S-matrix elements and off-shell tachyon action
with non-abelian gauge symmetry

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ABSTRACT

We propose a prescription for making off-shell and expanding the S-matrix elements that corresponds to a non-abelian gauge invariant tachyon action with arbitrary mass $m$. We apply this proposal for the S-matrix element of four tachyons and for the S-matrix element of two tachyons and two gauge fields, in both bosonic and superstring theories. In the superstring theory, the leading terms of the expansion are in full agreement with the symmetrised trace of the direct non-abelian generalization of tachyonic Born-Infeld action in which the potential is consistent with $V(T) = e^{\pi\alpha' m^2 T^2}$. In the bosonic theory, the leading terms are reproduced by the above action and some other gauge invariant couplings. These extra couplings are zero in the abelian case.
1 The idea

Decay of unstable branes is an interesting process which might shed new light in understanding properties of string theory in time-dependent backgrounds [1]-[28]. In particular, by studying the unstable branes in the boundary conformal field theory (BCFT), Sen has shown that the end of tachyon rolling in this theory is a tachyon matter with zero pressure and non-zero energy density [3]. These results is then reproduced in field theory by a tachyonic DBI action. The form of tachyon potential in this action at minimum of potential is then fixed by using the fact that the higher derivative corrections to the tachyonic DBI action are not important at this point, and the action should not have plane wave solution [4]. Possible application of the tachyonic DBI action to cosmology have been discussed in [29]. The tachyonic DBI action was originally proposed as an effective action for describing the dynamics of unstable D-branes of the world-sheet superstring theory for slowly varying tachyon field [30, 31]. This proposal was based on analyzing in details the S-matrix elements involving tachyon vertex operators [30]. In this paper we would like to gives more evidence on this proposal when there are $N$ coincident unstable D-branes.

The world volume theory of a single stable D9-brane\(^1\) in Type II superstring theory includes a $U(1)$ vector $A_a$. The leading order low-energy action for this field is the ten-dimensional $U(1)$ Yang-Mills theory. As usual in string theory, there are higher order $\alpha'$ corrections. As long as derivatives of the field strength is small compared to $\alpha'$, the action is known to all orders in $\alpha'$, i.e., the Born-Infeld action [38],[33]:

$$S = -T_p \int d^{p+1}x \sqrt{-\det(\eta_{ab} + 2\pi \alpha' F_{ab})}.$$  

Even though the above action can be understood on the general grounds of covariance and T-duality transformation, one may expand this action and confirms the coefficient of each terms in the expansion by comparing the S-matrix elements of this theory with the leading $\alpha'$ order terms of the string theory S-matrix elements. The non-leading terms, on the other hand, can fix the higher derivative corrections to above action [34].

One of the most remarkable aspects of D-brane physics is that the $U(1)$ gauge symmetry of an individual D-brane is enhanced to a non-abelian $U(N)$ symmetry for $N$ coincident D-branes [35]. When $N$ parallel D-branes approach each other, the ground state modes of string stretching between the D-branes become massless. These extra massless states carry the appropriate charges to then fill out $U(N)$ representations and the $U(1)^N$ of the individual D-branes is enhanced to $U(N)$. Hence, $A_a$ becomes a non-abelian gauge field of $U(N)$. In this case the leading low energy action for the gauge field is the ten-dimensional non-abelian $U(N)$ Yang-Mills theory. When the covariant derivative and the commutator of field strength are small compared to $\alpha'$, the action to all orders of $\alpha'$ is proposed to

\(^1\)For simplicity, we consider in this paper only space filling D-branes of both bosonic and superstring theories. The action for lower D-branes can be obtained by applying T-duality rules [32].
be the symmetrised trace of the direct non-abelian generalization of the above Born-Infeld action (NBI) \[36\]. In the superstring theory, the leading terms of the string theory S-matrix element of four gauge fields expanded at low energy are fully consistent with this NBI action \[37\]. In the bosonic theory, the leading terms of the same S-matrix element is consistent with the NBI action and some extra terms that are commutator of field strength \[38\], i.e.,

\[
\text{Tr}((4i\alpha'/3)F_{ab}[F^a_c, F^{bc}] + 2\alpha'F^{ab}F^{cd}[F_{ab}, F_{cd}]).
\]

Recent calculation in Witten’s cubic string field also confirms the above results \[39\].

The world-volume theory of a single unstable D-brane in bosonic (superstring) theory includes, in addition to the gauge field, a tachyon $T$ with mass $m$ that its on-shell value is $m^2 = -1/\alpha'$ ($m^2 = -1/(2\alpha')$). For $N$ coincident unstable D-branes the single tachyon enhanced to a tachyon in the adjoint representation of $U(N)$. The leading $\alpha'$ order non-abelian action in this case is $U(N)$ Yang-Mills theory couples to the tachyon. Ignoring the self coupling of tachyon, the action in the minimal coupling is

\[
-T_p \text{Tr} \left( \frac{(2\pi\alpha')}{2} m^2 T^2 + \frac{(2\pi\alpha')}{2} D_a T D^a T - \frac{(2\pi\alpha')^2}{4} F_{ab} F^{ba} \right),
\]

where the covariant derivative is $D_a T = \partial_a T - i[A_a, T]$. In general, $m$ does not satisfy the on-shell condition, i.e., the action is an off-shell action which includes gauge field and a tachyon(scalar) field with mass $m$. Now we would like to extend this action to a non-abelian gauge invariant action that includes all orders of $\alpha'$ and all possible ways of couplings. The only restriction is that we ignore the second and higher covariant derivatives of the tachyon(scalar) or covariant derivatives of gauge field strength.

There are different approaches to study this off-shell action. One is the method introduced in \[46\] which is based on derivative expansion of partition function. The other one is based on directly integrating out the massive modes of the string field theory to find an off-shell action that includes tachyon and gauge field \[39\]. Our approach is based on S-matrix elements, i.e., comparing the S-matrix elements which involve gauge and tachyon fields, in field theory, with the corresponding S-matrix elements in string theory to find the unknown coefficients of different couplings in field theory. To do this, one needs an expansion for the S-matrix elements of string theory that its leading terms produce the S-matrix elements of the field theory.

The guiding principle in finding the appropriate expansion for the string theory S-matrix elements is the fact that the action must carry the non-abelian gauge symmetry. We propose an expansion for the S-matrix elements that their leading order terms are correspond to an action which includes only gauge field strength, tachyon and first covariant derivative of tachyon. And the non leading terms are proposed to be correspond to the higher covariant derivative terms. The expansion is an $\alpha'$ expansion. In the superstring theory, the coefficient of each order of $\alpha'$ is also a Zeta function. These Zeta functions reflect the fact that the infinite number of massive poles in the amplitude are expanded. For instance, the S-matrix
element of \( n \) tachyons has an expansion like,

\[
A \sim a + b\zeta(n-2) + c\zeta(n-1) + d\zeta(n) + \cdots ,
\]

where \( a \) includes terms that have massless and tachyon poles, \( b \) includes contact terms that are the Mandelstam variables to power \( n-2 \) and are of order \((\alpha')^{n-2}\), \( c \) includes contact terms that are the Mandelstam variables to power \( n-1 \) and are of order \((\alpha')^{n-1}\), and so on.

