Black diholes in five dimensions

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Abstract

Using a generalized Weyl formalism, we show how stationary, axisymmetric solutions of the four-dimensional vacuum Einstein equation can be turned into static, axisymmetric solutions of five-dimensional dilaton gravity coupled to a two-form gauge field. This procedure is then used to obtain new solutions of the latter theory describing pairs of extremal magnetic black holes with opposite charges, known as black diholes. These diholes are kept in static equilibrium by membrane-like conical singularities stretching along two different directions. We also present solutions describing diholes suspended in a background magnetic field, and with unbalanced charges.
1. Introduction

The Bonnor dipole solution in Einstein–Maxwell theory has been known for some time [1], although it was not until recently that Emparan [2] found it could be interpreted as a pair of extremal Reissner–Nordström black holes with magnetic charges of equal magnitude but opposite signs. To maintain the black holes in static equilibrium, there are in general conical singularities, interpreted as struts or cosmic strings, pushing or pulling on the two black holes. Alternatively, they can be held apart by introducing a background magnetic field. Emparan called such a configuration a ‘black dihole’, and in his paper examined a number of their properties.

The analog of the Bonnor dipole in Kaluza–Klein theory was found by Gross and Perry [3]. They obtained it by Euclideanizing the Kerr solution $t \rightarrow i x^5$, adding on a flat time direction, and then dimensionally reducing along $x^5$. It was subsequently generalized to a solution of Einstein–Maxwell–dilaton theory with a general dilaton coupling in [4, 5]. (A way to derive this solution using the Weyl formalism can be found in [6].) These dilatonic dihole solutions can be interpreted as pairs of extremal dilatonic black holes with opposite charges.

It is possible to generalize the dihole solution in various ways. For example, a dihole carrying unbalanced charges in Einstein–Maxwell–dilaton theory was derived in [6]. Such a solution carries a net charge, unlike those considered previously. In [7], the non-extremal generalization of the dihole solution was constructed, and its various properties, such as its thermodynamics and the interaction between the black holes, were studied. Furthermore, the solution was embedded into string / M-theory, and a microscopic description of the entropy of a certain near-extremal dihole was found in terms of an effective string model consisting of interacting strings and anti-strings.

It would be of interest, particularly from the viewpoint of string or M-theory, to construct diholes in dimensions $D > 4$, and there has been a few attempts in this direction. One possibility would be to start from the higher-dimensional Euclidean Kerr solution [8] and add a flat time direction to it. However, this does not give a pair of Kaluza–Klein black holes upon dimensional reduction as one might hope, but rather a single spherical $(D - 4)$-brane [9, 10, 11]. In $D = 5$, for example, one obtains a Kaluza–Klein string wound in a circular loop. Clearly, a different approach is needed to find higher-dimensional dihole solutions.

In a separate development, the Weyl formalism was recently generalized by Emparan and Reall [12] to arbitrary dimensions $D \geq 4$. Space-times belonging to the generalized Weyl class admit $D - 2$ orthogonal commuting Killing vectors, and are specified by $D - 3$
axisymmetric solutions to the Laplace equation in flat three-dimensional space. A particular member of the generalized Weyl class is the five-dimensional Schwarzschild black hole. Moreover, it was pointed out in [12] that solutions describing superpositions of Schwarzschild black holes can also be obtained within this formalism. A five-dimensional generalization of the Israel–Khan solution, describing a collinear array of Schwarzschild black holes, was subsequently constructed in [13]. A charged version of this solution was also found, with each black hole having a fixed mass-to-charge ratio.

The fact that it is possible to construct solutions describing multiple black holes in five dimensions (possibly carrying charges of the same sign), strongly suggests that it would also be possible to find a five-dimensional analog of the dihole solution. Indeed, the aim of this paper is to present dihole solutions of five-dimensional dilaton gravity coupled to a two-form gauge field, following the strategy of [6]. As will be described in Sec. 2, the derivation makes use of the generalized Weyl formalism of [12]. A formal similarity is observed between the four-dimensional Ernst equation and the field equations coming from five-dimensional dilaton gravity theory when a generalized Weyl symmetry is assumed. This will allow us to turn known solutions of one system into new solutions of the other. Indeed, starting from the four-dimensional Kerr solution, we will show in Sec. 3 how this gives the desired five-dimensional dihole solution. We will also show how to rewrite this solution in a simpler form using C-metric-type coordinates.

In Sec. 4, we will discuss this solution in some detail. We show how the individual extremal dilaton black holes may be recovered, and that there are in general conical singularities in the space-time. These conical singularities are actually two-dimensional membranes extending along two different directions, similar to the situation in the two-black hole configuration of [13]. Instead of using conical singularities to achieve equilibrium, it is possible to use a background magnetic field to do so. We show how this is done by applying a five-dimensional analog of the Harrison transformation to the solution.

In Sec. 5, we derive a five-dimensional dihole solution carrying unbalanced charges following the strategy of [6]. This involves applying the solution-generating technique to the Kerr solution with a NUT parameter [14], instead of the usual Kerr solution. We then briefly analyze some of its properties. The paper ends off with a discussion of some possible avenues for future research.
2. Solution-generating technique

We shall consider five-dimensional gravity coupled to a dilaton field $\phi$ and two-form gauge field $B_{ab}$, with the action

$$\int d^5x \sqrt{-g} \left( R - \frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{12} e^{\alpha \phi} H_{abc} H^{abc} \right).$$

(2.1)

Here, $H_{abc} \equiv \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab}$ is the three-form field strength, and $\alpha$ is a parameter governing the coupling of the dilaton to the gauge field. Since changing the sign of $\alpha$ is equivalent to changing the sign of $\phi$, it is sufficient to consider only $\alpha \geq 0$. The case $\alpha = 2\sqrt{\frac{2}{3}}$ is particularly important as it arises from the low-energy effective action of string theory when compactified to five dimensions. By varying $g_{ab}$, $\phi$ and $B_{ab}$, we obtain from (2.1) the respective field equations:

$$R_{ab} = \frac{1}{2} \partial_a \phi \partial_b \phi + \frac{1}{4} e^{\alpha \phi} \left( H_{acd} H^{bd} - \frac{2}{9} g_{ab} H_{cde} H^{cde} \right),$$

