Cosmological constant and gravitational theory on D-brane

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In a toy model we derive the gravitational equation on a self-gravitating curved D-brane. The effective theory on the brane is drastically changed from the ordinal Einstein equation. The net cosmological constant on the brane depends on a tuning between the brane tension and the brane charges. Moreover, non-zero matter stress tensor exists if the net cosmological constant is not zero. This fact indicates a direct connection between matters on the brane and the dark energy.

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I. INTRODUCTION

Recent observations tell us the existence of the cosmological constant or/and dark energy [1,2,3], and its origin is definitely important now (See Ref. [4] for the review). Surprisingly its energy density is almost the same order of magnitude as the present mass density of the universe. This fact indicates a connection between the dark energy and matter. In this paper we will focus on it and show the direct connection between the presence of the matter and dark energy in the brane world context.

Our idea comes from the brane world based on D-brane. In Ref. [5], we have carefully investigated the gravitational theory on the self-gravitating D-brane in type IIB supergravity on $S^5$ which may be a realistic model for D-branes. The conclusion obtained there is that matters do not appear as a source term of the Einstein equation on the D-brane. In that analysis we supposed that the vacuum energy on the brane is zero. From that observation we can obtain the following naive conjecture: matters are localized on the brane in gravitational point of view if there is non-zero vacuum energy on the brane. That is, the presence of matter accompanied by the appearance of the vacuum energy. This might be regarded as an evidence of the tight connection between the matter and dark energy.

In the original model of type IIB supergravity on $S^5$, there are undesirable behavior of the bulk geometry due to the presence of the scalar fields related to the dilaton and the compactification from ten to five dimensions. Indeed the single brane model proposed by Randall and Sundrum [6] cannot be realized in the sense that the bulk geometry is not warped enough to make the volume of extra dimensions finite and the four dimensional Einstein gravity cannot be recovered even at low energy. Hence we will discuss in a toy model which is slightly simplified model keeping the essence for the above remarkable result. In our model, one can see that the bulk geometry looks like anti-de Sitter spacetime which is appropriate for the Randall-Sundrum II model.

The remainder of this paper is organized as follows. In Sec. II, we describe our toy model and write down the basic equations. In Sec. III we solve the bulk geometry with the appropriate boundary conditions on the brane in long-wave approximation [7]. Using the result obtained in Sec. II and III, we derive the gravitational theory on the brane in Sec. IV. The theory has quite different form from the ordinary theory in four dimensions. Finally we summarize the present work and give discussion in Sec. V.

II. MODEL

A. The action for toy model

We are interested in the gravitational theory on D3-branes. For simplicity we work with the toy model

$$S = \frac{1}{2\kappa^2} \int d^5 x \sqrt{-g} \left[ \frac{\alpha}{2} R - 2\Lambda - \frac{1}{2} |H|^2 \right]$$

$$- \frac{1}{2} (\nabla \chi)^2 - \frac{1}{2} |\tilde{F}|^2 - \frac{1}{2} |\tilde{G}|^2 + S_{brane} + S_{CS}, \quad (1)$$

where $H_{MNK} = \frac{2}{\kappa^2} \partial_M B_{NK}$, $F_{MNK} = \frac{1}{4} \partial_M C_{NK}$, $G_{K_i K_j K_k K_l} = \frac{1}{4} \partial_K D_{K_i K_j K_k K_l}$, $\tilde{F} = F + \chi H$ and $\tilde{G} = G + C \wedge H$. $M, N, K = 0, 1, 2, 3, 4$. $B_{MN}$ and $C_{MN}$ are 2-form fields, and $D_{K_i K_j K_k K_l}$ is the 4-form field. $S_{brane}$ is given by the Born-Infeld action $[8]$

$$S_{brane} = \beta \int d^4 x \sqrt{-\det(h + \tilde{F})}, \quad (2)$$

where $h_{\mu\nu}$ is the induced metric on the D-brane and

$$F_{\mu\nu} = B_{\mu\nu} + (-\beta)^{-1/2} F_{\mu\nu}, \quad (3)$$

and $F_{\mu\nu}$ is the $U(1)$ gauge field on the brane. $\mu, \nu = 0, 1, 2, 3$. $S_{CS}$ is Chern-Simons action