In the bosonic string theory, the S-matrix elements have some extra tachyonic pole that, as we will see later, they can not be reproduced by a gauge invariant action. Hence, one has to expand them to produce contact terms. The contact terms resulting from expansion of these poles have no Zeta function. So in the bosonic theory the expansion of the S-matrix elements is not an expansion in terms of Zeta function.

Our proposal for finding the expansion for the tachyon S-matrix elements that includes tachyon and gauge field vertex operators is the following:

1-Use the on-shell constraint on the Mandelstam variables to rewrite the S-matrix elements in the universal form.

2-Expand the S-matrix elements in the limit that the Mandelstam variables appearing in them go to zero.

To write the S-matrix elements in the universal form, one may compare the tachyon S-matrix elements with the corresponding scalar S-matrix elements in which the tachyon vertex operators are replaced by massless transverse scalar vertex operators [41, 42]. Then using the on-shell constraint on the Mandelstam variables, one can rewrite both S-matrix elements in a universal form. For example, consider the tachyon S-matrix elements that involves four tachyon vertex operators in superstring theory. This amplitude is given by [43]:

\[
A_{\text{tachyon}} \sim \alpha \frac{\Gamma(-2t)\Gamma(-2s)}{\Gamma(-1-2t-2s)} + \beta \frac{\Gamma(-2s)\Gamma(-2u)}{\Gamma(-1-2s-2u)} + \gamma \frac{\Gamma(-2t)\Gamma(-2u)}{\Gamma(-1-2t-2u)} ,
\]

where the Mandelstam variables are

\[
s = -\alpha'(k_1 + k_2)^2/2 ,
\]

\[
t = -\alpha'(k_2 + k_3)^2/2 ,
\]

\[
u = -\alpha'(k_1 + k_3)^2/2 .
\]

The coefficients \( \alpha, \beta, \gamma \) are the non-abelian group factors

\[
\alpha = \frac{1}{2} \left( \text{Tr}(\lambda_1\lambda_2\lambda_3\lambda_4) + \text{Tr}(\lambda_1\lambda_4\lambda_3\lambda_2) \right) ,
\]

\[
\beta = \frac{1}{2} \left( \text{Tr}(\lambda_1\lambda_3\lambda_4\lambda_2) + \text{Tr}(\lambda_1\lambda_2\lambda_4\lambda_3) \right) ,
\]

\[
\gamma = \frac{1}{2} \left( \text{Tr}(\lambda_1\lambda_4\lambda_2\lambda_3) + \text{Tr}(\lambda_1\lambda_3\lambda_2\lambda_4) \right) .
\]
The on-shell condition for the tachyons are $k_i^2 = 1/(2 \alpha')$, and the the Mandelstam variables satisfy the constraint $s + t + u = -1$. According to our proposal, to find the non-abelian expansion of (2), one should first compare it with the scalar S-matrix element that involve four scalar vertex operators to rewrite it in the universal form. The result for the scalar S-matrix element is $A_{\text{scalar}} = A_{s\text{scalar}} + A_{t\text{scalar}} + A_{u\text{scalar}}$ where

$$
A_{s\text{scalar}} \sim \zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \left( \alpha \frac{\Gamma(-2s)\Gamma(1 - 2t)}{\Gamma(-2s - 2t)} + \beta \frac{\Gamma(-2s)\Gamma(1 - 2u)}{\Gamma(-2s - 2u)} - \gamma \frac{\Gamma(1 - 2t)\Gamma(1 - 2u)}{\Gamma(1 - 2t - 2u)} \right),
$$

$$
A_{t\text{scalar}} \sim \zeta_1 \cdot \zeta_2 \cdot \zeta_4 \left( -\alpha \frac{\Gamma(1 - 2s)\Gamma(1 - 2t)}{\Gamma(1 - 2s - 2t)} + \beta \frac{\Gamma(-2u)\Gamma(1 - 2s)}{\Gamma(-2u - 2s)} + \gamma \frac{\Gamma(1 - 2t)\Gamma(1 - 2u)}{\Gamma(-2t - 2u)} \right),
$$

$$
A_{u\text{scalar}} \sim \zeta_1 \cdot \zeta_3 \cdot \zeta_4 \left( \alpha \frac{\Gamma(-2t)\Gamma(1 - 2s)}{\Gamma(-2t - 2s)} - \beta \frac{\Gamma(1 - 2s)\Gamma(1 - 2u)}{\Gamma(1 - 2s - 2u)} + \gamma \frac{\Gamma(1 - 2u)\Gamma(1 - 2s - 2t)}{\Gamma(-2u - 2t - 2s)} \right),
$$

where in this case the on-shell condition for the scalars are $k_i^2 = 0$, and the Mandelstam variables constrain to the relation $s + t + u = 0$. Now comparing the tachyon amplitude (2) with the scalar amplitude (5), and using the corresponding constraint on the Mandelstam variables, one can rewrite the gamma functions in both amplitudes in the universal form. The tachyon amplitude in the universal form is $A_{\text{tachyon}} = A_{s\text{tachyon}} + A_{t\text{tachyon}} + A_{u\text{tachyon}}$ where

$$
A_{s\text{tachyon}} \sim \alpha \frac{\Gamma(-2s)\Gamma(1 + s + u - t)}{\Gamma(u - s - t)} + \beta \frac{\Gamma(-2s)\Gamma(1 + s + t - u)}{\Gamma(t - s - u)} - \gamma \frac{\Gamma(1 + s + u - t)\Gamma(1 + s + t - u)}{\Gamma(1 + 2s)},
$$

$$
A_{t\text{tachyon}} \sim -\alpha \frac{\Gamma(1 + t + u - s)\Gamma(1 + s + u - t)}{\Gamma(1 + 2u)} + \beta \frac{\Gamma(-2u)\Gamma(1 + u + t - s)}{\Gamma(t - u - s)} + \gamma \frac{\Gamma(-2u)\Gamma(1 + s + u - t)}{\Gamma(s - u - t)},
$$

$$
A_{u\text{tachyon}} \sim \alpha \frac{\Gamma(-2t)\Gamma(1 + t + u - s)}{\Gamma(u - t - s)} - \beta \frac{\Gamma(1 + t + u - s)\Gamma(1 + s + t - u)}{\Gamma(1 + 2t)} + \gamma \frac{\Gamma(-2t)\Gamma(1 + s + t - u)}{\Gamma(s - t - u)}. 
$$

Now the non-abelian expansion for the scalar amplitude is the expansion at $s, t, u \to 0$. In
this expansion, one finds the following massless pole and tower of contact terms:

\[ A^{\text{tachyon}} = -4iT_p \left( \frac{(\alpha - \beta)(t - u)}{2s} + \frac{(\beta - \gamma)(s - t)}{2u} + \frac{(\alpha - \gamma)(s - u)}{2t} \right) + \zeta(2)(\alpha + \beta + \gamma) \left( 2ut + 2st + 2su - u^2 - t^2 - s^2 \right) + \cdots \]  \hspace{1cm} (6)

where we have also normalized the amplitude by the factor of \(-4T_p i\). The Zeta function is \(\zeta(2) = \pi^2/6\), and dots represents terms of order cubic and more in the Mandelstam variables. These terms are ordered in terms of higher Zeta functions, i.e., \(\zeta(3), \zeta(4), \cdots\).