(2.2)

$$\Box \phi - \frac{\alpha}{12} e^{\alpha \phi} H_{abc} H^{abc} = 0,$$

(2.3)

$$\nabla_a (e^{\alpha \phi} H^{abc}) = 0.$$  

(2.4)

We seek a solution to these equations that is static and axisymmetric with respect to two angular coordinates $\varphi$ and $\psi$, i.e., an $\mathbb{R} \times U(1) \times U(1)$ symmetry. The most general line element satisfying these conditions can be written as

$$ds^2 = -f dt^2 + l d\varphi^2 + k d\psi^2 + e^\mu (d\rho^2 + dz^2),$$

(2.5)

where $f$, $k$, $l$ and $\mu$ are functions of $\rho$ and $z$ only. This is an immediate generalization of the Weyl form for static, axisymmetric space-times in four dimensions [12]. Furthermore, we assume that the only non-zero component of the two-form gauge field is $B_{\varphi \psi} \equiv B$. Both $B$ and $\phi$ are also functions of $\rho$ and $z$ only.

Now, we begin with three of the six non-trivial equations coming from (2.2):

$$2e^{\mu} DR_{tt} = (D f_\rho)_\rho + (D f_z)_z - D f^{-1} (f_\rho^2 + f_z^2) = \frac{2 f}{3 kl} e^{\alpha \phi} (B_\rho^2 + B_z^2),$$

(2.6)

$$-2e^{\mu} DR_{\varphi \varphi} = (D l_\rho)_\rho + (D l_z)_z - D l^{-1} (l_\rho^2 + l_z^2) = -\frac{1}{3} e^{\alpha \phi} (B_\rho^2 + B_z^2),$$

(2.7)

$$-2e^{\mu} DR_{\psi \psi} = (D k_\rho)_\rho + (D k_z)_z - D k^{-1} (k_\rho^2 + k_z^2) = -\frac{1}{3} e^{\alpha \phi} (B_\rho^2 + B_z^2),$$

(2.8)
where we have defined $D^2 \equiv fkl$, and denoted $f_\rho \equiv \partial f/\partial \rho$, $f_z \equiv \partial f/\partial z$, etc., for brevity. Considering the combination $e^\mu D^{-1}(klR_{tt} - fkr_{\varphi\varphi} - fIR_{\psi\psi})$, we obtain

$$D_{\rho\rho} + D_{zz} = 0.$$  \hfill (2.9)

A solution is $D = \rho$. Substituting this back into (2.6) and (2.8) respectively, we obtain

$$f_{\rho\rho} + f_{zz} + \rho^{-1} f_\rho - f^{-1}(f_\rho^2 + f_z^2) = \frac{2}{3}\rho^{-2} f^2 e^{\alpha \phi}(B_\rho^2 + B_z^2),$$ \hfill (2.10)

$$k_{\rho\rho} + k_{zz} + \rho^{-1} k_\rho - k^{-1}(k_\rho^2 + k_z^2) = -\frac{1}{3}\rho^{-2} f k e^{\alpha \phi}(B_\rho^2 + B_z^2).$$ \hfill (2.11)

Also, the dilaton and two-form gauge field equations respectively become

$$\phi_{\rho\rho} + \phi_{zz} + \rho^{-1} \phi_\rho = \frac{1}{2}\alpha \rho^{-2} f e^{\alpha \phi}(B_\rho^2 + B_z^2),$$ \hfill (2.12)

$$B_{\rho\rho} + B_{zz} - \rho^{-1} B_\rho = -f^{-1}(B_\rho f_\rho + B_z f_z) - \alpha(\phi_\rho B_\rho + \phi_z B_z).$$ \hfill (2.13)

If we define

$$f^2 \equiv f e^{\alpha \phi}, \quad w \equiv i\sqrt{\frac{4 + 3\alpha^2}{12}} B,$$ \hfill (2.14)

then (2.10), (2.12) and (2.13) imply that

$$\tilde{f}(f_{\rho\rho} + f_{zz} + \rho^{-1} \tilde{f}_\rho) - \tilde{f}_\rho^2 - \tilde{f}_z^2 + \rho^{-2} \tilde{f}^4(w_\rho^2 + w_z^2) = 0,$$ \hfill (2.15)

$$\tilde{f}(w_{\rho\rho} + w_{zz} - \rho^{-1} w_\rho) + 2(w_\rho \tilde{f}_\rho + w_z \tilde{f}_z) = 0.$$ \hfill (2.16)

These two equations are equivalent to the Ernst equation (c.f. Eqn. (2.12a) and (2.12b) of [15]) coming from the four-dimensional vacuum Einstein equation, if we consider a stationary, axisymmetric line element of the form

$$ds^2 = -\tilde{f}(dt - w d\varphi)^2 + \rho^2 \tilde{f}^{-1} d\varphi^2 + e^\nu (d\rho^2 + dz^2).$$ \hfill (2.17)

This is the crucial correspondence which would allow us to obtain solutions to the action (2.1), starting from solutions to pure Einstein gravity. Note, however, that this correspondence in general gives a $B$ that is imaginary. To obtain a real expression for $B$, the original solution must admit a parameter which can be analytically continued so that $w$ becomes imaginary while leaving $\tilde{f}$ real.

Supposing we have found suitable solutions $\tilde{f}$ and $w$ to the equations (2.15) and (2.16), the next step is to solve for $k$ and $\phi$. (The latter would then give us $f$ by the first equation of (2.14).) Note, from (2.12) and (2.15), that

$$\phi = \frac{6\alpha}{4 + 3\alpha^2} \ln \tilde{f},$$ \hfill (2.18)
up to the addition of a harmonic function \( \tilde{\phi} \), satisfying \( \tilde{\phi}_{\rho\rho} + \tilde{\phi}_{zz} + \rho^{-1} \tilde{\phi}_\rho = 0 \). For the choice \( \tilde{\phi} = 0 \), we have

\[
f = \tilde{f} \frac{\rho}{4 + 3\alpha^2}.
\]

(2.19)

Similarly, we deduce from (2.11) and (2.12) that

\[
k = e^{h - \frac{2\phi}{3\alpha}} = e^{h - \frac{4}{4 + 3\alpha^2}},
\]

(2.20)

where \( h \) is another arbitrary harmonic function. A suitable choice for \( h \) will be made below.