$$S_{CS} = \gamma \int d^4 x \sqrt{-h} \epsilon^{\mu\nu\rho\sigma} \left[ \frac{1}{4} F_{\mu\nu} C_{\rho\sigma} + \frac{\chi}{8} F_{\mu\nu} F_{\rho\sigma} ight]$$

$$+ \frac{1}{24} D_{\mu\nu\rho\sigma}. \quad (4)$$

In general $\gamma \neq \beta$. We will show the difference is essential to recover the ordinary four dimensional Einstein gravity.
on the brane. Setting
\[2\Lambda + \frac{5\kappa^2}{6} \gamma^2 = 0 \quad (5)\]
as well as \(\beta = \gamma\), we obtain the results in Ref. [3].

In our toy model (1), there are no scalar fields related to the dilaton and compactification while the model in Ref. [2] includes these fields. This is because we can see that the scalar fields do not play an important role but make the calculations complicated in Ref. [5]. Instead we introduced the bulk cosmological constant \(\Lambda\). On the other hand, the form fields are indispensable. As seen soon the background solution could be anti-de Sitter like spacetime due to the current simplification of the action and it is guaranteed that the four dimensional Einstein gravity is recovered on the brane.

For simplicity we set \(H_{\mu\nu\alpha} = 0\) and \(\tilde{F}_{\mu\nu\alpha} = 0\).

**B. Basic equations**

In this subsection we write down the basic equations and boundary conditions. Since we are interested in the effective theory on the brane, it is better for our purpose to adopt the bulk metric

\[ds^2 = dy^2 + g_{\mu\nu}(y, x) dx^\mu dx^\nu, \quad (6)\]

and perform \((1+4)\)-decomposition. \(y\) is the coordinate orthogonal to the brane.

The “evolutional” equations to the \(y\)-direction are

\[\partial_y K = R - \kappa^2 \left( (5) T_\mu^\nu - \frac{4}{3} (5) T_\mu^M \right) - K^2, \quad (7)\]

\[\partial_y \tilde{K}_\nu = \tilde{R}_\nu - \kappa^2 \left( (5) T_\nu^\mu - \frac{1}{4} \tilde{R} (5) T_\alpha^{\nu\alpha} \right) - \tilde{K}^2. \quad (8)\]

\[\partial^2_y \chi + 2D^2 \chi + K \partial_y \chi - \frac{1}{2} H_{y\alpha\beta} F_{y\alpha\beta} = 0, \quad (9)\]

\[\partial_y X^{y\mu\nu} + K X^{y\mu\nu} + \frac{1}{2} F_{y\alpha\beta} \tilde{G}^{y\alpha\beta\mu\nu} = 0, \quad (10)\]

\[\partial_y \tilde{F}^{y\mu\nu} + K \tilde{F}^{y\mu\nu} - \frac{1}{2} H_{y\alpha\beta} \tilde{G}^{y\alpha\beta\mu\nu} = 0, \quad (11)\]

\[\partial_y \tilde{G}_{y\alpha_1 \alpha_2 \alpha_3 \alpha_4} = K \tilde{G}_{y\alpha_1 \alpha_2 \alpha_3 \alpha_4}, \quad (12)\]

where \(X^{y\mu\nu} := H^{y\mu\nu} + \chi F^{y\mu\nu}\) and the energy-momentum tensor is

\[\kappa^2 (5) T_{MN} = \frac{1}{2} \left[ \nabla_M \nabla_N \chi - \frac{1}{2} g_{MN} (\nabla \chi)^2 \right] + \frac{1}{4} \left[ H_{MKL} H_N^{KL} - g_{MN} H^2 \right] + \frac{1}{4} \left[ \tilde{F}_{MKL} \tilde{F}_N^{KL} - g_{MN} \tilde{F}^2 \right] + \frac{1}{96} \tilde{G}_{MKL} \tilde{K}_K \tilde{K}_\ell \tilde{K}_N \tilde{K}_1 \tilde{K}_2 \tilde{K}_3 \tilde{K}_4 - \Lambda g_{MN}. \quad (13)\]

\(K_{\mu\nu}\) is the extrinsic curvature, \(K_{\mu\nu} = \frac{1}{2} \partial_y g_{\mu\nu}\). \(\tilde{K}_\nu\) and \(\tilde{R}_\nu\) are the traceless parts of \(K_\nu\) and \(R_\nu\), respectively.

The constrains are

\[-\frac{1}{2} \left[ R - \frac{3}{4} K^2 + \tilde{K}_\nu \tilde{K}_\nu \right] = \kappa^2 (5) T_{yy}, \quad (14)\]

\[D_\nu K_\mu - D_\mu K_\nu = \kappa^2 (5) T_{\nu\mu}, \quad (15)\]

\[D^\alpha X_{y\alpha\mu} = 0, \quad (16)\]

\[D^\alpha F_{y\alpha\mu} = 0, \quad (17)\]

\[D^\alpha \tilde{G}_{y\alpha_1 \alpha_2 \alpha_3 \alpha_4} = 0, \quad (18)\]

where \(D_\mu\) is the covariant derivative with respect to \(g_{\mu\nu}\). The junction conditions at the brane located \(y = 0\) are

\[\left[ K_{\mu\nu} - h_{\mu\nu} K \right]_{y=0} = -\frac{k^2}{2} \beta (h_{\mu\nu} - T_{\mu\nu}) + O(T^2_{\mu\nu}) \quad (19)\]

\[H_{y\mu\nu}(0, x) = -\kappa^2 \beta F_{\mu\nu}, \quad (20)\]

\[\tilde{F}_{y\mu\nu}(0, x) = -\frac{k^2}{2} \gamma \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \quad (21)\]

\[\tilde{G}_{y\mu\nu\alpha\beta}(0, x) = -\kappa^2 \gamma \epsilon_{\mu\nu\alpha\beta}, \quad (22)\]

\[\partial_y \chi(0, x) = -\frac{k^2}{8} \gamma \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \quad (23)\]

where

\[T_{\mu\nu} = F_{\mu\alpha} F^{\alpha\beta} - \frac{1}{4} h_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}. \quad (24)\]

The boundary conditions are essential for self-gravitating brane.

**III. LONG-WAVE APPROXIMATION**

To derive the effective gravitational theory on the brane, the effects from the bulk are also important and it is necessary to solve the bulk fields. In this paper we employ the gradient expansion scheme as Ref. [2]. We assume the following hierarchy in the order of magnitude

\[|T_{\mu\nu}^2| < |T_{\mu\nu}(1 - \beta/\gamma)| < (1 - \beta/\gamma)^2 \ll |T_{\mu\nu}| < |1 - \beta/\gamma| (25)\]
Note that we assume $\beta \leq \gamma < 0$. We will derive the gravitational theory up to the order of $O(T_{\mu\nu}(\gamma - \beta))$, that is, we will omit $O(T_{\mu\nu}^2)$ terms for simplicity. As we will see soon, $R_{\mu\nu}(h) = O(\gamma - \beta)$.

The bulk metric is written in Gaussian-normal coordinate again

$$
\text{ds}^2 = dy^2 + g_{\mu\nu}(y, x) dx^\mu dx^\nu. \tag{26}
$$

The induced metric on the brane will be denoted by $h_{\mu\nu} := g_{\mu\nu}(0, x)$ and then

$$
g_{\mu\nu}(y, x) = a^2(y) \left[ h_{\mu\nu}(x) + g_{\mu\nu}^{(1)}(y, x) + \cdots \right]. \tag{27}
$$