Now in field theory we calculate the same S-matrix element. Using the free action, one finds that \(k_i^2 = -m^2\), and the Mandelstam variable become

\[ s = -\alpha'(k_1 + k_2)^2/2 = -\alpha'(-2m^2 + 2k_1 \cdot k_2)/2 \]
\[ t = -\alpha'(k_2 + k_3)^2/2 = -\alpha'(-2m^2 + 2k_2 \cdot k_3)/2 \]
\[ u = -\alpha'(k_1 + k_3)^2/2 = -\alpha'(-2m^2 + 2k_1 \cdot k_3)/2 \]  \hspace{1cm} (7)

The on-shell condition constrains the Mandelstam variables in the relation

\[ s + t + u = 2\alpha'm^2 \]  \hspace{1cm} (8)

Note that \(m\) can have any value. Using the Feynman rules for evaluating S-matrix elements in field theory, one realizes that the massless poles in (6) are reproduced by the leading action (1). The contact terms of order \(\zeta(2)\) are reproduced by the following couplings\(^2\):

\[ -(2\pi\alpha')^2 T_p \text{Str} \left( \frac{m^4}{8} T^4 + \frac{m^2}{4} T^2 D_a T D^a T - \frac{1}{4} (D_a T D^a T)^2 \right) \]  \hspace{1cm} (9)

In reaching to this result we have used several times the kinematic relations (7) and (8). It is easy to see that the leading action (1) and above couplings are different terms of the following symmetrised trace non-abelian tachyonic BI action

\[ \mathcal{L}^{\text{BI}} = -T_p \text{Str} \left( V(T) \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab} + 2\pi\alpha' D_a T D_b T)} \right) \]  \hspace{1cm} (10)

where the tachyon potential has the expansion i.e.,

\[ V(T) = 1 + \pi\alpha'm^2T^2 + \frac{1}{2}(\pi\alpha'm^2T^2)^2 + \cdots \]  \hspace{1cm} (11)

The higher order terms in (6) which have coefficient of higher Zeta function i.e., \(\zeta(n)\) with \(n > 2\) are expected to be related to the higher covariant derivatives of \(T\) which are not included in (10).

\(^2\)We didn’t try gauge invariant couplings like \(T^2 DDT DDT\). We expect these terms appear in the next order which includes second covariant derivative terms.
As another example consider the tachyon S-matrix element of four tachyon vertex operators in the bosonic string theory. The amplitude, in the universal form, in this case has an extra tachyonic pole relative to the amplitude in the superstring case \[42\], i.e.,

\[
A_{b}\mathrm{bosonic\ string} = \frac{1}{1 + 2s} A_{s}\mathrm{superstring},
\]
\[
A_{u}\mathrm{bosonic\ string} = \frac{1}{1 + 2u} A_{u}\mathrm{superstring},
\]
\[
A_{t}\mathrm{bosonic\ string} = \frac{1}{1 + 2t} A_{t}\mathrm{superstring}. \tag{12}
\]

Writing \(1/(1 + 2x) = -2x/(1 + 2x) + 1\), one may rewrite \(A_{s}\) as

\[
A_{s}\mathrm{bosonic\ string} \sim \frac{1}{1 + 2s} \left( \frac{\Gamma(1 - 2s)\Gamma(1 + s + u - t)}{\Gamma(u - s - t)} + \frac{\beta (1 - 2s)\Gamma(1 + s + t - u)}{\Gamma(t - s - u)} + \frac{\gamma (1 + s + t - u)\Gamma(1 + s + u - t)}{\Gamma(2s)} \right) + A_{s}\mathrm{superstring},
\]

similarly for \(A_{u}\) and \(A_{t}\). To find the expansion which corresponds to the non-abelian gauge theory, the gamma functions in the big parentheses which contains only massive poles must be expanded at \(s, t, u \to 0\). Now it raises a question that since we are interested in an action which includes massless as well as tachyon, should we expand the above tachyon pole or should we keep it as it stands and reproduce it by field theory? Replacing expansion for the gamma function in above equation, one finds

\[
A_{s}\mathrm{bosonic\ string} \sim \frac{1}{1 + 2s} \left( \alpha (u - s - t) + \beta (t - s - u) + \gamma (2s) + \cdots \right) + A_{s}\mathrm{superstring}, \tag{13}
\]

where dots represents terms cubic or more in the Mandelstam variables. The above leading terms are zero in the abelian case, so it is impossible to produce the above tachyonic pole by a field theory which includes coupling between three tachyons, i.e., \(T D_{a} T D^{a} T\) or \(T^{3}\). Hence, one has to expand also the tachyon pole at \(s \to 0\) to produce some contact terms of four tachyons and hopes that the contact terms are reproduced by a non-abelian gauge invariant contact terms in field theory. This is what has been done in the massless case \[38, 33, 39\] to find the gauge invariant action. As we will see later, however, the tachyon S-matrix elements, in general, have other tachyonic poles that one should not expand them. Naming the contact terms resulting from expanding the tachyon pole in (13), which only appear in the bosonic amplitude, as \(A_{\text{extra}}\), one can write them as:

\[
A_{\text{extra}} = -4i T_{p} \left( 2(\alpha - \beta)(u - t) - 2s(\alpha - \beta)(u - t) - 4\gamma s^2 + 2s^2(\alpha + \beta) \right) - 4i T_{p} \left( 2(\gamma - \alpha)(s - u) - 2t(\gamma - \alpha)(s - u) - 4\beta t^2 + 2t^2(\alpha + \gamma) \right) - 4i T_{p} \left( 2(\beta - \gamma)(t - s) - 2u(\beta - \gamma)(t - s) - 4\alpha u^2 + 2u^2(\beta + \gamma) \right) + \cdots, \tag{14}
\]
where dots represents contact terms which includes Mandelstam variables in cubic form or more. Unlike the \( S^{\text{superstring}} \) part, because of the expansion of the tachyon pole at \( s \to 0 \), not all these contact terms have one Zeta function. They are of order \( O(\alpha'^3) \), and are expected to be related to second and more covariant derivatives of the tachyon. Now consider the following non-abelian gauge invariant couplings:

\[
\mathcal{L}_1^{\text{extra}} = (2\alpha')^2 T_p \text{Tr} \left\{ \pi i F^{ab} D_a T D_b T + m^2 T D_a T D^a T - m^2 T D_a T D^a T T \right\} . \tag{15}
\]

