA, i.e. \( \phi = 10.5 \pi/16 \), and find that Eq.(26) evaluates to

\[ \frac{2 \Phi_{\gamma}}{\gamma^2} = 8.62 \times 10^{-10} \text{cm}. \]

Then, with \( \gamma_f = 1.96 \times 10^6 \) (1 TeV) and \( \gamma_i = 1.96 \times 10^4 \) (10 GeV),

\[ \Delta \varepsilon = 1.086 \times 10^{-6} \text{cm} = 1.086 \times 10^{-8} \text{m}. \]

This can be compared with the nominal invariant emittance \( \varepsilon = 3 \times 10^{-5} \text{m} \) of the SLAC Linear Collider leading to

\[ \frac{\Delta \varepsilon}{\varepsilon} = 3.62 \times 10^{-4}. \]

This is a very small emittance growth and gives no cause for concern with the parameters assumed.

4. DISCUSSION

The above results are rather insensitive to the injection energy since reducing \( \gamma_i \) even to 1 in Eq.(25) only increases \( \Delta \varepsilon \) by about 10%. In fact, this increase would be somewhat less, because at lower injection energies we should use \( \theta_{\max} \) from Eq.(4) rather than Eq.(5). The logarithm in Eq.(26) would then be replaced by

\[ \ln \left( \frac{2\gamma \lambda_D \gamma}{\lambda_C} \right) \]

which decreases with decreasing energy.
If we use the SLC emittance as a reference there is considerable freedom in the choice of SWA parameters without appreciable degradation of performance due to multiple scattering in the plasma. This also implies that, if injected beams of appreciably lower emittance were available with the required intensity, there are potential benefits to be obtained in terms of luminosity and beam power before emittance growth becomes a serious constraint.

In the present evaluation no account has been taken of possible fluctuations of plasma density and focusing forces, nor of the influence of contamination of the plasma by high-Z impurities. With the comparison used here it is unlikely that such effects will be significant, but it may be necessary to scrutinise them more carefully if extremely low-emittance beams were contemplated. Furthermore, the assumption that the quantum fluctuations of synchrotron radiation produced by the focusing fields have a negligible influence on emittance growth requires proper verification.
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