Having obtained \( \phi, k, f \) and \( B \), the final step is to solve for \( \mu \). Using the three remaining equations of (2.2), namely those for \( R_{\rho\rho}, R_{\rho z} \) and \( R_{zz} \), we obtain

\[
\rho^{-1} \mu_\rho = \frac{1}{2} f^{-2}(f \rho^2 - f_z^2) + \frac{1}{2} k^{-2}(k_\rho^2 - k_z^2) + \frac{1}{2} f^{-1}k^{-1}(f \rho k_\rho - f_z k_z) - \rho^{-1} f^{-1} f_\rho - \rho^{-1} k^{-1} k_\rho
\]

\[
+ \frac{1}{2} (\phi_\rho^2 - \phi_z^2) + \frac{1}{2} \rho^{-2} f e^{\alpha \phi}(B_\rho^2 - B_z^2),
\]

(2.21)

\[
\rho^{-1} \mu_z = f^{-2} f_\rho f_z + k^{-2} k_\rho k_z + \frac{1}{2} f^{-1}k^{-1}(f \rho k_z + f_z k_\rho) - \rho^{-1} f^{-1} f_z - \rho^{-1} k^{-1} k_z
\]

\[
+ \phi_\rho \phi_z + \rho^{-2} f e^{\alpha \phi} B_\rho B_z.
\]

(2.22)

These two equations can be integrated to obtain \( \mu \). This therefore completes the procedure whereby a stationary, axisymmetric solution (2.17) to the four-dimensional vacuum Einstein equation may be converted into a static, axisymmetric solution of five-dimensional dilaton gravity coupled to a two-form gauge field. It is the analog of the result found in [6] in four dimensions.

### 3. Derivation of dihole solution

A natural starting point would be the four-dimensional Kerr solution with mass \( m \) and angular momentum \( a \), after performing the analytic continuation \( a \to ia \). It turns out that this solution is most conveniently written in terms of prolate spheroidal coordinates \( (p,q) \) as [15, 16]

\[
\tilde{f} = \frac{\sigma^2 p^2 - a^2 q^2 - m^2}{(\sigma p + m)^2 - a^2 q^2}, \quad w = \frac{2ima(\sigma p + m)(1 - q^2)}{\sigma^2 p^2 - a^2 q^2 - m^2},
\]

(3.1)

where \( \sigma \equiv \sqrt{m^2 + a^2} \). If necessary, it can be rewritten in terms of Weyl coordinates \( (\rho,z) \) using the relations

\[
p = \frac{1}{2\sigma}(R_1 + R_3), \quad q = \frac{1}{2\sigma}(R_1 - R_3),
\]

(3.2)
where we have defined
\[ R_1 \equiv \sqrt{\rho^2 + (z + \sigma)^2}, \quad R_3 \equiv \sqrt{\rho^2 + (z - \sigma)^2}. \] (3.3)

(This choice of numbering would become clearer below.) With this \( \tilde{f} \) and \( w \), we can immediately deduce \( f \), \( \phi \) and \( B \) using (2.19), (2.18) and the second equation of (2.14), respectively.

Now, to ensure that the line element (2.5) has a chance of being asymptotically flat, the harmonic function \( h \) is taken to be [12]

\[ h = \ln [R_1 + (z + \sigma)]. \] (3.4)

After integrating (2.21) and (2.22) to obtain \( \mu \), we finally arrive at the line element:

\[
ds^2 = -H_{4+3\alpha^2}^8 dt^2 + H_{4+3\alpha^2}^8 \left[ \frac{(\sigma p + m)^2 - a^2 q^2}{\sigma^2 (p^2 - q^2)} \right]^{\frac{12}{4+3\alpha^2}} \frac{1}{2K_0 R_1} (d\rho^2 + dz^2) + H_{4+3\alpha^2}^{-4} \left\{ [R_1 + (z + \sigma)]^{-1} \rho^2 d\phi^2 + [R_1 + (z + \sigma)] d\psi^2 \right\},
\] (3.5)

where \( K_0 \) is a dimensionless constant to be determined below, and

\[ H = \frac{\sigma^2 p^2 - a^2 q^2 - m^2}{(\sigma p + m)^2 - a^2 q^2}. \] (3.6)

The dilaton and two-form gauge field are respectively given by

\[
\phi = \frac{6\alpha}{4 + 3\alpha^2} \ln H, \quad B_{\varphi \psi} = \sqrt{\frac{12}{4 + 3\alpha^2}} \frac{2am(\sigma p + m)(1 - q^2)}{\sigma^2 p^2 - a^2 q^2 - m^2}. \] (3.7)

The rod structure of this solution along the \( z \)-axis can be deduced following [12], and is shown in Fig. 1(a). As can be seen, the choice of \( h \) in (3.4) has produced semi-infinite rods corresponding to the \( \varphi \) and \( \psi \) coordinates, a property of asymptotically flat space-times such as the five-dimensional Schwarzschild solution [12]. Furthermore, the rod structure corresponding to the time coordinate shows that there are two ‘black’ objects with horizons at \( z = \pm \sigma \). The fact that these rods have shrunk down to points indicates that they are extremally charged. The one at \( z = -\sigma \) has a horizon with topology \( S^3 \); thus it is an extremal black hole. On the other hand, the one at \( z = \sigma \) has a horizon with topology \( S^2 \times S^1 \), and so it is an extremal black ring.

To obtain instead a system describing two extremal black holes, we shall modify this solution to one with the rod structure as shown in Fig. 1(b). The difference is that part of
the rod from \( z = 0 \) to \( \sigma \) has been moved from the \( \varphi \) to the \( \psi \) coordinate. This is achieved quite simply by adding an appropriate harmonic function to \( h \) in (3.4). For the choice

\[
h = \ln \left( \frac{[R_1 + (z + \sigma)](R_2 - z)}{R_3 - (z - \sigma)} \right),
\]

where \( R_2 \equiv \sqrt{\rho^2 + z^2} \), we arrive at the new solution:

\[
ds^2 = -H^{\frac{8}{4+3\alpha^2}}dt^2 + H^{\frac{8}{4+3\alpha^2}} \left[ \frac{(\sigma p + m)^2 - a^2q^2}{\sigma^2(p^2 - q^2)} \right]^{\frac{12}{4+3\alpha^2}} \frac{Y_{12}Y_{23}}{R_1R_2R_3Y_{13}} \times \frac{1}{4K_0} (d\rho^2 + dz^2) + H^{-\frac{4}{4+3\alpha^2}} \left\{ \frac{R_3 - (z - \sigma)}{[R_1 + (z + \sigma)](R_2 - z)} \rho^2\varphi^2 \right\} + Y_{12} \frac{[R_1 + (z + \sigma)](R_2 - z)}{R_3 - (z - \sigma)} d\psi^2,
\]

where

\[
Y_{12} \equiv \rho^2 + (z + \sigma)z + R_1R_2,
Y_{23} \equiv \rho^2 + (z - \sigma)z + R_3R_3,
Y_{13} \equiv \rho^2 + (z^2 - \sigma^2) + R_1R_3,
\]

and with \( H, \phi \) and \( B_{\varphi\psi} \) unchanged as in (3.6) and (3.7).

