Formation of cosmological mass condensation within a FRW universe: exact general relativistic solutions

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Abstract

Within the framework of an exact general relativistic formulation of gluing manifolds, we consider the problem of matching an inhomogeneous overdense region to a Friedmann-Robertson-Walker background universe in the general spherical symmetric case of pressure-free models. It is shown that, in general, the matching is only possible through a thin shell, a fact ignored in the literature. In addition to this, in subhorizon cases where the matching is possible, an intermediate underdense region will necessarily arise.

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Large scale structure formation is a challenging field from both theoretical and observational points of view. Motivated by the observational data on the existence of large scale voids in the universe, Sato and co-workers considered in a series of papers an underdense spherical region immersed in a Friedmann-Robertson-Walker (FRW) background universe applying the thin shell formalism of general relativity[1]. On the other hand, the appearance of holes with densities less than the average cosmological density around rich spherical clusters of galaxies has been seen by some authors [2-4]. In this paper, we model such inhomogeneous large scale structures as a Lemaitre-Tolman-Bondi (LTB) manifold with different density profiles glued to a homogeneous pressure-free FRW background from which a sphere of dust matter is removed. Our calculation is based on an exact general relativistic formulation of gluing manifolds. We may therefore consider our contribution as a generalization of the work done by Olson and Silk[3]. Their calculation was made basically within Newtonian dynamics without being cautious about the matching conditions.

Consider first a spherical inhomogeneous region containing dust matter, represented by a LTB cosmological model embedded in a pressure-free FRW background universe with the uniform density $\rho_b$. We choose the LTB metric to be written in the synchronized comoving coordinates in the form[5]

$$ds^2 = -dt^2 + \frac{R^2}{1 + E(r)} dr^2 + R^2(r,t)(d\theta^2 + \sin^2 \theta d\phi^2).$$  

(1)

The overdot and prime denote partial differentiation with respect to $t$ and $r$, respectively, and $E(r)$ is an arbitrary real function such that $E(r) > -1$. Then the corresponding Einstein field equations turn out to be

$$\dot{R}^2(r,t) = E(r) + \frac{2M(r)}{R^2},$$  

(2)

$$4\pi \rho(r,t) = \frac{M'(r)}{R^2 R''}. $$  

(3)

To be as general as possible, we let the density $\rho(r,t)$ to be a general function of $r$, i.e., it is not necessarily a monotonic function of radial distance from the center of the region with mass condensation. $M(r)$ is defined by

$$M(r) = 4\pi \int_0^{R(r,t)} \rho(r,t) R^2 dR.$$  

(4)

Furthermore, in order to avoid shell crossing of dust matter during their radial motion, we must have $R'(r,t) > 0$. The solution to the above equations shows that an overdense spherical inhomogeneity with $E(r) < 0$ within $R$ evolves just like a closed universe, namely it reaches to a maximum radius at a certain time, then the expansion ceases and undergoes a gravitational collapse so that a bound object forms in such a way.

Now, in the literature on large scale structures of the universe it is always tacitly
assumed that the junction of an overdense region to a background FRW universe is a continuous one and there is no need of a singular hypersurface along which the gluing is made[6]. Let us denote by $\Sigma$ the (2+1)-dimensional timelike boundary of the two distinct spherically symmetric regions glued together. We will show that this is in general not true and in the most cases of interest $\Sigma$ cannot be just a boundary surface but is a singular hypersurface carrying energy momentum.

We now write down the appropriate junction equation on $\Sigma$ expressing the jump of the angular component of extrinsic curvature tensor $K_\theta^\theta$ across $\Sigma$ as [7,8]

$$\epsilon_{in}\sqrt{1 + \left(\frac{dR}{d\tau}\right)^2 - \frac{8\pi \bar{p}}{3} R^2} - \epsilon_{out}\sqrt{1 + \left(\frac{dR}{d\tau}\right)^2 - \frac{8\pi \rho_b}{3} R^2} \equiv 4\pi\sigma R,$$  

(5)

where $\equiv$ means that both sides of the equality are evaluated on $\Sigma$, $\tau$ is the proper time of the comoving observer on $\Sigma$, and $\sigma$ is the surface energy density of the shell $\Sigma$. The sign functions are fixed according to the convention $\epsilon_{in}(\epsilon_{out}) = +1$ for $R$ increasing in the outward normal direction to $\Sigma$, while $\epsilon_{in}(\epsilon_{out}) = -1$ for decreasing $R$. An average density for the inhomogeneous region is also defined as

$$\overline{\rho} = \left. \frac{M(r)}{\frac{4\pi}{3} R^3} \right|_{\Sigma}.$$  

(6)

Note that in general the matching is only possible if a thin layer is formed on the boundary of the two manifolds, i.e. $\sigma \neq 0$, except for the case where $\overline{\rho} = \rho_b$, and $\epsilon_{in} = \epsilon_{out}$.