In the above $g_{\mu\nu}^{(1)}(0, x) = 0$ and $a(0) = 1$. In a similar way, the extrinsic curvature is expanded as

$$
K_{\nu}^\mu = K_{\nu}^{(0)} + K_{\nu}^{(1)} + K_{\nu}^{(2)} + \cdots. \tag{28}
$$

### A. 0th order

It is easy to obtain the zeroth order solutions. Without derivation we present them.

$$
K_{\nu}^{(0)} = -\frac{1}{\ell} \delta_{\nu}^\mu, \tag{29}
$$

$$
R_{\mu\nu}^{(0)}(h) = 0, \tag{30}
$$

$$
g_{\mu\nu}^{(0)} = a^2(y) h_{\mu\nu}(x) = e^{-\frac{\chi}{2} y} h_{\mu\nu}(x), \tag{31}
$$

where

$$
\frac{1}{\ell} = -\frac{1}{6} \kappa^2 \gamma. \tag{32}
$$

$\ell$ is the curvature scale of anti-deSitter like spacetimes. This represents the Randall-Sundrum tuning.

In addition,

$$
\tilde{G}_{y_1y_2a_3a_4} = -a^4 \kappa^2 \gamma \epsilon_{y_1y_2a_3a_4}, \tag{33}
$$

where $\epsilon_{y_1y_2a_3a_4}$ is the Levi-Civita tensor with respect to the induced metric $h_{\mu\nu}$ on the brane\textsuperscript{[16]}. The warp factor $a(y)$ behaves well for the localization of gravity on the brane, that is, we do not encounter a serious problem of the localization in our previous work\textsuperscript{[15]}.

### B. 1st order

The first order equations for $\tilde{F}_{\mu\nu}$ and $H_{\mu\nu}$ are

$$
\partial_{y}^{(1)} \tilde{F}_{\mu\nu} - \frac{1}{2a^4} H_{\gamma\alpha\beta} \tilde{G}_{\gamma\alpha\rho\mu\nu} h^{\alpha\rho} h^{\beta} = 0 \tag{34}
$$

and

$$
\partial_{y}^{(1)} H_{\mu\nu} + \frac{1}{2a^4} \tilde{F}_{\gamma\alpha\beta} \tilde{G}_{\gamma\alpha\rho\mu\nu} h^{\alpha\rho} h^{\beta} = 0. \tag{35}
$$

Together with the junction conditions the solutions are given by

$$
\tilde{F}_{\mu\nu}^{(1)} = -\kappa^2 \gamma a^{-6} F_{\mu\nu}, \tag{36}
$$

and

$$
R_{\mu\nu}^{(1)} = \frac{\kappa^2}{2} \gamma a^{-6} \epsilon_{\mu\nu\rho\sigma} F_{\alpha\beta} h^{\rho\alpha} h^{\sigma\beta}. \tag{37}
$$

Using these results the evolutional equation for the traceless part of the extrinsic curvature is

$$
\partial_{y}^{(1)} K_{\nu}^{\mu} = -K_{\nu}^{(2)} + \frac{1}{a^2} \tilde{F}_{\nu}^{(1)}(h) - \kappa^2 \gamma a^{-16} T_{\nu}^{(2)}, \tag{38}
$$

where $R_{\mu\nu}(h) = h^{\mu\alpha} R_{\alpha\nu}(h)$ is the Ricci tensor with respect to $h_{\mu\nu}$ and $T_{\nu}^{(2)} = h^{\alpha\nu} T_{\alpha\nu}$. The solution is

$$
K_{\nu}^{(1)}(y, x) = -\frac{\ell}{2a^2} \tilde{F}_{\nu}^{(1)}(h) + \frac{1}{2} \kappa^2 \gamma a^{-16} T_{\nu}^{(2)} + \chi_{\nu}^{(2)}(x)/a^4, \tag{39}
$$

where $\chi_{\nu}^{(2)}$ is the “integration of constant” and expresses the holographic CFT stress tensor\textsuperscript{[8]}. Since it is not affect the result below, we will omit $\chi_{\nu}^{(2)}$ hereafter.

The trace part of the extrinsic curvature can be evaluated from the Hamiltonian constraint as

$$
K^{(1)}(y, x) = -\frac{\ell}{6a^2} R(h). \tag{40}
$$

Finally, we obtain

$$
[K_{\nu}^{(1)} - \delta_{\nu}^{\mu} K]^{(1)} = -\frac{\ell}{2a^2} G_{\nu}^{(1)}(h) + \frac{1}{2} \kappa^2 \gamma a^{-16} T_{\nu}^{(2)}. \tag{41}
$$

From junction condition, it is easy to see that the Einstein equation up to 1st order becomes

$$
G_{\mu\nu}(h) = -\frac{\kappa^2}{\ell} (\gamma - \beta) h_{\mu\nu}. \tag{42}
$$

The stress energy tensor of the gauge field does not appear in this order. Thus we must discuss the next order to see how the ordinary gravitational equation recovers.

We calculate the metric at first order, $g_{\mu\nu}^{(1)}$, which will be used in the computation of the second order Ricci tensor $[R_{\mu\nu}^{(2)}]$. The result is

$$
g_{\mu\nu}^{(1)} = \frac{\ell^2}{2}(1 - a^{-2}) \left[ R_{\mu\nu}(h) - \frac{1}{6} h_{\mu\nu} R(h) \right]
+ \frac{3}{8} a^{-16} T_{\mu\nu}. \tag{43}
$$

### C. 2nd order

First of all, using Eq. $\text{(43)}$, we compute the second order Ricci tensor.
\[ [R_\nu^\mu]^{(2)} = \frac{\ell^2}{4} a^{-2}(1 - a^{-2}) \left[ D_\alpha D_\nu R^\mu_\alpha + D_\alpha D_\mu R^\nu_\alpha - D^2 R^\nu_\alpha - \frac{2}{3} \delta_\nu^\mu D_\nu R + \frac{1}{6} \delta_\nu^\mu D^2 R - 2 R^\mu_\alpha R^\nu_\alpha + \frac{1}{3} R_\nu^\mu R \right] \]
\[ + \frac{3}{16} a^{-2}(1 - a^{-16}) \left[ D_\alpha D_\nu T^{\alpha\mu} + D_\alpha D^\mu T^\alpha_\nu - D^2 T_\mu^\nu - 2 T^\mu_\alpha R^\nu_\alpha \right] \]
\[ \simeq a^{-2} \left[ \kappa^4 \frac{(\gamma - \beta)^2}{6} (a^2 - 1) \delta_\nu^\mu + \frac{3 \kappa^2}{8 \ell} (\gamma - \beta) (a^{-16} - 1) T^\mu_\nu - \frac{3}{16} (a^{-16} - 1) \left( D_\alpha D_\nu T^{\alpha\mu} + D_\alpha D^\mu T^\alpha_\nu - D^2 T_\mu^\nu \right) \right] . \] (44)

From the first to second line in the r.h.s we used \( R_{\mu\nu}(h) \simeq \frac{\kappa^4}{3} (\gamma - \beta) h_{\mu\nu} \). The second term in Eq. (44) is computed as
\[ \kappa^2 \left[ (5) T^\mu_\nu - \frac{1}{4} \delta_\nu^\mu (5) T^a_a \right]^{(2)} = -\kappa^4 \gamma (\gamma - \beta) (2a^{-18} - a^{-16}) T^\mu_\nu . \] (45)

Since we need the higher order solutions for \( \tilde{F}_{\mu\nu} \) and \( H_{\mu\nu} \), we derive them in the appendix. Then the evolutionary equation for \( \tilde{K}_\nu^\mu \) is
\[ \partial_\gamma \tilde{K}_\nu^\mu = \left[ R_\nu^\mu \right]^{(2)} - K \tilde{K}_\nu^\mu - \tilde{K}_\nu^\mu - \kappa^4 \gamma (\gamma - \beta) (2a^{-18} - a^{-16}) T^\mu_\nu . \] (46)

Its solution is easily found and the value on the brane becomes
\[ \tilde{K}_\nu^\mu = -\frac{3 \ell}{28} \left[ D_\alpha D_\nu T^\alpha_\mu + D_\alpha D_\mu T^\alpha_\nu - D^2 T^\mu_\nu \right] \]
\[ + \frac{\ell}{21} \kappa^4 \gamma (\beta - \gamma) T^\mu_\nu . \] (47)

From the Hamiltonian constraint
\[ \left[ R(h) \right]^{(2)} + \frac{3}{2} \kappa K + \frac{3}{4} K^2 - \tilde{K}_\nu^\mu \tilde{K}_\nu^\mu = \frac{1}{2} \kappa^4 \gamma (\beta - \gamma) \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} , \] (48)

we can compute the trace part of second order extrinsic curvature \( \tilde{K}^{(2)}_\nu \)
\[ \tilde{K}^{(2)}_\nu = -\frac{\ell}{2} \tilde{R}_\nu^\mu T^\mu_\nu - \frac{\ell}{12} \kappa^4 \gamma (\beta - \gamma) \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \]
\[ - \frac{\ell}{24} \left( \tilde{R}_\nu^\mu \tilde{R}^\mu_\nu - \frac{1}{3} \tilde{R}^2 \right) \]
\[ \simeq \frac{\kappa^4 \ell}{18} (\gamma - \beta)^2 + \frac{\kappa^4}{12} \gamma (\gamma - \beta) \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} . \] (49)

### IV. LOW ENERGY GRAVITATIONAL THEORY ON D-BRANE

#### A. General case

Now we are ready to derive the gravitational equation on the brane. From the junction condition, the effective equation is given by
\[ G_{\mu\nu}(h) = -\frac{\kappa^4}{7} (\gamma - \beta) h_{\mu\nu} - \frac{\kappa^4}{12} (\gamma - \beta)^2 h_{\mu\nu} \]
\[ + \frac{3 \kappa^2}{7 \ell} \left( 1 - \frac{\gamma}{\beta} \right) \tilde{T}^\mu_\nu \]
\[ + \frac{3 \kappa^2}{4 \ell} \left( 1 - \frac{\gamma}{\beta} \right) F_{\alpha\beta} F^{\alpha\beta} h_{\mu\nu} + \tau_{\mu\nu} . \] (50)

where we set \( B_{\mu\nu} = 0 \) and defined
\[ \tilde{T}^\mu_\nu := F_{\mu\alpha} F^{\nu\alpha} - \frac{1}{4} h_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} . \] (51)

\( \tau_{\mu\nu} \) is defined by
\[ \tau_{\mu\nu} = \frac{3}{14 \beta} \left( D_\alpha D_\nu T^{(F)}_\mu + D_\alpha D_\mu T^{(F)}_\nu - D^2 T^{(F)} \right) . \] (52)

Note that \( \tau_{\mu\nu} \) is traceless.

The first and second terms in the r.h.s of Eq. (50) are regarded as the (positive) vacuum energy. The third term is the energy-momentum tensor of the gauge field with the coupling which depends on \( \beta, \gamma, \ell \).

When \( \gamma = \beta \), the equation becomes vacuum Einstein equation
\[ G_{\mu\nu}(h) \simeq 0 . \] (53)

This is the result obtained in our previous study. In the above we used \( \tau_{\mu\nu} \propto (\gamma - \beta) \).

#### B. Homogeneous and isotropic universe

Let us examine the gravitational equation on the brane by considering the homogeneous and isotropic universe. The gauge field on the brane will be regarded as the radiation fluid. In this case, \( F^2 \simeq 0 \) and then the gravitational equation becomes
\[ G_{\mu\nu}(h) \simeq -\lambda_{\text{eff}} h_{\mu\nu} + S_{\mu\nu}^{\text{eff}} . \] (54)
where
\[ \lambda_{\text{eff}} := \frac{\kappa^2}{\ell} (\gamma - \beta) + \frac{\kappa^4}{12} (\gamma - \beta)^2, \]  
(55)

and
\[ S_{\mu\nu}^{\text{eff}} := \frac{3}{7} \frac{\kappa^2}{\ell} \left( 1 - \frac{\gamma}{\beta} \right) T_{\mu\nu} + \tau_{\mu\nu}. \]  
(56)

Since \( S_{\mu\nu}^{\text{eff}} \) is traceless, it behaves like radiation in homogeneous and isotropic universe.
\[ S_{00}^{\text{eff}} \approx \frac{1}{17} \frac{\kappa^2}{\ell} (1 - \gamma \beta) \rho, \]  
(57)

where \( \rho = \langle F \rangle \). Consequently we can observe that the conventional cosmology is recovered if and only if the non-zero (positive) vacuum energy exists.

V. SUMMARY AND DISCUSSION

In this paper, we derive the effective gravitational theory on a self-gravitating D-brane in the toy model. The brane is described by (the bosonic part of) Born-Infeld action and then we expect naively that the gauge field on the brane acts as the source term of the four dimensional Einstein gravity. The result is as follows. If the net vacuum energy is tuned to zero on the brane, the gauge field does not act as the source term in the gravitational theory on the brane. On the other hand, if the net positive vacuum energy is on the brane, the gauge field plays the source term on the brane. This implies that the existence of the cosmological constant on the brane is essential for the realization of the four dimensional Einstein gravity on the brane. This might give a new insight into a direct connection between matters and dark energy.

Finally we briefly comment on the unfamiliar term \( h_{\mu\nu} F^2 \) in Eq. (50). It is cosmology where gauge fields affect the spacetime in observational point of view. Under the fluid approximation in the homogeneous and isotropic universe, however, \( F^2 \approx 0 \). Thus the existence of such unfamiliar term is not inconsistent with observations. To judge our current model the study on fermionic part is desired.

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APPENDIX A: DERIVATION OF EQ. (45)

In the next order the equations for \( \tilde{F}_{y\mu\nu} \) and \( \tilde{H}_{y\mu\nu} \) becomes
\[ \partial_y \tilde{F}_{y\mu\nu} - \frac{3}{\ell} \tilde{H}_{y\alpha\beta} \epsilon^{\alpha\beta}_{\mu\nu} = 0 \]  
(A1)

and
\[ \partial_y \tilde{H}_{y\mu\nu} + \frac{3}{\ell} \tilde{F}_{y\alpha\beta} \epsilon^{\alpha\beta}_{\mu\nu} = 0. \]  
(A2)

The junction conditions are
\[ \tilde{F}_{y\alpha\beta} (0, x) = 0 \]  
(A3)

and
\[ \tilde{H}_{y\alpha\beta} (0, x) = \kappa^2 (\gamma - \beta) F_{\alpha\beta}. \]  
(A4)

It is easy to obtain the solutions as
\[ \tilde{F}_{y\alpha\beta} = \frac{\kappa^2}{4} (\gamma - \beta) (a^{-6} - a^6) \epsilon_{\mu}^{\alpha\beta} F_{\mu\nu} \]  
(A5)

and
\[ \tilde{H}_{y\alpha\beta} = \frac{\kappa^2}{2} (\gamma - \beta) (a^{-6} + a^6) F_{\mu\nu}. \]  
(A6)

Then we obtain the expression of Eq. (45).

[9] In the present study, the terms up to the order of $O(\mathcal{F}^2)$ are enough. The higher order correction will be not significant in our current approximation.
[10] $G_{\gamma\alpha\gamma\beta\gamma\delta\alpha\beta\gamma\delta\alpha\beta\gamma\delta}$ can be solved in full order: $G_{\gamma\alpha\gamma\beta\gamma\delta\alpha\beta\gamma\delta\alpha\beta\gamma\delta} = -\kappa^2 \gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}(g)$. Here $\gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}(g)$ is the Levi-Civita tensor with respect to $g_{\mu\nu}$.