Now, the solution as written in the Weyl form (3.9) has the disadvantage that the \( R_i \) contain square roots, making calculations somewhat cumbersome. Before proceeding any further, it would be useful to find alternative coordinates that would simplify the form of the solution by getting rid of these square roots.

If the rod structure had just two ‘characteristic points’ where the rods terminate, as in Fig. 1(a), then such a coordinate system will be provided by the prolate spheroidal coordinates (3.2). But since the rod structure of our final solution has three characteristic points

\[
\begin{array}{ccc}
  z = -\sigma & z = +\sigma & t \\
  & \bullet & \\
  & \varphi & \\
  & \bullet & \\
  \psi & & \\
  & \bullet & \\
  & \varphi & \\
  & \bullet & \\
  \psi & & \\
\end{array}
\]

(a)  \hspace{3cm} (b)  

Figure 1: Rod structures of (a) the extremal black hole / black ring system, and (b) the two-extremal black hole system.
(at \( z_1 = -\sigma, \ z_2 = 0, \) and \( z_3 = \sigma \)), a different coordinate system has to be adopted. It
turns out to be possible to introduce coordinates similar to that used in the standard form
of the C-metric, whose rod structure also has three characteristic points (see, e.g., [17]). The
transformation in this case is [18]

\[
\rho = \frac{2\sigma \sqrt{xy(1 - x^2)(y^2 - 1)}}{(x - y)^2}, \quad z = \frac{\sigma(x + y)(1 - xy)}{(x - y)^2},
\]

(3.11)

where the new coordinates \((x, y)\) assume the finite range \(0 \leq x \leq 1\) and \(-1 \leq y \leq 0\). Under
this transformation, \(R_i\) take the simple algebraic form:

\[
R_1 = -\sigma \frac{xy - x - y - 1}{x - y}, \quad R_2 = \sigma \frac{xy + 1}{x - y}, \quad R_3 = -\sigma \frac{xy + x + y - 1}{x - y}.
\]

(3.12)

This is the crucial simplification made possible by using the C-metric-type coordinates.

In these new coordinates, the line element (3.9) becomes

\[
ds^2 = -H^{\frac{8}{4 + 3\alpha^2}} dt^2 + H^{\frac{8}{4 + 3\alpha^2}} \left[ \frac{(\sigma p + m)^2 - a^2 q^2}{\sigma^2 (p^2 - q^2)} \right]^{\frac{12}{4 + 3\alpha^2}}
\]

\[
\times \frac{\sigma}{2K_0} \left( \frac{1}{(x - y)^2} \left( \frac{dx^2}{x(1 - x^2)} + \frac{dy^2}{y(y^2 - 1)} \right) \right)
\]

\[
+ H^{-\frac{4}{4 + 3\alpha^2}} \frac{2\sigma}{(x - y)^2} \{ y(y^2 - 1) d\varphi^2 + x(1 - x^2) d\psi^2 \},
\]

(3.13)

with \((p, q)\) given in terms of \((x, y)\) by

\[
p = \frac{1 - xy}{x - y}, \quad q = \frac{x + y}{x - y}.
\]

(3.14)

The black holes at \((\rho, z) = (0, -\sigma)\) and \((0, \sigma)\) are respectively located at \((x, y) = (0, -1)\) and
\((1, 0)\) in the new coordinate system, while the origin \((\rho, z) = (0, 0)\) is at \((1, -1)\). Asymptotic
infinity is at \((0, 0)\).

4. Physical properties

We now examine some physical properties of the solution (3.13), beginning with its
asymptotic structure. If we introduce coordinates \((r, \theta)\) given by

\[
x = \frac{2\sigma}{K_0 r^2} \cos^2 \theta, \quad y = -\frac{2\sigma}{K_0 r^2} \sin^2 \theta,
\]

(4.1)
and take the limit $r \to \infty$, the line element becomes

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + K_0 \sin^2 \theta \, d\varphi^2 + K_0 \cos^2 \theta \, d\psi^2).$$  \tag{4.2}$$

This is Minkowski space in spherical polar coordinates, with manifest $U(1) \times U(1)$ rotational symmetry. However, to ensure that both $\varphi$ and $\psi$ have the usual periodicity of $2\pi$, we must set $K_0 = 1$. With this choice, (3.13) becomes an asymptotically flat solution. From the asymptotic behavior of $g_{tt}$, we deduce that the mass of this solution is $\frac{16m}{4 + 3a^2}$.

When $m = 0$, (3.13) reduces to the vacuum space-time

$$ds^2 = -dt^2 + \frac{\sigma}{(x - y)^2} \left( \frac{dx^2}{2x(1 - x^2)} + \frac{dy^2}{2y(y^2 - 1)} + 2y(y^2 - 1) \, d\varphi^2 + 2x(1 - x^2) \, d\psi^2 \right).$$  \tag{4.3}$$

The spatial part of this space-time is but the Euclideanized C-metric, for some specific choice of the mass and acceleration parameters. The rod structure of (4.3) is given by that in Fig. 1(b), but without the two point sources belonging to the time coordinate. From the remaining rods corresponding to the $\varphi$ and $\psi$ coordinates, we see that there are two ‘outer’ semi-infinite axes, and two ‘inner’ finite ones. Since we have demanded that the outer axes have the usual periodicity, conical singularities necessarily appear along the two inner ones [12]. It can be checked that they both have periods $4\pi$ [13], corresponding to struts in the space-time. As explained in [13], these struts are actually membranes, and have the topology of a sphere. Furthermore, these two topological spheres are orthogonal to each other, since they are aligned along the two different rotational axes.

The three points, $z_1$, $z_2$, and $z_3$, where the two axes meet up are fixed points of both $U(1)$’s. It is thus possible to introduce at these points, objects whose constant-radius surfaces have $S^3$ topology, such as black holes. Indeed, this background was first used in [12] to construct an asymptotically flat three-black hole solution. It was subsequently utilized in [13] to construct a two-black hole solution with equal charge-to-mass ratio (which was also generalized to an $N$-black hole system). Here, we will show that the solution (3.13) actually corresponds to a two-black hole solution with opposite charge, i.e., a dihole.

Clearly, the solution is magnetically charged with respect to $H_{abc}$. The asymptotic behavior of $B_{\varphi \psi}$ reveals that it describes a dipole configuration, with a moment proportional to $ma$. (Since changing the sign of $a$ is equivalent to reversing the orientation of the dipole, we may take $a \geq 0$ without any loss of generality.) The non-dilatonic case $\alpha = 0$ is therefore a $U(1) \times U(1)$-symmetric five-dimensional generalization of Bonnor’s magnetic dipole solution [1].
To show that there are two black holes located at \((x, y) = (0, -1)\) and \((1, 0)\), we have to change to suitable coordinates which blow up each of these regions, while sending the other off to infinity. Let us concentrate on the left black hole at \((x, y) = (0, -1)\) first. An appropriate choice of coordinates is \((r, \theta)\) given by

\[
x = \frac{r^2}{2\sigma} \cos^2 \theta, \quad y = -1 + \frac{r^2}{\sigma} \sin^2 \theta,
\]

in the limit \(a \to \infty\) such that \(r \sin \theta\) and \(r \cos \theta\) remain finite. We obtain:

\[
d s^2 = -\left(1 + \frac{m}{r^2}\right)^{-\frac{8}{4+3\alpha^2}} dt^2 + \left(1 + \frac{m}{r^2}\right)^{\frac{4}{4+3\alpha^2}} \left[dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2 + \cos^2 \theta \, d\psi^2)\right],
\]

\[
\phi = -\frac{6\alpha}{4 + 3\alpha^2} \ln \left(1 + \frac{m}{r^2}\right),
\]

\[
B_{\varphi\psi} = \sqrt{\frac{12}{4 + 3\alpha^2}} 2m \cos^2 \theta.
\]

This is just the solution for an extremal dilatonic black hole [19], but with a conical singularity attached to it along the \(\varphi\)-axis, with a period of \(4\pi\). This conical singularity is an artifact of the background space-time (4.3), since the latter has a conical singularity with exactly the same period. Its presence will modify the calculation of the ADM mass of this black hole, in that an extra factor of two will appear in the integral of energy-density over a spatial hypersurface [20]. The mass is obtained to be \(\frac{8m}{4+3\alpha^2}\).

To recover the right black hole at \((x, y) = (1, 0)\), the corresponding transformation is

\[
x = 1 - \frac{r^2}{\sigma} \cos^2 \theta, \quad y = -\frac{r^2}{2\sigma} \sin^2 \theta.
\]

We obtain the solution:

\[
d s^2 = -\left(1 + \frac{m}{r^2}\right)^{-\frac{8}{4+3\alpha^2}} dt^2 + \left(1 + \frac{m}{r^2}\right)^{\frac{4}{4+3\alpha^2}} \left[dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2 + 4 \cos^2 \theta \, d\psi^2)\right],
\]

\[
\phi = -\frac{6\alpha}{4 + 3\alpha^2} \ln \left(1 + \frac{m}{r^2}\right),
\]

\[
B_{\varphi\psi} = \sqrt{\frac{12}{4 + 3\alpha^2}} 2m \sin^2 \theta.
\]

There is now a conical singularity attached to the black hole along the \(\psi\)-axis with a period of \(4\pi\), as expected. Note that this black hole has opposite charge to the other one. But its
mass is also \( \frac{8m}{4+3\alpha^2} \), giving a total mass of \( \frac{16m}{4+3\alpha^2} \) for the system, in agreement with the result derived above.

In a similar vein, we may perform the transformation

\[
    x = 1 - \frac{4r^2}{\sigma} \cos^2 \theta, \quad y = -1 + \frac{4r^2}{\sigma} \sin^2 \theta,
\]

(4.8)
to blow up the region around the origin \((x, y) = (1, -1)\). We obtain the line element

\[
    ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + 4\sin^2 \theta \, d\varphi^2 + 4\cos^2 \theta \, d\psi^2),
\]

(4.9)
which is flat Minkowski space but with double the usual periodicity for both the angles \( \varphi \) and \( \psi \), as expected. There is nothing special in this region, apart from the two conical singularities meeting at the \( U(1) \times U(1) \) fixed point \( r = 0 \).

There is another limit one could consider, namely the near-horizon region of each black hole. For the left one, the transformation is the same as in (4.4), except that instead of taking the limit of large \( a \), we take the limit of small \( r \). We obtain the line element:

\[
    ds^2 = g(\theta)^{\frac{8}{4+3\alpha^2}} \left[ -\left( \frac{r^2}{Q} \right)^{\frac{8}{4+3\alpha^2}} \, dt^2 + \left( \frac{Q}{r^2} \right)^{\frac{4}{4+3\alpha^2}} (dr^2 + r^2 d\theta^2) \right] + g(\theta)^{-\frac{8}{4+3\alpha^2}} \left( \frac{Q}{r^2} \right)^{\frac{4}{4+3\alpha^2}} r^2 (4\sin^2 \theta \, d\varphi^2 + \cos^2 \theta \, d\psi^2),
\]

(4.10)
where we have defined \( Q \equiv m(m + \sigma)/\sigma \) and

\[
    g(\theta) = \sin^2 \theta + \frac{a^2}{\sigma^2} \cos^2 \theta.
\]

(4.11)
The latter is a deformation factor which determines how the metric deviates from spherical symmetry. A similar result may be obtained for the right black hole. Thus this shows that, even at finite distance, there are two, albeit distorted, black holes present in the space-time. The distortion suffered by each black hole is due to the forces exerted by the other black hole.

The two black holes are joined up by the inner \( \varphi \)- and \( \psi \)-axes. The former is parameterized by \( y = -1 \) with \( 0 < x < 1 \), while the latter is parameterized by \( x = 1 \) with \( -1 < y < 0 \). Note that each segment now has the topology of a disk [13]. The two topological disks span orthogonal directions, and join up at the origin \((x, y) = (1, -1)\). By calculating the proper distance along these two segments, it may be seen that \( a \) determines the separation.
of the black holes. For large $a$, this distance is proportional to $\sqrt{a}$. In the coincidence limit $a \to 0$, the magnetic charge cancels out and we obtain a five-dimensional dilatonic generalization of the Darmois solution [16]. The latter may be thought of as a superposition of two Schwarzschild black holes at the same point, but its physical interpretation remains rather obscure.

The non-dilatonic case $\alpha = 0$ has to be treated separately, since the proper distance calculated is always infinite. This is due to the well-known fact that the Reissner–Nordström black hole has an infinite throat in the extremal limit. However, $a$ still gives an indication of the separation of the black holes, as can be seen, for example, from the dependence of the dipole moment on it.

Now recall that $K_0$ was chosen above so that the outer $\varphi$- and $\psi$-axes have the usual periodicity. This implies that conical singularities necessarily appear along the inner $\varphi$- and $\psi$-axes. Indeed, the conical excess along both axes can be calculated to be

$$\delta = 2\pi \left[ 2 \left( 1 + \frac{m^2}{a^2} \right)^{\frac{a}{4+3\alpha^2}} - 1 \right],$$

which shows that there are two orthogonal disk-like struts joining the black holes. They provide the stress required to counterbalance the attraction between the black holes along the two orthogonal directions, thus yielding a static configuration. On the other hand, if we had chosen

$$K_0 = \frac{1}{4} \left( \frac{a^2}{m^2 + a^2} \right)^{\frac{12}{4+3\alpha^2}},$$

the inner axes would have the usual periodicity, but conical singularities would appear along the outer axes. It can be checked that both axes would have a deficit angle of

$$\delta = 2\pi \left[ 1 - \frac{1}{2} \left( \frac{a^2}{m^2 + a^2} \right)^{\frac{6}{4+3\alpha^2}} \right].$$

This may be interpreted as a pair of semi-infinite ‘cosmic membranes’ (the higher-dimensional generalization of cosmic strings) pulling on the black holes in orthogonal directions to maintain equilibrium.

Apart from struts or cosmic membranes, it is possible to use a background magnetic field, appropriately aligned along the dipole, to provide the necessary forces to keep the system in equilibrium. Such a magnetic field can be introduced into the solution (3.13) by means of a Harrison-type transformation. In four dimensions, such a transformation was
generalized to dilaton gravity in [21]. The corresponding transformation in the present case is
\[ \tilde{f}' = \Lambda \tilde{f}, \]
\[ B'_{\varphi \psi} = -\sqrt{\frac{12}{4 + 3\alpha^2}} \frac{1}{\mathcal{B} \Lambda} \left( 1 + \sqrt{\frac{4 + 3\alpha^2}{12} \mathcal{B} \varphi \psi} \right), \]
\[ e^{\mu'} = \Lambda^{\frac{\mu}{4 + 3\alpha^2}} e^\mu, \] (4.15)
where
\[ \Lambda \equiv \left( 1 + \sqrt{\frac{4 + 3\alpha^2}{12} \mathcal{B} \varphi \psi} \right)^2 + \mathcal{B}^2 \rho^2 \tilde{f}^{-2}, \] (4.16)
and \( \mathcal{B} \) is a constant that determines the background magnetic field strength. The proof of (4.15) involves writing this transformation in terms of \((\tilde{f}', \omega')\), and showing that it is a solution to (2.15) and (2.16). The behavior of \( \mu \) under this transformation can then be deduced from (2.21) and (2.22).

Applying the transformation (4.15) to the dihole solution, we obtain a solution that again takes the form (3.13), (3.7), but with
\[ H = \left\{ (\mathcal{B}^2 - a^2 q^2 - m^2) + 4\mathcal{B}am(\sigma p + m)(1 - q^2) \right. \\
+ \mathcal{B}^2 (1 - q^2) \left[ (\sigma p + m)^2 - a^2 \right] + a^2 \sigma^2 (p^2 - 1)(1 - q^2) \left\} \right/ \left( (\sigma p + m)^2 - a^2 q^2 \right), \]
\[ B_{\varphi \psi} = \sqrt{\frac{12}{4 + 3\alpha^2}} \left\{ 2am(\sigma p + m) + \mathcal{B} \left[ (\sigma p + m)^2 - a^2 \right] \right. \\
\left. + \mathcal{B}^2 (1 - q^2) \left[ (\sigma p + m)^2 - a^2 \right] \right\} (1 - q^2) \left/ \left[ H (\sigma p + m)^2 - a^2 q^2 \right] \right. \]. (4.17)

An appropriate constant has been added to \( B_{\varphi \psi} \) so that it reduces to the one in (3.7) when \( \mathcal{B} = 0 \). This solution describes a dihole configuration suspended in a background magnetic field. To see this, note that it has the same asymptotic limit (after performing the transformation (4.1) with \( K_0 = 1 \)) as the solution
\[ ds^2 = H^{-\frac{8}{4 + 3\alpha^2}} (-dt^2 + dr^2 + r^2 d\theta^2) + H^{-\frac{4}{4 + 3\alpha^2}} r^2 (\sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2), \] (4.18)
with
\[ H = 1 + \frac{1}{4} \mathcal{B}^2 r^4 \sin^2 2\theta, \]
\[ \phi = \frac{6\alpha}{4 + 3\alpha^2} \ln H, \]

*An analogous transformation for a magnetically charged one-form gauge field in five dimensions can be found in [11].
\[ B_{\varphi\psi} = \frac{1}{2} \sqrt{\frac{3}{4 + 3\alpha^2}} H^{-1} Br^4 \sin^2 2\theta. \]  

(4.19)

This is a five-dimensional analog of the dilatonic Melvin universe [21]. It describes a \(U(1) \times U(1)\) symmetric magnetic field in an otherwise empty universe.

It is possible to find the near-horizon limit of this dihole solution using the same transformation as above. For the left black hole, we again obtain a line element of the form (4.10), but with the deformation factor:

\[ g(\theta) = \sin^2 \theta + \frac{1}{\sigma^2} [a + 2Bm(m + \sigma)]^2 \cos^2 \theta. \]  

(4.20)

Thus the presence of the background field will modify the shape of the horizons, as to be expected. It can also be checked that the conical excess along the inner \(\varphi\)- and \(\psi\)-axes is now

\[ \delta = 2\pi \left[ 2 \left( 1 + \frac{m^2}{a^2} \right)^{\frac{6}{4 + 3\alpha^2}} \left( 1 + \frac{2Bm(m + \sigma)}{a} \right)^{-\frac{12}{4 + 3\alpha^2}} - 1 \right]. \]  

(4.21)

By adjusting \(B\) appropriately, it is possible to set this to the natural value \(2\pi\), i.e., the conical excess that is present in (4.3) even when there are no black holes or background fields. This happens at the critical field strength

\[ B_{\text{crit}} = \frac{\sigma - a}{2m(m + \sigma)}. \]  

(4.22)

and represents the situation in which the background magnetic field strength precisely cancels out the gravitational and magnetic attraction between the two black holes. Note that in this case, \(g(\theta) = 1\) and so the black holes are perfectly spherical, another consequence of the forces being balanced out.

We remark that it is also possible to tune the background field strength to make \(\delta = 0\) and hence remove all the conical singularities from the space-time. However, this still leads to a non-trivial distortion factor in (4.20), and can not be regarded as a natural situation in which the forces are canceling out.

Finally, we recall that in five dimensions, a solution that is magnetically charged with respect to the three-form field strength \(H_{abc}\) may be dualized into another solution that is electrically charged with respect to a two-form field strength \(F_{ab}\). This duality transformation takes the form

\[ \phi' = -\phi, \quad F_{ab} = e^{\alpha\phi} (\ast H)_{ab}, \]  

(4.23)
with the new solution extremizing the action

\[ \int d^5x \sqrt{-g} \left( R - \frac{1}{2} \partial_\alpha \phi' \partial^\alpha \phi' - \frac{1}{4} \epsilon^{\alpha \beta} F_{\alpha \beta} F^{\alpha \beta} \right). \]  (4.24)

Applying this transformation to (4.17), we obtain an electrically charged dihole solution in a background electric field. The resulting electric potential is given by

\[ A_t = \sqrt{\frac{12}{4 + 3\alpha^2}} \left\{ \frac{2maq}{(\sigma p + m)^2 - a^2q^2} \left[ \frac{1}{2} \partial_\alpha \phi' \partial^\alpha \phi' \right]^2 - 2Bq \left[ \sigma p - 2m + Bma(3 - q^2) \right] \right\}. \]  (4.25)

In the asymptotic limit, it reduces to the electric dual of the Melvin universe (4.18), (4.19).

5. Diholes with unbalanced charges

Instead of starting with the four-dimensional Kerr solution, as in (3.1), one can start with the Demiański–Newman solution \[14\] which contains an additional NUT parameter \(l\). Repeating the procedure described in Sec. 3 and 4, we obtain the three-parameter solution (with \(K_0 = 1\):

\[ ds^2 = -H^{\frac{8}{4 + 3\alpha^2}} \frac{d\tau^2}{\sigma^2} + H^{\frac{8}{4 + 3\alpha^2}} \left[ \frac{(\sigma p + m)^2 - (aq + l)^2}{\sigma^2(p^2 - q^2)} \right]^{\frac{12}{4 + 3\alpha^2}} \]
\[ \times \left( \frac{d\sigma^2}{2(x - y)^2} \left( \frac{dx^2}{x(1 - x^2)} + \frac{dy^2}{y(y^2 - 1)} \right) \right) \]
\[ + H^{-\frac{4}{4 + 3\alpha^2}} \frac{\sigma}{(x - y)^2} \{ y(y^2 - 1) d\varphi^2 + x(1 - x^2) d\psi^2 \}, \]  (5.1)

where now \(\sigma \equiv \sqrt{m^2 + a^2 - l^2}\), and

\[ H = \frac{\sigma^2 p^2 - a^2 q^2 - m^2 + l^2}{(\sigma p + m)^2 - (aq + l)^2}, \]
\[ \phi = \frac{6\alpha}{4 + 3\alpha^2} \ln H, \]
\[ B_{\varphi \psi} = \sqrt{\frac{12}{4 + 3\alpha^2}} \frac{2a(m \sigma p + m^2 - l^2)(1 - q^2) + 2l \sigma^2(p^2 - 1)q}{\sigma^2 p^2 - a^2 q^2 - m^2 + l^2}. \]  (5.2)

As usual, \((p, q)\) are related to \((x, y)\) by (3.14). This solution clearly reduces to (3.13), (3.6) and (3.7) when \(l = 0\). However, unlike the previous solution, it carries a net magnetic charge \(-\sqrt{\frac{12}{4 + 3\alpha^2}} 4l\). We may assume \(l \geq 0\) without loss of generality, corresponding to a negatively charged solution.
To recover the individual black holes, we change variables as in (4.4) and (4.6), to obtain

\[
\begin{align*}
\text{d} s^2 &= -\left(1 + \frac{m + l}{r^2}\right)^{-\frac{8}{4+3\alpha^2}} \text{d}t^2 + \left(1 + \frac{m + l}{r^2}\right)^{\frac{4}{4+3\alpha^2}} \left[ \text{d}r^2 + r^2 (\text{d}\theta^2 + 4 \sin^2 \theta \text{d}\varphi^2 + \cos^2 \theta \text{d}\psi^2) \right], \\
\phi &= -\frac{6\alpha}{4+3\alpha^2} \ln \left(1 + \frac{m + l}{r^2}\right), \\
B_{\varphi\psi} &= \sqrt{\frac{12}{4+3\alpha^2}} 2(m + l) \cos^2 \theta; \\
\end{align*}
\]

(5.3)

and

\[
\begin{align*}
\text{d} s^2 &= -\left(1 + \frac{m - l}{r^2}\right)^{-\frac{8}{4+3\alpha^2}} \text{d}t^2 + \left(1 + \frac{m - l}{r^2}\right)^{\frac{4}{4+3\alpha^2}} \left[ \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2 + 4 \cos^2 \theta \text{d}\psi^2) \right], \\
\phi &= -\frac{6\alpha}{4+3\alpha^2} \ln \left(1 + \frac{m - l}{r^2}\right), \\
B_{\varphi\psi} &= \sqrt{\frac{12}{4+3\alpha^2}} 2(m - l) \sin^2 \theta, \\
\end{align*}
\]

(5.4)

respectively. This shows that there is a black hole with mass \(m + l\) at \((x, y) = (0, -1)\), and another one with mass \(m - l\) at \((1, 0)\). Thus, \(l\) controls the distribution of mass (and hence charge) between the two black holes, although the total mass (charge) is still conserved. In the case \(l = m\), one of the black holes disappears, leaving just a single black hole with mass \(\frac{16m}{4+3\alpha^2}\).

As with the \(l = 0\) case, the two black holes are held apart by struts stretching between them. The conical excess along the inner \(\varphi\)- and \(\psi\)-axes is

\[
\delta = 2\pi \left[ 2 \left(1 + \frac{m^2 - l^2}{a^2}\right)^{-\frac{6}{4+3\alpha^2}} - 1 \right].
\]

(5.5)

Note that \(\delta\) takes the natural value \(2\pi\) when \(m = l\), as to be expected when there is only one black hole left in the space-time.

We may suspend the solution (5.1), (5.2) in a background magnetic field \(B\), by applying the Harrison transformation (4.15) to it. The resulting solution again takes the form (5.1) and (5.2), but with

\[
H = \left\{ (\sigma^2 p^2 - a^2 q^2 - m^2 + l^2) + 4B \left[ a(m \sigma p + m^2 - l^2)(1 - q^2) + l a^2 (p^2 - 1) q \right] \\
+ B^2 (1 - q^2) \left[ (\sigma p + m)^2 - a^2 - l^2 \right]^2 + \left( a - \frac{2lq}{1 - q^2} \right)^2 \sigma^2 (p^2 - 1)(1 - q^2) \right\}
\]
\[ B_{\varphi\psi} = \sqrt{\frac{12}{4 + 3\alpha^2}} \left\{ 2 \left[ a(m\sigma p + m^2 - l^2)(1 - q^2) + l\sigma^2(p^2 - 1)q \right] 
\quad + B(1 - q^2) \left[ \left( (\sigma p + m)^2 - a^2 - l^2 \right)^2 + \left( a - \frac{2lq}{1 - q^2} \right)^2 \sigma^2(p^2 - 1)(1 - q^2) \right] \right\} 
\left/ H \left[ (\sigma p + m)^2 - (aq + l)^2 \right] \right. \]  

(5.6)

It has the same asymptotic limit as the Melvin universe (4.18), (4.19).

The conical excess along the inner $\varphi$- and $\psi$-axes is then

\[ \delta = 2\pi \left[ 2 \left( 1 + \frac{m^2 - l^2}{a^2} \right)^{\frac{6}{4 + 3\alpha^2}} \left( 1 + \frac{2B(m\sigma p + m^2 - l^2)}{a} \right)^{-\frac{12}{4 + 3\alpha^2}} - 1 \right], \]

(5.7)

which takes the natural value $2\pi$ when the background field assumes the strength

\[ B_{\text{crit}} = \frac{\sigma - a}{2(m\sigma + m^2 - l^2)}. \]

(5.8)

However, unlike the $l = 0$ case above, it is not possible to let the outer $\varphi$- and $\psi$-axes continue taking on their natural periodicity of $2\pi$. Instead, they must take the values

\[ \Delta\varphi = 2\pi(1 + Bl)^{\frac{12}{4 + 3\alpha^2}}, \]
\[ \Delta\psi = 2\pi(1 - Bl)^{\frac{12}{4 + 3\alpha^2}}, \]

(5.9)

which implies that there is a conical excess along the $\varphi$-axis, and a conical deficit along the $\psi$-axis. This is due to the asymmetric distribution of the charge, and is similar to the situation in four dimensions [6].

6. Discussion

In this paper, we have developed a solution-generating technique and used it to obtain a five-dimensional dihole solution of dilaton gravity coupled to a two-form gauge field. The main properties of this solution were then studied. In particular, it was shown that there are membrane-like conical singularities in the space-time keeping the black holes in static equilibrium. We also presented a dihole solution suspended in a background magnetic field, as well as one with unbalanced charges.

There are a few possible extensions of this work. For example, it would be interesting to study its thermodynamic properties and find a microscopic description for it following the
methods of [22], as was done in [7] in the four-dimensional case. However, this requires one to first obtain the non-extremal dihole solution, perhaps using a five-dimensional generalization of Sibgatullin’s method [23]. Such a solution, even if it could be constructed, is likely to be very complicated like its four-dimensional counterpart [7].

There is also the question how the five-dimensional dihole solution can be embedded and interpreted in string or M-theory. When (3.13) is uplifted to ten dimensions in the usual way, it appears to describe a non-standard type of 5-brane – anti-5-brane configuration. Normally, a 5-brane is assumed to have \( SO(4) \) symmetry in the transverse directions, and a space-time containing two 5-branes would therefore be expected to have maximal \( SO(3) \) symmetry. However, our solution has only \( U(1) \times U(1) \) symmetry, and this is tied in to the presence of the two conical membranes stretching between the 5-brane and anti-5-brane.

This leads one to wonder if there exists another dihole solution with \( SO(3) \) instead of \( U(1) \times U(1) \) symmetry. However, since such a space-time will not belong to the generalized Weyl class [12], the procedure used to obtain the dihole solutions in this paper would simply not apply. In view of this, a radically different approach might be needed (see, e.g., [24]).

It would also be of interest to try to find dihole solutions in six or higher dimensions. Unfortunately, such solutions will not belong to the generalized Weyl class, since black holes in \( D \geq 6 \) dimensions do not admit \( D - 2 \) commuting Killing vectors [12]. Again, a different approach would be needed to construct such solutions, and we do not have anything to add in this respect.

Finally, recall that in the derivation of the dihole solution in Sec. 3, an intermediate solution consisting of an extremal black hole surrounded by an extremal black ring was constructed. It is possible to check that they carry opposite charges, which cancel out to give a system with zero net charge. This extremal black ring is in fact equivalent, in the non-dilatonic case, to one obtained recently in [25], and it has a singular event horizon with vanishing area. It might be worthwhile to study this system in its own right.

References