Now, similar to the approach used by Berezin et al [8], we solve Eq. (5) for $\left(\frac{dR}{d\tau}\right)^2$ to obtain

$$\left(\frac{dR}{d\tau}\right)^2 = \left(4\pi^2 \sigma^2 (\xi - 1)^2 + \frac{8\pi}{3} \rho_b\right) R^2 - 1,$$  

(7)

where $\xi$ is defined by

$$\xi \equiv \frac{\rho_b - \overline{\rho}}{6\pi\sigma^2}.$$  

(8)

Noting that the surface energy density $\sigma$ on $\Sigma$ is positive, we substitute Eq. (7) back into Eq. (5) to get

$$\epsilon_{in}\left|\xi + 1\right| - \epsilon_{out}\left|\xi - 1\right| = 2.$$  

(9)

It is easily seen that the sign functions $\epsilon_{in/out}$ are determined by the value of $\xi$:

$$(\epsilon_{in}, \epsilon_{out}) = \begin{cases} (+1, +1) & \text{for } \xi > 1, \\ (+1, -1) & \text{for } |\xi| < 1, \\ (-1, -1) & \text{for } \xi < -1. \end{cases}$$  

(10)
Now, $\epsilon_{out}$ may be related to different parameters of FRW universe. It is known that \[9\]

$$\epsilon_{out} = \text{sgn} \left( \frac{dr_b}{d\chi_b} + v_b H_b R \right),$$

(11)

where $H_b$ is the Hubble parameter of the background, $v_b$ is the peculiar velocity of $\Sigma$ relative to the background. The two coordinates $r_b$ and $\chi_b$ are related to each other as

$$r_b(\chi_b) = \begin{cases} 
\sin \chi_b & (k = +1, \text{ closed universe}), \\
\chi_b & (k = 0, \text{ flat universe}), \\
\sinh \chi_b & (k = -1, \text{ open universe}).
\end{cases}$$

(12)

Depending on the density distribution within the shell, the following cases may now be distinguished:

(I) Junction of an overdense region to an background FRW universe, i.e., $\rho(r, \tau) > \rho_b$. Obviously $\xi < 0$. Therefore, according to Eq. (10), we must have $\epsilon_{out} = -1$. This leads to the following cases:

1. $k = 0, -1; R < H_b^{-1}$. In both these cases it is easily seen from Eqs. (11),(12) that $\epsilon_{out} = +1$. Therefore, there is no matching of an overdense region smaller than the Hubble radius to a background FRW universe with $k = 0, -1$.

2. $k = 0, -1; R > H_b^{-1}$, and the peculiar velocity $v_b < 0$. It is seen from Eq. (11) that $\epsilon_{out} = -1[9]$. Therefore, overdense structures with a radius greater than the Hubble radius and $v_b < 0$ may be glued to a background FRW with $k = 0, -1$. It is trivially seen that there is no matching for the case of $v_b > 0$.

(II) $\bar{\rho} < \rho_b; k = 0, -1$. We have therefore an overdense inhomogeneous region surrounded continuously by an underdense region $\rho(r, t) < \rho_b$, i.e., a void described by the same LTB metric. Now, note that the junction of the LTB region at the underdense void part of it to the background FRW is also described by Eq. (5), while $\bar{\rho}$ is the overall average density related to the total mass enclosed within the boundary separating the hole region of physical radius $R$ from the uniform background.

1. $R < H_b^{-1}$. It is seen from Eq. (11) that $\epsilon_{out} = +1$, irrespective of the sign of the peculiar velocity. Therefore, according to Eq. (10), the matching is possible with $\xi > 1$ via a thin shell having $\sigma \neq 0$.

2. $R > H_b^{-1}$. In this case for the negative peculiar velocity we have $\epsilon_{out} = -1$. Therefore the matching is possible with $0 < \xi < 1$, i.e., $\sigma \neq 0$. If the peculiar velocity is positive we are faced with a case exactly the same as (1).
(III) \( \bar{\rho} = \rho_b; k = 0, -1 \). Now, the void completely compensates the overdense region, so that the overall mean density is equal to the background density.

(1) \( R < H_b^{-1} \). It turns out that \( \sigma = 0 \), i.e., the matching is possible without formation of a thin shell. This situation reminds us of the Einstein-Strauss type model [10] where the overdense region is surrounded by the vacuum shell to compensate the mass excess.

(2) \( R > H_b^{-1} \). For the negative peculiar velocity, we have again \( \epsilon_{out} = -1 \). Both junctions with \( \sigma = 0 \) and \( \neq 0 \) are possible, as it is easily seen from Eq. (5). The case \( \epsilon_{in} = -1 \) corresponds to \( \sigma = 0 \) meaning no thin shell, and the case \( \epsilon_{in} = +1 \) is only possible through a thin shell. The case of positive peculiar velocity is similar to the (1) case.

(IV) \( \bar{\rho} > \rho_b; k = 0, -1 \). This is similar to the matching where no void is in between, as discussed in the cases considered in (I).

Table I summarizes the above results.

Table I. Classification of junctions of LTB spherical inhomogeneity to the flat and open FRW background. An extensive version of the paper with the detailed calculation and critical review of the relevant literature is going to be published in [11].

<table>
<thead>
<tr>
<th>( \bar{\rho} )</th>
<th>Junction impossible</th>
<th>Junction impossible</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R &gt; H_b^{-1}, v_p &lt; 0 )</td>
<td>Thin shell ( \xi &lt; 1 )</td>
<td>No thin shell ( \epsilon_{in} = -1 )</td>
</tr>
<tr>
<td>( R &lt; H_b^{-1} ) or ( R &gt; H_b^{-1}, v_p &gt; 0 )</td>
<td>No thin shell ( \epsilon_{in} = +1 )</td>
<td>Thin shell ( \xi &gt; 1 )</td>
</tr>
<tr>
<td>( \bar{\rho} = \rho_b )</td>
<td>Thin shell ( \xi &gt; 0 )</td>
<td>Thin shell ( \xi &gt; 1 )</td>
</tr>
<tr>
<td>( \bar{\rho} &lt; \rho_b )</td>
<td>Thin shell ( 0 &lt; \xi &lt; 1 )</td>
<td>Thin shell ( \xi &gt; 1 )</td>
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References