The tachyon kinetic term that result from expanding the action (10) and the first term above give the vertex function for two external tachyons and one internal gauge field. The propagator for internal gauge field can also be read from the gauge field kinetic term that is in (10). They are

\[
V_{ij}^a(T_1T_2) = (2\pi\alpha')^2 T_p \left( (\lambda_1 \lambda_2)_{ij} - (\lambda_2 \lambda_1)_{ij} \right) (1 - 2s)(k_1^a - k_2^a) ,
\]

\[
(G_A)^{ab}_{ij,kl} = \frac{\delta^{ab} \delta_{jk} \delta_{il}}{(2\pi\alpha')^2 T_p s} i . \tag{16}
\]

Now the \( s \)-channel Feynman diagram \( V(T_1T_2) G_A V(T_3T_4) \) produces the massless pole in the first line of (6) and the first two terms in first line of (14). Similarly for the \( t \)-channel and \( u \)-channel. The other contact terms in (14) are reproduced by the four tachyons couplings in (15). The \( A^{\text{superstring}} \) part of the bosonic amplitude is obviously consistent with tachyonic BI action (10).

In the rest of this paper, we would like to extend above calculation to the tachyon S-matrix element that involves two tachyon and two gauge field vertex operators. We perform this calculation in both superstring and bosonic string theory. By comparing them with the scalar S-matrix elements of two gauge fields and two scalar vertex operators, we rewrite the amplitudes in the universal forms and find the non-abelian expansion for the amplitudes. In the superstring case that we perform the calculations in the next section, we show that the S-matrix element in the non-abelian limit has both tachyonic and massless poles, and infinite tower of contact terms. The tachyon and massless poles, and the leading contact terms are naturally reproduced by the non-abelian tachyonic BI action (10). In the bosonic case that we perform the calculations in section 3, the amplitude has two kind of tachyonic poles. In the non-abelian limit, one of them must be expanded and the other one again naturally is reproduced by the non-abelian gauge theory which is the tachyonic BI action (10), the non-abelian terms in (15), and some other gauge invariant couplings. Section 4 is devoted for some discussions on the results.
2 Amplitude in superstring theory

In this section, using the world-sheet conformal field theory technique, we evaluate 4-point function of two gauge and two tachyon vertex operators on the world-volume of \( N \) coincident unstable D-branes. This amplitude is given by the following correlation function:

\[
A \sim \text{Tr} \left( V^\text{gauge}(2k_1, \zeta_1) : V^\text{gauge}(2k_2, \zeta_2) : V^\text{tachyon}(2k_3) : V^\text{tachyon}(2k_4) : \right),
\]

where \( \zeta_i \) are polarization of the gauge field and \( k_i \) are momentum of states. We have used the doubling trick to work with only holomorphic functions on the boundary of world-sheet [40]. The vertex operators in superstring theory are:

\[
V^\text{gauge}_{-1}(k, \zeta) = \lambda \int dx (\zeta \cdot \psi(x)) e^{-\phi(x)} e^{ik \cdot X(x)},
\]

\[
V^\text{tachyon}_0(k) = \lambda \int dx (ik \cdot \psi(x)) e^{ik \cdot X(x)},
\]

where the on-shell condition for gauge field is \( k^2 = 0 = \zeta \cdot k \) and for tachyon is \( k^2 = 1/(2\alpha') \). In above equation, \( \lambda \) is matrix in the adjoint representation of the \( U(N) \) group. Now using different world-sheet propagators [40], and using the Wick theorem, one can easily calculate the correlators in (17). Performing these correlations, one finds that the integrand has \( SL(2, R) \) symmetry. One should fix this symmetry by fixing position of three vertices in the real line. Different fixing of these positions give different ordering of the four vertices in the boundary of the world-sheet. One should add all non-cyclic permutation of the vertices to get the correct scattering amplitude. So one should add the amplitudes resulting from the fixing \( (x_1 = 0, x_2, x_3 = 1, x_4 = \infty), (x_1 = 0, x_2, x_4 = 1, x_3 = \infty), (x_1 = 0, x_3, x_4 = 1, x_2 = \infty), (x_1 = 0, x_3, x_2 = 1, x_4 = \infty), (x_1 = 0, x_4, x_2 = 1, x_3 = \infty), (x_1 = 0, x_4, x_3 = 1, x_2 = \infty) \). After these gauge fixing, one ends up with only one integral which gives the beta function. The result is

\[
A^\text{tachyon} \sim \frac{1}{2} \zeta_1 \cdot \zeta_2 \left( -\alpha \frac{\Gamma(-2s)\Gamma(1/2 - 2t)}{\Gamma(-1/2 - 2s - 2t)} - \beta \frac{\Gamma(-2s)\Gamma(1/2 - 2u)}{\Gamma(-1/2 - 2s - 2u)} + \gamma \frac{\Gamma(1/2 - 2u)\Gamma(1/2 - 2t)}{\Gamma(1 + 2s)} \right)
+ 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_4 \left( \alpha \frac{\Gamma(-2s)\Gamma(1/2 - 2t)}{\Gamma(1/2 - 2s - 2t)} - \beta \frac{\Gamma(-2s)\Gamma(1/2 - 2u)}{\Gamma(-1/2 - 2s - 2u)} + \gamma \frac{\Gamma(-1/2 - 2u)\Gamma(1/2 - 2t)}{\Gamma(-2t - 2u)} + 3 \leftrightarrow 4 \right).
\]

The Mandelstam variables satisfy the on-shell constraint \( s + t + u = -1/2 \). To find the expansion corresponding to the non-abelian gauge symmetry, we should compare it with the S-matrix element of two gauge and two scalar vertex operators which is the following:

\[
A^\text{scalar} \sim \zeta_3 \cdot \zeta_4 \left( -\frac{\alpha}{\beta} \frac{\Gamma(-2s)\Gamma(1 - 2t)}{\Gamma(-2s - 2t)} - \frac{\beta}{\alpha} \frac{\Gamma(-2s)\Gamma(1 - 2u)}{\Gamma(-2s - 2u)} \right).
\]

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In this case the Mandelstam variables satisfy the constraint \( s + t + u = 0 \). Now it is not difficult to see that the gamma functions in both tachyon and scalar S-matrix elements can be written in the following universal form:

\[
A^{\text{tachyon}} \sim \frac{1}{2} \zeta_1 \cdot \zeta_2 \left( -\alpha \frac{\Gamma(-2s)\Gamma(1-t+u+s)}{\Gamma(u-s-t)} - \beta \frac{\Gamma(-2s)\Gamma(1-u+t+s)}{\Gamma(t-s-u)} \right)
\]

\[
+ \gamma \frac{\Gamma(-2s)\Gamma(1-t+u+s)}{\Gamma(1+2s)} \right) + 3 \leftrightarrow 4 \right) .
\]

Writing the amplitude in the universal form, one finds its non-abelian expansion by sending the Mandelstam variables in it to zero, i.e., \( s, t, u \to 0 \). We are not interested in the contact terms which have the Zeta function \( \zeta(3) \) and more, since they are of higher order of \( \alpha' \). So in the first big parentheses above we should keep the following terms of the gamma expansion:

\[
\frac{\Gamma(-2s)\Gamma(1-t+u+s)}{\Gamma(u-t-s)} = \frac{1}{2} + \frac{t-u}{2s} + \zeta(2)(s^2 - (u-t)^2) + \cdots ,
\]

\[
\frac{\Gamma(-2s)\Gamma(1-u+t+s)}{\Gamma(t-u-s)} = \frac{1}{2} + \frac{u-t}{2s} + \zeta(2)(s^2 - (u-t)^2) + \cdots ,
\]

\[
\frac{\Gamma(1-u+s+t)\Gamma(1-t+u+s)}{\Gamma(1+2s)} = 1 - \zeta(2)(s^2 - (u-t)^2) + \cdots ,
\]

and in the second parentheses have should keep the following terms:

\[
\frac{\Gamma(-2s)\Gamma(1-t+u+s)}{\Gamma(1+u-t-s)} = -\frac{1}{2s} - \zeta(2)(-t+u+s) + \cdots ,
\]

\[
\frac{\Gamma(-2s)\Gamma(-u+t+s)}{\Gamma(t-u-s)} = -\frac{1}{2s} + \frac{1}{-u+s+t} + \zeta(2)(-t+u+s) + \cdots ,
\]

\[
\frac{\Gamma(-u+t+s)\Gamma(1-t+u+s)}{\Gamma(1+2s)} = \frac{1}{-u+s+t} - \zeta(2)(-t+u+s) + \cdots .
\]
Replacing the above expansion for the gamma function in (18), one finds the following leading terms:

\[
A^{\text{tachyon}} = 4i(2\pi\alpha')T_p \left\{ \frac{1}{2} \zeta_1 \cdot \zeta_2 \left( -\frac{\alpha + \beta}{2} + \gamma \right) - \frac{\alpha - \beta}{2s} \left( \frac{1}{2}(t - u)\zeta_1 \cdot \zeta_2 + 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_4 \right) - \frac{\gamma - \beta}{u - t - s} \left( 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_4 \right) - \zeta(2)(\alpha + \beta + \gamma) \left( \frac{1}{2} \zeta_1 \cdot \zeta_2 (s^2 - (u - t)^2) + 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_4 (-t + u + s) \right) + \cdots \right\}.
\]

plus term with 3 $\leftrightarrow$ 4. We have normalized the amplitude at this point by the factor $4i(2\pi\alpha')T_p$. In above equation, dots represents contact terms that have $\zeta(3)$ or higher Zeta functions.

Now in field theory, using the fact that particle 1, 2 are massless gauge fields, and 3, 4 are tachyon with mass $m$, the Mandelstam variables become:

\[
s = -\alpha'(k_1 + k_2)^2/2 = -\alpha'(2k_1 \cdot k_2)/2 ,
\]
\[
t = -\alpha'(k_2 + k_3)^2/2 = -\alpha'(-m^2 + 2k_2 \cdot k_3)/2 ,
\]
\[
u = -\alpha'(k_1 + k_3)^2/2 = -\alpha'(-m^2 + 2k_1 \cdot k_3)/2 .
\]

Conservation of momentum constrains them in the relation
\[
s + t + \nu = \alpha'm^2.
\]

Expansion of the DBI action (10) has the following terms:

\[
\mathcal{L}^{\text{DBI}} = -T_p \text{STr} \left( 1 + (2\pi\alpha')(\frac{1}{2}D_a T D^{aT} + \frac{1}{2}m^2T^2) - \frac{(2\pi\alpha')^2}{4}F_{ab}F^{ba} + (2\pi\alpha')^3 \left( -\frac{1}{8}m^2T^2F_{ab}F^{ba} + \frac{1}{2}F_{ab}F^{bc}D_a T D^a T - \frac{1}{8}D_a T D^{aT}F_{bc}F^{cb} \right) + \cdots \right) .
\]

It is a simple exercise to verify that the commutator $[A_a, T][A^a, T]$ of the tachyon kinetic term reproduces the terms in the first line of (19). The Feynman tree diagram with one vertex $\partial_a T[A^a, T]$ from the tachyon kinetic term, the other vertex $\partial_a A^b[A^a, A^b]$ from the kinetic term of the gauge field, and the gauge field as propagator reproduces the massless pole in (19). The Feynman tree diagram with the two vertices $\partial_a T[A^a, T]$ from the tachyon kinetic term and tachyon as propagator, i.e., $G_T = i/(\pi\alpha'T_p(2u - \alpha'm^2)) = i/(\pi\alpha'T_p(u - s - t))$, reproduces the tachyonic pole in (19). Finally, the contact terms in the second line above are exactly reproduce the couplings in the third line of (19).

### 3 Amplitude in Bosonic string theory

The vertex operators in the bosonic string theory are:

\[
V^{\text{gauge}}(k, \zeta) = \int dx \left( \zeta \cdot \partial X \right) e^{ik \cdot X(x)} ,
\]

10
\[ V^{\text{tachyon}}(k) = \int dx e^{ik \cdot X(x)}, \]

where the on-shell condition for gauge field is again \( k^2 = 0 = \zeta \cdot k \) and for tachyon is \( k^2 = 1/(\alpha') \). Straightforward calculation as in the superstring theory gives the following result:

\[
A^{\text{tachyon}} \sim \left(-\frac{1}{2} \zeta_1 \cdot \zeta_2 + 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_3 \right)
\times \left( \alpha B(-1 - 2s, -2t) + \beta(-1 - 2s, -2u) + \gamma B(-2u, -2t) \right)
+ 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_4 \left( \alpha B(-1 - 2s, 1 - 2t) + \beta B(-1 - 2s, -1 - 2u) \right)
- \gamma B(-1 - 2u, 1 - 2t) \right) + 3 \leftrightarrow 4. \tag{22}
\]

The Mandelstam variables satisfy \( s + t + u = -1 \). As can be seen there are massless and tachyonic poles in all \( s, u \) and \( t \) channels. A non-trivial question is: which of these should be expanded and which should be reproduced by the non-abelian gauge theory? To answer this question, one has to find a limit that reduces the above amplitude to a series that its leading terms, which has tachyonic poles, massless poles and contact terms, are reproduced by a non-abelian gauge theory. According to our proposal for the expansion, one has to compare it with the scalar S-matrix that involves two gauge and two scalar vertex operators. The amplitude in this case is given by

\[
A^{\text{scalar}} \sim \zeta_3 \cdot \zeta_4 \left( -\frac{1}{2} \zeta_1 \cdot \zeta_2 + 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_3 \right)
\times \left( \alpha B(-1 - 2s, 1 - 2t) + \beta(-1 - 2s, 1 - 2u) + \gamma B(1 - 2u, 1 - 2t) \right)
+ 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_4 \left( \alpha B(-1 - 2s, 2 - 2t) + \beta B(-1 - 2s, -1 - 2u) \right)
- \gamma B(-2u, 2 - 2t) \right) + 3 \leftrightarrow 4 \right) .
\]

In this case the Mandelstam variables satisfy \( s + t + u = 0 \). Comparing the tachyon and scalar S-matrix elements, one realizes that the amplitude in the universal form is the following:

\[
A^{\text{tachyon}} \sim \left(-\frac{1}{2} \zeta_1 \cdot \zeta_2 + 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_3 \right) \left( \alpha B(-1 - 2s, 1 - t + u + s) \right)
+ \beta(-1 - 2s, 1 - u + t + s) + \gamma B(1 - u + t + s, 1 - t + u + s) \right)
+ 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_4 \left( \alpha B(-1 - 2s, 2 - t + u + s) + \beta B(-1 - 2s, -u + s + t) \right)
- \gamma B(-u + t + s, 2 - t + u + s) \right) + 3 \leftrightarrow 4 .
\]
In the universal form, expansion corresponding to the non-abelian gauge theory is the
to the superstring case. We call it $A^{\text{universal}}$. The non-abelian limit of this part is
expansion at $s,t,u$.

To study the non-abelian limit of the amplitude, it is convenient to separate the ampli-
tude into two parts. One is the amplitude that has the same dependency on the momenta
as in the superstring case. We call it $A^{\text{superstring}}$. The non-abelian limit of this part is
consistent with the non-abelian DBI action (10). The other part has all other terms that
appear only in the bosonic theory. We call it $A^{\text{extra}}$. Hence

$$A = A^{\text{superstring}} + A^{\text{extra}},$$

where $A^{\text{extra}}$ has the following terms:

$$2\alpha'\zeta_1 \cdot k_3 \zeta_2 \cdot k_3 \left( \frac{\Gamma(-2s)\Gamma(1-t+u+s)}{\Gamma(u-t-s)} \right)$$

$$+ \beta \frac{\Gamma(-2s)\Gamma(1-u+t+s)}{\Gamma(t-u-s)} - \gamma \frac{\Gamma(1-u+t+s)\Gamma(1-t+s+u)}{\Gamma(1+2s)}$$

$$+ 2\alpha'\zeta_1 \cdot k_3 \zeta_2 \cdot k_4 \left( \frac{\alpha(-t+u+s)}{\Gamma(1+u-t-s)} \right)$$

$$+ \beta(t-u-s) \frac{\Gamma(-2s)\Gamma(-u+s+t)}{\Gamma(t-u-s)} + \gamma(-t+u+s) \frac{\Gamma(-u+t+s)\Gamma(1-t+u+s)}{\Gamma(1+2s)}$$

$$+ \frac{1}{1+2s} \left\{ \frac{\Gamma(1-2s)\Gamma(1-t+u+s)}{\Gamma(1-u+t+s)} \right\} + 3 \leftrightarrow 4.$$

Expanding the gamma functions and the tachyonic pole $1/(1+2s)$ at $s,t,u \to 0$, after some
simple algebra, one finds the following leading terms:

$$A^{\text{extra}} = -4i(2\pi\alpha')T_p \left\{ \frac{-\alpha - \beta}{2s} \left( \alpha'\zeta_1 \cdot k_2 \zeta_2 \cdot k_1 (t-u) + \alpha'\zeta_1 \cdot k_3 \zeta_2 \cdot k_4 (\alpha - \beta) \right) \right.$$
\begin{align*}
&+ \left( s \zeta_1 \cdot \zeta_2 + \alpha' k_1 \cdot \zeta_2 k_2 \cdot \zeta_1 \right) (\alpha - \beta)(t - u) \\
&+ \left( s \zeta_1 \cdot \zeta_2 + \alpha' k_1 \cdot \zeta_2 k_2 \cdot \zeta_1 \right) s (\alpha + \beta - 2\gamma) + 3 \leftrightarrow 4 + \cdots \right) .
\end{align*}
\tag{24}

The above leading terms are zero for the abelian case, i.e., when \(\alpha = \beta = \gamma = 1\). In reaching to the above result we have used the conservation of momentum and the identities:
\[
\frac{(-t + u + s)(1 - t + u + s)}{(1 + 2s)(-u + t + s)} = \frac{t - s - u}{1 + 2s} + \frac{s + u - t}{-u + t + s} ,
\]
\[
\frac{2s(-t + u + s)}{(1 + 2s)(-u + t + s)} = \frac{-2s}{1 + 2s} + \frac{2s}{-u + t + s} .
\]

Note that the poles like \(1/( -u + s + t)\) which appear in (23) are disappeared in the expanded amplitude (24). Now consider the non-abelian tachyonic DBI action (10), the couplings (15), and the following couplings
\[
L_{\text{extra}}^2 = (2\pi \alpha')^2 T_p \text{Tr} \left\{ \frac{4 \pi i}{3} F^{ab} F_{ac} F_b^c + m^2 T F_{ab} T F^{ab} - m^2 T F_{ab} F^{ab} T
\right.
\]
\[
- D_a T D^a T F_{bc} F^{bc} + D_a T F_{bc} T D^a T F^{bc} \right\} .
\tag{25}
\]

The gauge field kinetic term that result from expanding the action (10) and the first term above give the following vertex function for two external gauge fields and one internal gauge field:
\[
V_{ij}^a(A_1 A_2) = (2\pi \alpha')^2 i T_p \left\{ (\lambda_1 \lambda_2)_{ij} - (\lambda_2 \lambda_1)_{ij} \right\} \left\{ \zeta_1 \cdot \zeta_2 (k_1^a - k_2^a) + 2k_2 \cdot \zeta_1 k_2^a - 2k_1 \cdot \zeta_2 k_1^a
\right.
\]
\[
- 2s \zeta_1 \cdot \zeta_2 (k_1^a - k_2^a) - 2k_1 \cdot \zeta_2 k_1^a \cdot \zeta_1 (k_1^a - k_2^a) \right\} .
\]

Now the s-channel Feynman diagram \(V(A_1 A_2)G_A V(T_3 T_4)\), where \(G_A\) and \(V(T_3 T_4)\) are given in (16), produces the massless pole in the second line of (19) and the massless pole in the first line of (24). It also produces some contact terms. These contact terms and the contact terms \(AATT\) resulting from the first term in (15) reproduce the contact terms in the first four lines of (24). The contact terms in the last line of (24) are reproduced by the other couplings in (25). On the other hand, the first term in (15) can produce, in general, vertex function with one gauge and one tachyon as external states and one tachyon as internal state. So with two of these vertex functions one produces a S-matrix element for two tachyons and two gauge fields. However, imposing the on-shell condition for the external gauge field, one finds that the result is zero. This is again consistent with the string theory result (24) which has no tachyonic pole.

4 Discussion

In this paper we proposed a prescription for expanding the S-matrix elements involving tachyon and gauge field vertex operators that correspond to tachyon action with arbitrary
mass and non-abelian gauge symmetry. The prescription has two steps: 1- Write the S-matrix elements in the universal form. 2- Expand them as the Mandelstam variables in them go to zero. We applied this prescription for the S-matrix element of four tachyon vertex operators and for the S-matrix element of two tachyons and two gauge fields. We then find the gauge invariant action that is consistent with the leading terms of the S-matrix element in this expansion. In the superstring theory, the non-abelian gauge invariant action is the symmetrised trace of the direct non-abelian generalization of the tachyonic DBI action (10). In the bosonic theory, the action has some extra gauge invariant couplings that are zero in the abelian case.

In principle, finding non-abelian gauge invariant tachyon action from on-shell S-matrix elements has less ambiguity than finding the tachyon action with abelian gauge symmetry. In the latter case, the calculation has the ambiguity that one can replace \( \alpha' \partial_a \partial^a T f(T, A) \) by on-shell mass of tachyon \( -T f(T, A) \), whereas, in the non-abelian case the same term appears as \( \alpha' D_a D^a T f(T, A) \) which can not be replaced by the tachyon mass because it represents coupling to gauge field as well.

In our proposal for finding the expansion for the S-matrix elements, one has to write the S-matrix elements in the universal form and then expand them. The result is exactly the same for tachyon and for the massless scalar fields. Hence the proposal is a specific prescription of how to sent the S-matrix elements to off-shell physics, i.e., there is no reference to the mass of tachyon in the string theory amplitudes. This implies that, unlike the on-shell S-matrix elements, the off-shell S-matrix elements have no ambiguity at all between \( \alpha' \partial_a \partial^a T f(T, A) \) and \( -T f(T, A) \), because these two are the same only in on-shell physics. In the field theory, on the other hand, one has the freedom to choose either \( \alpha' D_a D^a T f(T, A) \) or \( \alpha' m^2 T f(T, A) \) in one specific S-matrix element. However, the consistency with other S-matrix element may fix this arbitrariness. For example, one may change the coefficient of \( T^4 \) in the tachyon potential (11), in expense of adding coupling \( m^2 T^3 D_a D^a T \) to the action (9). This ambiguity, however, can be fixed by analyzing the S-matrix element of four tachyons and one gauge field vertex operator

The leading contact terms of the off-shell S-matrix element of four tachyons (6) is of order \( (\alpha')^2 \) which reproduces by the tachyonic DBI action (10). The non-leading terms of (6) that have the Zeta function \( \zeta(n) \) with \( n > 2 \) are of order \( (\alpha')^n \). They are expected to be related to the higher derivative terms in field theory. In fact the terms that have coefficient \( \zeta(3) \) in the expansion of (6) are the following:

\[
A(\alpha'^3) \sim 4\zeta(3) \left( \alpha(s^2u + ut^2 - su^2 - u^2t) \right)
\]

Note that there is no ambiguity in having term like \( T^2 D_a D^a T D_b D^b T \). Mass of the tachyon does not appear as the coefficient of this term. So if this term appears for the tachyon, it should also appear in the massless case. However, using the fact that the massless scalars can be added into the action by using the T-duality rules, such a term is not consistent with the gauge symmetry. Hence there is no such coupling for tachyon either.

\[3\]
\[ + \beta(u^2 t + s^2 t - ut^2 - st^2) + \gamma(st^2 + su^2 - s^2 t - s^2 u) \].

These contact terms are zero in the abelian case, i.e., when \( \alpha = \beta = \gamma = 1 \). So these terms should be reproduced by some combination of non-abelian gauge invariant coupling like \( DTDDTDDTDT \) which vanishes in the abelian case. This means in field theory at order \( (\alpha')^3 \) there is no term without covariant derivative, i.e., \( \zeta(3)(\alpha'm^2)^3T^4 \). All contact terms at this order must be reproduced by higher covariant derivative terms. The \( (\alpha')^4 \) contact terms of amplitude (6) are non-zero in the abelian case. So one can not immediately conclude that there is no term without covariant derivative. However, the coefficient of these terms is \( \zeta(4) \), so it is very unlikely that there is a term like \( \zeta(4)(\alpha'm^2)^4T^4 \) in field theory. If there is such a contact term, then it should be added to the coefficient of \( T^4 \) in the expansion (11). In that case, the tachyon potential could not be written in a closed form. On the other hand, if the higher order terms in the expansion of the string theory amplitude are reproduced only by higher covariant derivative of the tachyon field, then (11) is the correct expansion for the tachyon potential. In this case, using the fact that the on-shell tachyon potential should vanish at the minimum of the potential, one may expect the following form for the tachyon potential\(^4\):

\[
V(T) = e^{\pi \alpha m^2 T^2}.
\] (26)

When the second covariant derivative of tachyon is negligible, then the non-abelian tachyonic DBI action with the above potential is effectively describing the \( N \) coincident unstable D-branes in superstring theory. The restriction to one unstable D-brane is trivial. Now if one considers the tachyonic DBI action with this potential, one should not expect that the action must not have plane wave solution at the minimum of the tachyon potential. This because plane wave is not a slowly varying field. This is unlike the unstable branes considered in [4] that the higher derivative terms are negligible at the minimum of their on-shell tachyon potential.

The tachyon S-matrix elements in the universal form have no reference to the mass of the tachyon vertex operator. It indicates that there is nothing special about tachyon vertex operators. In fact, one can rewrite S-matrix element of any massive scalar vertex operator in the bosonic theory in the class,

\[
V = \lambda \int dx (\zeta_i \partial^n X^i) e^{ik \cdot X},
\] (27)

where \( n \geq 0 \), in the universal form. In above equation \( \zeta_i \) is polarization of the scalar states. The on-shell condition for the momentum is \( k^2 = -(n - 1)/\alpha' \). For example, the S-matrix element of four of these vertex operators is \( A = A_s + A_u + A_t \) where

\[
A_s \sim \zeta_1 \zeta_2 \zeta_3 \zeta_4 \left( \frac{\Gamma(-1-2s)\Gamma(2n-1-2t)}{\Gamma(2n-2-2s-2t)} + \beta \frac{\Gamma(-1-2s)\Gamma(2n-1-2u)}{\Gamma(2n-2-2s-2u)} \right).
\]

\(^4\)For the superstring theory, the on-shell tachyon potential (26) is the same as the potential one finds in the partition function approach [46].
than one, i.e., the massless gauge field and the above massive scalar. When the massive scalars are more
that we have found has arbitrary mass. Hence, the action is also the effective action for
four tachyons in the universal form (12). On the other hand, the gauge invariant action
above can be dropped. In this case, the above S-matrix element is exactly the amplitude
where the Mandelstam variables are those in (3). They satisfy the on-shell relation \( s+t+u = 2n-2 \). Using this relation, one can rewrite the amplitude in the universal form, i.e.,

\[
\begin{align*}
A_s & \sim \frac{\zeta_1 \zeta_3 \zeta_4}{1+2s} \left\{ \frac{\alpha \Gamma(-2s)\Gamma(1+s+u-t)}{\Gamma(u-s-t)} + \frac{\beta \Gamma(-2s)\Gamma(1+s+t-u)}{\Gamma(t-s-u)} \right\} , \\
A_u & \sim \frac{\zeta_1 \zeta_3 \zeta_4}{1+2u} \left\{ \frac{-\alpha \Gamma(1+t+u-s)\Gamma(1+s+u-t)}{\Gamma(1+2u)} + \frac{\beta \Gamma(-2u)\Gamma(1+u+t-s)}{\Gamma(t-u-s)} \right\} , \\
A_t & \sim \frac{\zeta_1 \zeta_4 \zeta_3}{1+2t} \left\{ \frac{\alpha \Gamma(1+s+u-t)\Gamma(-2t)\Gamma(1+s+t-u)}{\Gamma(u-t-s)} - \frac{\beta \Gamma(1+t+u-s)\Gamma(1+s+t-u)}{\Gamma(1+2t)} \right\} .
\end{align*}
\]

If one considers \( D_{24} \)-brane, then the scalar polarization is 1, and the polarization factors
above can be dropped. In this case, the above S-matrix element is exactly the amplitude
for four tachyons in the universal form (12). On the other hand, the gauge invariant action
that we have found has arbitrary mass. Hence, the action is also the effective action for
the massless gauge field and the above massive scalar. When the massive scalars are more
than one, i.e., \( D_p \)-brane with \( p < 24 \), the scalar field takes the index \( T^i \). It is not difficult
to insert this index in the effective action (10), (15), and (25). The only thing that one has
to do for on-shell massive field is to replace the on-shell mass of the field into the action.
In particular the potential (26) for massive field has no maximum, and its minimum is at
zero. Whereas, the potential for tachyon has maximum at zero and minimum at infinity.
Only in the latter case there is the interesting physics of tachyon condensation.

One may ask the question: Is it possible to rewrite the tachyon S-matrix element and
the S-matrix element of some other massive scalar vertex operators in the universal form?
For example consider the following vertex operator:

\[ V = \lambda \int dx (\zeta_{ij} \partial X^i \partial X^j) e^{ik \cdot X}, \]

where \( k^2 = -1/\alpha' \). The S-matrix element of four of this vertex operator has, among other things, the following term:

\[ A \sim \text{Tr}(\lambda_1 \lambda_2 \lambda_3 \lambda_4) \text{Tr}(\zeta_1^T \zeta_2 \zeta_4 \zeta_3^T) \frac{\Gamma(1 - 2s) \Gamma(3 - 2t)}{\Gamma(4 - 2s - 2t)}, \]

where the Mandelstam variables are constrain in the relation \( s + t + u = 2 \). One can not rewrite the the above S-matrix element and the corresponding S-matrix element of four tachyons in a universal form. That means even number of tachyons and only the scalar states in the class (27) have similar couplings. Other massive scalars have many different couplings that tachyon does not have.

We have seen that, in bosonic theory, to the order that we have considered in this paper, the tachyon pole in the universal amplitude has to be expanded to be consistent with non-abelian gauge symmetry. That means in the gauge theory one can not have coupling between two transverse scalars and one tachyon, or between three tachyons, i.e., \( TD_a TD^a T \) or \( T^3 \). That means the off-shell extension of the S-matrix element of three tachyons should have at least four momenta. Our prescription for finding the gauge invariant coupling involving the tachyon field does not work when there are odd number of tachyons, e.g., the S-matrix element of three transverse scalars is zero, whereas, the S-matrix element of three tachyons is non-zero. Hence they can not be written in the universal form. Similarly, the S-matrix element of one graviton and one scalar vertex operator is zero\(^5\), whereas, the S-matrix element of one graviton and one tachyon is non-zero. The off-shell extension of this coupling should be related again to higher derivative terms. This stems from the analysis done in [44] that shown the S-matrix element of two gravitons agrees with field theory if one expand the tachyonic pole in the amplitude. In general, one may expect that the off-shell extension of any S-matrix element of odd number of tachyons have no term without momenta. In that case, these S-matrix elements have no contribution to the tachyon potential. It would be interesting then to compare the S-matrix elements of odd number of tachyons with some other massive vertex operators that both can be written in the universal form. In that case, one would be able to extend the S-matrix elements to the off-shell physics.

Finally, the boundary conformal field theory (BCFT) with the specific marginal boundary term presented in [47, 2] is exactly solvable, and the couplings between gravity and the unstable brane are consistent with the way that the gravity appears in the tachyonic DBI

\(^5\)In order to write the scalar S-matrix elements and the tachyon S-matrix elements in the universal form, one must discard the couplings between scalar and the closed string which represent the functional dependency of the closed string fields on the transverse scalar fields [45].
action (10) with the following tachyon potential [22, 48]

\[ V(T) = \frac{1}{\cosh(\sqrt{\pi T}/\alpha)} , \]  

(28)

where \( \alpha = \sqrt{2} \) for bosonic theory and \( \alpha = 1 \) for the superstring theory. As noted in [22], the quadratic term of this potential gives the mass of tachyon only in the superstring theory. The marginal operator has the interpretation of an array of stable D-branes in the imaginary time, which is the unstable brane. A prescription is proposed in [49] on how to relate the scattering amplitude of closed strings from one stable D-brane in real time to the scattering amplitude from the array of D-branes in the imaginary time, i.e., the unstable branes of BCFT. In this discussion the open string of the stable D-brane has the interpretation of deformation of the unstable branes. The proposal gives zero result for the S-matrix elements that have only world-volume channels, e.g., s-channel. To have a bulk channel, e.g., t-channel, the simplest non-trivial calculation would be the evaluation of the disk level S-matrix element of two closed string gravitons and two open string tachyons. It would be interesting then to evaluate this S-matrix elements and then translate them to the scattering from unstable branes of the BCFT.

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