Modal cut-off and the $V$–parameter in photonic crystal fibers

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We address the long-standing unresolved problem concerning the $V$–parameter in a photonic crystal fiber (PCF). Formulate the parameter appropriate for a core-defect in a periodic structure we argue that the multi-mode cut-off occurs at a wavelength $\lambda^*$ which satisfies $V_{SIF}(\lambda^*) = \pi$. Comparing to numerics and recent cut-off calculations we confirm this result.

In photonic crystal fibers (PCFs) an arrangement of air-holes running along the full length of the fiber provides the confinement and guidance of light. The air-holes of diameter $d$ are typically arranged in a triangular lattice[1] with a pitch $\Lambda$ (see insert in Fig. 2), but e.g. honey-comb[2] and kagome[3, 4] arrangements are other options. By making a defect in the lattice, light can be confined and guided along the fiber axis. The guidance mechanism depends on the nature of the defect and the air-hole arrangement. For the triangular lattice with a silica-core light is confined by total-internal reflection[1] whereas for an air-core a photonic-bandgap confines light to the defect.[5] For recent reviews we refer to Ref. 6 and references therein.

Both type of PCFs have revealed surprising and novel optical properties. In this work we consider the silica-core PCF (see insert in Fig. 2) which was the one first reported.[1] This structure provides the basis of a variety of phenomena including the endlessly single-mode behaviour,[5] large-mode area PCFs,[6] as well as highly non-linear PCF with unique dispersion properties.[7–10]

Properties of standard fibers are often parametrized by the so-called $V$–parameter and the entire concept is very close to the heart of the majority of the optical fiber community (see e.g. Refs. 12, 13). The cut-off properties and the endlessly single-mode phenomena of PCFs can also be qualitatively understood within this framework.[1–4, 9, 13] However, the proper choice of the correct length scale for the $V$–parameter has, until now, remained unsolved as well as the value of $V^*$ that marks the second-order cut-off. In this Letter we clarify this problem and also put recent work on multi-mode cut-off[16, 17] into the context of the $V$–parameter.

The tradition of parametrizing the optical properties in terms of the $V$–parameter stems from analysis of the step-index fiber (SIF). The SIF is characterized by the core radius $\rho$, the core index $n_c$, and the cladding index $n_{cl}$ which all enter into the parameter $V_{SIF}$ given by

$$V_{SIF}(\lambda) = \frac{2\pi \rho}{\lambda} \sqrt{n_c^2 - n_{cl}^2}. \quad (1)$$

Because of its inverse dependence on the wavelength $\lambda$, this quantity is often referred to as the normalized frequency. However, in a more general context, this is somewhat misleading (especially if $n_c$ and/or $n_{cl}$ has a strong wavelength dependence) and in this Letter we would like to emphasize a more physical interpretation. To do this, we first introduce the numerical aperture NA (or the angle of divergence $\theta$) given by

$$NA = \sin \theta = \sqrt{n_c^2 - n_{cl}^2} \quad (2)$$

which follows from use of Snell’s law for critical incidence at the interface between the $n_c$ and $n_{cl}$ regions (see e.g. Refs. 12, 13). Next, we introduce the free-space wave-number $k = 2\pi/\lambda$ and its transverse projection $k_\perp = k \sin \theta$. The $V$–parameter can now simply be written as

$$V_{SIF} = k_\perp \rho. \quad (3)$$

From this form it is obvious why the parameter carries information about the number of guided modes; the natural parameter describing the transverse intensity distribution is nothing but $k_\perp \rho$. Furthermore, for the second-order cut-off wavelength $\lambda^*$ the usual value $V_{SIF}(\lambda^*) = V_{SIF}^* \approx 2.405$ follows naturally from the solution of the first zero of the Bessel function, i.e. $J_0(V_{SIF}^*) = 0$.

In general, for wave-propagation in confined structures the number $k_\perp \rho$ has a very central role. The transmission cross-section of a narrow slit[19] is an example and counterparts of the electro-magnetic problem can also be seen in e.g. electronic systems like the quantum-point contact where $k_\perp \rho$ also determines the number of modes (see e.g. Ref. 24). In the context of PCFs it is also natural to consider a $V$–parameter which was done already in the seminal work by the Bath–group[1] and in the subsequent work on endlessly single-mode properties[7] and

![FIG. 1: Schematics of the cross-section of a PCF. The dashed line illustrates the field-amplitude of a second-order mode with a single node.](image)
effective $V$–values. However, in attempt of adopting Eq. (1) to PCFs one is faced with the problem of choosing a value for $\rho$ and in Refs. [7, 11] it was emphasized that one may choose any transverse dimension. In this Letter, we point out that the problem is not a matter of defining a core-radius, but rather one should look for the natural length-scale of the problem; the air-hole pitch $\Lambda$. This choice was also suggested in Ref. [5] though considered an arbitrary choice. Regarding the second-order cut-off it was in Refs. [14] suggested that $VPCF \approx 2.5$ but it was also concluded that the arbitrary choice of the length scale means that the particular number for $VPCF$ also becomes somewhat arbitrary. In this Letter, we demonstrate that this is not the case and that a very simple and elegant solution exists.

To show this, we introduce the following $V$–parameter for a PCF

$$VPCF(\lambda) = \frac{2\pi \Lambda}{\lambda} \sqrt{n_c^2(\lambda) - n_{cl}^2(\lambda)}$$

(4)

where $n_c(\lambda) = c\beta/\omega$ is the “core index” associated with the effective index of the fundamental mode and similarly $n_{cl}(\lambda)$ is the effective index of the fundamental space-filling mode in the triangular air-hole lattice. The second-order cut-off occurs at a wavelength $\lambda^*$ where the effective transverse wavelength $\lambda_{\perp} = 2\pi/k_{\perp}$ allows a mode with a single node (see schematics in Fig. 1) to fit into the defect region, i.e. $\lambda_{\perp}^* \approx 2\Lambda$. Writing Eq. (4) in terms of $k_{\perp}$ the corresponding value of $VPCF^*$ easily follows

$$VPCF^* = k_{\perp}^* \Lambda = \frac{2\pi}{\lambda_{\perp}^*} \Lambda = \pi.$$  

(5)

Though this derivation may seem somewhat heuristic we shall compare to numerical results and show that the very central number $\pi$ is indeed the correct value.

For the numerical comparison we need to calculate both $VPCF(\lambda)$ and the second-order cut-off $\lambda^*$. For the $V$–parameter we use a fully-vectorial plane-wave method [21] to calculate $n_c(\lambda)$ and $n_{cl}(\lambda)$ for various air-hole diameters. For the material refractive index we use $n = 1$ for the air-holes and $n = 1.444$ for the silica. Ignoring the frequency dependence of the latter, the wave equation becomes scale-invariant [22] and all the results to be presented can thus be scaled to the desired value of $\Lambda$. Regarding the cut-off, one of us recently suggested a phase diagram for the single and multi-mode operation regimes [17] which was subsequently followed up in more detail by Kuhlme et al. [18]. From highly accurate multipole solutions of Maxwell’s equations, it was numerically found that the single/multi-mode boundary can be accounted for by the expression [18]

$$\lambda^*/\Lambda \simeq \alpha (d/\Lambda - d^*/\Lambda)\gamma.$$  

(6)

Here, $\alpha \simeq 2.80 \pm 0.12$, $\gamma \simeq 0.89 \pm 0.02$, and $d^*/\Lambda \simeq 0.406$. This phase-boundary is shown by the solid line in panel (a) of Fig. 2 and it has recently been confirmed experimentally based on cut-off measurements in various

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**FIG. 2:** Panel (a) shows the single/multi-mode phase diagram. The solid line shows the phase-boundary of Kuhlme et al. [18] [Eq. (6)] and the circles indicate solutions to $VPCF(\lambda^*) = \pi$ [Eqs. (4,5)]. Panel (b) shows numerical results for PCFs with varying hole diameter ($d/\Lambda = 0.43, 0.44, 0.45, 0.475, 0.50, 0.55, 0.60, 0.65$, and $0.70$ from below). The full lines show results for the $V$–parameter [Eq. (4)], the circles indicate the corresponding cut-off wavelengths [Eq. (6)], and the dashed line shows $VPCF^*$ [Eq. (5)].
For \( d/\Lambda < d^*/\Lambda \) the PCF has the remarkable property of being so-called endlessly single-mode\(^7\) and for \( d/\Lambda > d^*/\Lambda \) the PCF supports a second-order mode at wavelengths \( \lambda/\Lambda < \lambda^*/\Lambda \) and is single-mode for \( \lambda/\Lambda > \lambda^*/\Lambda \).

In panel (b) of Fig. 2 we show numerical results for various values of \( d/\Lambda \). The full lines show results for the \( V \)–parameter, Eq. (4), the circles indicate the corresponding cut-off wavelengths, Eq. (6), and the dashed line shows \( V_{PCF} \), Eq. (5). First of all we notice that the cut-off results of Kuhlmey et al.,\(^{18}\) Eq. (6), agrees with a picture of a constant \( V \)–value \( V^*_{PCF} \) below which the PCF is single-mode. This similarity with SIFs indicates that the cut-off in SIFs and PCFs rely on the same basic physics. Furthermore, it is also seen that the cut-off points are in excellent agreement with the value \( V^*_{PCF} = \pi \), Eq. (6), and this also supports the idea of \( \Lambda \) as the natural length scale for the \( V \)–parameter.

In conclusion we have shown that the multi-mode cut-off in PCFs can be understood from a generalized \( V \)–parameter and that the single-mode regime is characterized by \( V_{PCF} < V^*_{PCF} = \pi \).

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References: