Defect formation in the early universe

Arttu Rajantie *
DAMTP, CMS, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, UK
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Abstract
Topological defects are common in many everyday systems. In general, they appear if a symmetry is broken at a rapid phase transition. In this article, I explain why it is believed that they should have also been formed in the early universe and how that would have happened. If topological defects are found, this will provide a way to study observationally the first fractions of a second after the Big Bang, but their apparent absence can also tell us many things about the early universe.

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1 Introduction

Imagine sitting at a round dinner table, with a plate in front of each guest and a spoon at half way between each two plates. This setting is symmetric between left and right, and you could therefore choose to use either the spoon on your left or the one on your right. However, as soon as your neighbours make their choice, this symmetry disappears, and you have to conform to that choice. This is a classic example of spontaneous symmetry breaking (SSB) (see Ref. [1]).

However, it may happen that the people on your right choose to use the spoons on their left and vice versa, leaving you with no spoon at all. This is an example of a topological defect. They occur generally in systems with SSB, because that requires a choice between several identical possibilities, and the choice is often different in different parts of the system.

Spontaneous symmetry breaking is a very common phenomenon in many physical systems. In the Standard Model of particle physics, it is in a sense responsible for the masses of the elementary particles, although the associated symmetry is of a more abstract nature, as it is a local gauge symmetry rather than a global symmetry. The Standard Model actually happens to be a special case with no topological defects, but this, nevertheless, suggests that its extensions would generally predict the existence of topological defects. Depending on the details, these defects could be domain walls, cosmic strings or magnetic monopoles.

Both in the early universe and in our dinner table example, the symmetry was initially unbroken. It got broken when the first guest picked up a spoon, or when the temperature of the universe decreased below a certain critical temperature. At the dinner table, it is easy to see that if table is large enough, topological defects are bound to form [2], but the process of defect formation was more complicated in the early universe. The purpose of this article is to explain it.

We can also ask if these defects could have survived until today and if and how they could be observed in that case. So far, we have no observational evidence for their existence. This alone can tell us many things about the early universe. On the other hand, defects have not been ruled out, and if they are found, that would have a huge impact on our understanding of the universe.

In order to draw any conclusions from the absence or possibly the existence of defects, it is crucial that we understand the process of defect formation properly. Fortunately, we do not have to rely solely on theoretical calculations, because topological defects are also formed at phase transitions in certain condensed matter systems such as superfluids and superconductors. This phenomenon is theoretically very similar to its cosmological counterpart, and we can use

*E-mail: a.k.rajantie@damtp.cam.ac.uk
this analogy to do ‘cosmological experiments’ [3]. On a more fundamental level, these same experiments can be used to test our understanding of non-equilibrium dynamics of quantum field theories, which will be very important for particle physics.

In this article, I review the theory of defect formation at phase transitions, emphasizing those aspects that are relevant for the early universe. I will also discuss the importance of defects to cosmology, but readers who want to know more about that are advised to read Refs. [4, 5, 6]. More discussion on defect formation at phase transitions can be found in Refs. [7, 8, 9].

Throughout the paper, I will use the natural units $c = \hbar = k_B = 1$. This means that everything is expressed in units of GeV (giga electron volt). Table 1 shows how they can be converted into SI units.

### Table 1: Conversion between natural and SI units.

<table>
<thead>
<tr>
<th>Unit</th>
<th>Natural units</th>
<th>SI units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>1 GeV</td>
<td>$1.60 \times 10^{-10}$ J</td>
</tr>
<tr>
<td>Temperature</td>
<td>1 GeV</td>
<td>$1.16 \times 10^{13}$ K</td>
</tr>
<tr>
<td>Mass</td>
<td>1 GeV</td>
<td>$1.78 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Distance</td>
<td>1 GeV$^{-1}$</td>
<td>$1.97 \times 10^{-16}$ m</td>
</tr>
<tr>
<td>Time</td>
<td>1 GeV$^{-1}$</td>
<td>$6.65 \times 10^{-25}$ s</td>
</tr>
</tbody>
</table>

2 Scalar fields and global symmetries

Apart from gravity, all physics at microscopic scales is described by quantum field theories (see Ref. [10]). For the moment, we will restrict our discussion to scalar fields, which is the simplest type of quantum fields. A scalar field is Lorentz invariant, which means that its value does not depend on the reference frame. It simply has some value at each point in spacetime, and this value is independent of the orientation and velocity of the observer. These values can be real numbers, but we can also think of complex scalar fields, or vector or matrix valued scalar fields.

Because the field is Lorentz invariant, there is a large amount of freedom in constructing scalar field theories. Most of this freedom is in the choice of the potential $V(\phi)$, which gives the energy density of a constant field with value $\phi$. For a real scalar field $\phi$, the potential leads to the classical equation of motion

$$\ddot{\phi} - \nabla^2 \phi = -V'(\phi),$$

where the double dot indicates a second time derivative. Apart from the gradient term $\nabla^2 \phi$, Eq. (1) is identical to the Newton equation for a particle in the same potential $V$. Therefore, we can understand the behaviour of the scalar field by thinking of a ball moving in a one-dimensional landscape.

In the vacuum state, the field is constant in space and time, and its value corresponds to the global minimum of $V(\phi)$. The value of $\phi$ in the vacuum is known as the vacuum expectation value (vev).

Let us now imagine a potential that has reflection symmetry around $\phi = 0$. Typically, this would be a polynomial in $\phi^2$, for instance

$$V(\phi) = \frac{1}{4} \lambda (\phi^2 - v^2)^2.$$  

The value of the potential does not change if we flip the sign of $\phi$. If $v^2$ is negative, the vacuum state is at $\phi = 0$. However, if $v^2 > 0$, this point becomes a local maximum, and two minima appear at $\phi = \pm v$. The field is now in the same situation as a guest at the dinner discussed at the beginning of this article. It will have to choose one of the two minima, and this breaks the reflection symmetry spontaneously.

If we have, instead of a real field, a complex one, we can think of a ball on a two-dimensional surface, which corresponds to the complex plane. We will be mostly interested in the case in which the potential depends only on the absolute value of $\phi$, not on its phase angle. Then, this two-dimensional picture is symmetric around the origin. A potential like this is shown in Fig. 1. A symmetry with respect to rotations around one axis, such as this, is known as U(1).
We can easily think of scalar fields with more components, although it is then more difficult to visualize the motion. In that case, the mechanical analogue is a ball moving in a higher-dimensional landscape, which we will call the *internal space*.

Whenever a symmetry is broken, the potential has several minima, which are related to each other by the symmetry, and each of them corresponds to a possible vacuum state. If the symmetry is continuous, these minima form a continuous valley with a perfectly flat bottom. If we give the ball a push in this direction, it will roll around the set of possible minima, no matter how weak the push was. This has the consequence that the quantum theory has states with arbitrarily low energies, or in other words, massless particles. These particles are known as *Goldstone bosons*.

Our previous dinner table example is not very helpful for visualizing the breaking of a U(1) symmetry, because it only dealt with a choice from two possibilities. In the case of U(1), we have a continuous set of possible vacua. Perhaps the closest analogue in everyday life is the time of the day. If we did not interact with other people, it would not matter to which time we set our watches. At any time of the day, we could choose any possible value from the whole range of possibilities from 00:00 to 23:59. Because 24:00 is the same as 00:00, we can think of a circle of possibilities. This can be interpreted as a U(1) symmetry.

However, our interaction with other people breaks this symmetry. The whole society has to set their watches to the same time in order to work properly, and this time plays the role of the vacuum expectation value. In the UK it was chosen that the clocks are set to 12 o’clock when the sun is at its highest above Greenwich, but in principle this choice was arbitrary.

### 3 Gauge fields and local gauge symmetries

The symmetries we have discussed so far have been global symmetries: The system is only symmetric under rigid rotations, where the field is rotated by the same amount everywhere. In our time analogy, this corresponds to changing the time by the same amount in the whole country. This is, of course, done every spring and autumn, when the UK changes from Greenwich Mean Time (GMT) to British Summer Time (BST) and back, and because of this symmetry, it does not...
cause any problems. On the other hand, if you forget to move your clock, you will most probably be missing appointments and encountering other problems.

The Standard Model of particle physics actually possesses an higher symmetry that allows rotations of the fields at different points by different amount without changing their physical content. A rotation like this is known as a local gauge transformation, and the corresponding symmetry as local gauge symmetry or gauge invariance. It is made possible by the gauge fields, which also carry the electromagnetic, weak and strong interactions. In this way, this symmetry dictates many properties of these interactions.

We can use the time analogy to illustrate gauge fields. As we noted, you cannot use a separate time convention from the rest of the country without having problems, but worldwide, different countries do use different time conventions. This is made possible by an extra structure, namely the time zones. We know that when we travel from the UK to, say, New York, we have to move our watches five hours backwards, and this is analogous to what the gauge field tells us in the case of a gauge field theory. If we want to compare the scalar field to its value at some other point, we have to first rotate it by the amount indicated by the gauge field.

Every March, when the UK moves clocks one hour forward and changes to BST, we carry out something similar to a local gauge transformation. Many countries do not move their clocks at the same time, but that does not cause problems as long as travellers are aware what the current time difference is on any given day. Analogously, if we carry out a local gauge transformation in a gauge field theory, the gauge field is changed in order to compensate for the rotation of the scalar field.

The simplest example of a gauge field theory is the Abelian Higgs model. It consists of a complex scalar field $\phi$ with a U(1) symmetry and a gauge field $A$, which is a real vector-valued field. Because we cannot directly compare field values at two different points, we have to replace gradients by covariant derivatives

$$\nabla \phi \to D \phi = \nabla \phi + ieA\phi,$$

where $e$ is known as the gauge coupling. (In non-Abelian theories, which we will discuss later, the gauge coupling is usually denoted by $g$.) Physically, this replacement gives the scalar particles an electric charge $e$, with the gauge field playing the role of the vector potential. The electric field is given by $E = -\partial A/\partial t$, and the magnetic field by $B = \nabla \times A$.

In the broken phase, where $\phi$ has a non-zero value, the system becomes essentially a superconductor. Any field configuration with a constant and non-zero $\phi$ and $B$ would have an infinite energy density. This is known as the Meissner effect and is a characteristic property of superconductors (see Ref. [11]). Real superconductors can be described with a similar model, in which the role of $\phi$ is played by Cooper pairs that consist of two electrons. In particle physics and cosmology, we are would think of the Abelian Higgs model as a relativistic theory and $\phi$ as a fundamental field. Then, the Meissner effect corresponds to a non-zero photon mass and is also known as the Higgs mechanism. In contrast to the global case, there are no massless Goldstone bosons, because the rotation around the vacuum manifold can always be compensated by the gauge field. It is therefore often said that the gauge field “eats” the Goldstone boson.

In terms of our time zone analogy, the presence of a non-zero magnetic field would correspond to a hypothetical situation, in which the amount by which travellers need to move their clock when they cross a border is not determined globally by time zones, but is specified for each border separately. Imagine that when you travel from the UK to France, you are told to move you clock one hour forward. When you travel further to Spain, you are again told to move your clock forward by one hour. Finally, when you travel from Spain to the UK, you are told to move your clock one hour backward. As a net effect, when you return home, your watch would show one hour more than the clocks you have in your home. In reality, this does not happen because each country follows a specific time zone. To the extent that the choice of the time zone corresponds to SSB, we can understand this as an example of the Meissner effect.

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1 Actually, the gauge field is a four-vector $A_\mu$, but we ignore the time component here. This can be done consistently without changing the physical content of the theory, and is known as the temporal gauge.
4 Particle physics

In particle physics, we encounter more complicated gauge symmetries than U(1). Weak and strong interactions are described by groups known as SU(2) and SU(3), correspondingly. These groups are called non-Abelian, because in these cases the gauge field is matrix valued. The product of two matrices depends on the order, i.e., $AB \neq BA$, and in algebra, an operation with this property is called non-Abelian or non-commutative. Nevertheless, the basic principles are the same as with the Abelian U(1) group.

The Standard Model of particle physics is based on SU(3), SU(2) and U(1) groups, and gives an extremely accurate description of strong, weak and electromagnetic interactions in terms of gauge fields, quarks, leptons and a scalar field known as the Higgs field.

The Higgs field is a complex two-component scalar field, which is sensitive to both U(1) and SU(2) groups. In the vacuum, the Higgs field has a non-zero value ($v \approx 246$ GeV), which breaks these groups spontaneously, but a residual U(1) subgroup survives. This U(1) group gives rise to electrodynamics in the same way as in the Abelian Higgs model. As a consequence, the Higgs mechanism makes the W and Z bosons massive, but leaves the photon massless.

The Standard Model has been highly successful experimentally, and in fact, there are still no experimental results that disagree with it. Nevertheless, it is in a sense unnecessarily complicated, because it has three separate gauge groups, each with its own coupling constant, and the way the quarks and leptons transform under gauge transformations does not seem very natural. In fact, if one calculates how the theory behaves at high energies, one finds that at around $10^{16}$ GeV [12], the strong, weak and electromagnetic interactions become equally strong. Furthermore, if one thinks of these gauge groups as subgroups of a larger gauge group, it turns out that it is also possible to explain the gauge transformation properties of quarks and leptons. There are several possible ways of doing this, and they are known as grand unified theories (GUT). The simplest example is based on a group known as SU(5).

In GUTs, the grand unified gauge group is split into the SU(3), SU(2) and U(1) subgroups by SSB. In the SU(5) theory, that is achieved by postulating a new, 24-component scalar field $\Phi$, which has a non-zero vev (of the order $10^{16}$ GeV). Unfortunately these energies are far beyond the reach of any experiments. There are other GUTs in which the U(1) group appears only at lower energies, such as the SO(10) theory with the symmetry breaking pattern

$$SO(10) \rightarrow SU(4) \times SU(2) \times SU(2)$$
$$\rightarrow SU(3) \times SU(2) \times U(1),$$  \hspace{1cm} (4)

where the energy scale of the latter symmetry breaking could be of the order $10^{12}$ GeV [13].

5 Topological defects

So far, we have been mainly discussing the vacuum states, but it is often also possible to find other time-independent states. Topological defects are one class of these, and they arise in models with spontaneously broken symmetries, if the choice of vacuum is different in different directions and these choices do not match completely.

We already encountered one very simple example of a topological defect in the introduction. There, the defect arose because the symmetry breaking involved a choice from a discrete set of possibilities. Defects like these are called domain walls, because in three dimensions, such a defect would form a surface separating regions of different vacua. For a real scalar field with the potential in Eq. (2) the corresponding solution is the “kink”,

$$\phi(x, y, z) = v \tanh \sqrt{\lambda} vz.$$  \hspace{1cm} (5)

This solution has a domain wall at $z = 0$, where the field vanishes and the symmetry is restored. On different sides of the wall, the fields approach different vacua asymptotically. The kink is stable, because it is not possible to deform the field into the vacuum configuration without changing its value everywhere in half of the space simultaneously. However, if a kink encounters an anti-kink, they annihilate.

In the case of a U(1) symmetry, the set of possible vacua, i.e. the vacuum manifold $\mathcal{M}$, is a circle (see Fig. 1). Let us now consider a closed curve, which we will denote by $C$, in space. At
Figure 2: A cosmic string, or a vortex line, in three dimensions. The direction of the complex scalar field $\phi$, indicated by the small arrows on the cross-sectional plane, rotates by a full $360^\circ$ around the vortex. The field vanishes in the vortex core.

Each point on the curve $C$, the field $\phi$ has some value, which we can think of the position of a ball in the potential in Fig. 1. Let us now move along the curve $C$ and ask how the ball moves in the potential.

In vacuum, $\phi$ has the same value everywhere, and the ball would therefore simply stay still in the same place, but this is not always true. In any case, when we have moved around the whole curve $C$ and come back to our starting point, the ball must also have returned to its original place, but along the way, it may have travelled around the circle of minima. The number of times it moved around the circle is known as the \textit{winding number}.

Let us now assume that this winding number is not zero. If we now deform the curve $C$ continuously, the path of the ball in the potential must also change continuously. If the ball stays on the vacuum manifold, the winding number cannot change, because that would require a discontinuous change. On the other hand, if we shrink the curve $C$ to a point, the path can consist of one point only and therefore must have winding number zero. The only way to avoid a contradiction is that the path of the ball has left the vacuum manifold at one point, but this means that somewhere inside the curve $C$, there must have been a point at which $\phi$ was not in vacuum. This point is known as a \textit{vortex}.

In circular cylindrical coordinates, a typical vortex has the rotation-invariant form

$$\phi(r, \varphi, z) = ve^{iwn}\ f(r),$$

where $n_W$ is the winding number and $f(r)$ is a function that vanishes at $r = 0$ and approaches 1 at infinity. In three dimensions, the vortices are one-dimensional vortex lines (see Fig. 2), and in the cosmological context, they are also known as \textit{cosmic strings}.

We have seen that the time zones correspond to a spontaneous breakdown of a U(1) symmetry, and there are also vortices in this analogy, namely the North and South Poles. Around each pole,
the time zone changes by a full 24 hours around any closed path that encircles a pole. The time zone is not well defined at the pole itself, and the symmetry is therefore restored there.

For a three-component real scalar field, the vacuum manifold is a sphere. In that case, it is possible that the field values on a closed surface \( S \) in space wrap around the sphere, analogously to how the field values wrapped around the circle on the curve \( C \) if the winding number was non-zero. Then the same argument shows that there must be a point inside the surface at which the field leaves the vacuum manifold. This is a *monopole*. A typical monopole has the spherically symmetric “hedgehog” form,

\[
\Phi^a(x) = v f(|x|) \frac{x^a}{|x|},
\]

where \( v \) is a constant. The function \( f(r) \) is again a function that vanishes at \( r = 0 \) and approaches unity at infinity, which means that the field is on the vacuum manifold far away from the origin. In \( \Phi^a \) the index \( a \) indicates the direction in the internal space, whereas in \( x^a \) it indicates the direction in coordinate space. In this sense, \( \Phi \) and \( x \) are parallel.

With a four-component scalar field, there is a further possibility that the field configuration in the whole three-dimensional coordinate space wraps around the vacuum manifold. That is known as a *texture* [14]. It is easy to see that a texture is not a stable object, because the energy of any scalar field configuration is generally of the form

\[
E = \int d^3 x \left[ (\nabla \phi)^2 + V(\phi) \right].
\]

In a texture, the field is everywhere on the vacuum manifold, and the latter term vanishes. The energy of a texture is completely due to the spatial variation of the field \( \phi \). If we double the size of the field configuration, the first term doubles, and therefore a texture would shrink to a point and disappear in order to minimize its energy.

In fact, only domain walls are truly localized objects in global theories, because the energies of vortices and monopoles diverge logarithmically and linearly with the system size, respectively. This is again due to the gradient term in Eq. (8). The only way to have a configuration with a finite energy is to have an equal number of vortices and antivortices, or monopoles and antimonopoles. Even then, a vortex and an antivortex would have a logarithmic interaction, which binds them in a pair. For a monopole-antimonopole pair, the interaction is linear, and therefore they would be confined just like quarks.

Topological defects also exist in gauge field theories, but their properties are somewhat different. Far from the defect, where the scalar field is on the vacuum manifold, the gauge field can cancel the gradient contribution to the energy in Eq. (8), because the gradient \( \nabla \phi \) gets replaced by the covariant derivative \( D \phi \) (see Eq. (3)). In the case of the Abelian Higgs model, this happens when

\[
A = -\frac{1}{e} \nabla \theta,
\]

where \( \theta = \text{arg} \phi \) is the phase angle of the scalar field. Thus, the (cross-sectional) energy of the vortex is finite, but it still does not vanish completely, because the energy density is non-zero near the vortex core, where the field is away from the vacuum manifold. In any case, this means that the energy of a defect is localized and finite in gauge field theories. Their interactions are also much weaker than those of global defects: Interactions of vortices are exponentially suppressed at long distances, and monopoles have a Coulomb type interaction.

It also follows from Eq. (9) that a gauged vortex carries magnetic flux [15]. The phase angle changes by an integer multiple of \( 2\pi \) around a vortex, but this means that the contour integral of the gauge field around the vortex must be a multiple of \( 2\pi/e \). This contour integral is nothing but the magnetic flux through the contour, and therefore the flux is proportional to the winding number

\[
\Phi = \oint d\mathbf{r} \cdot \mathbf{A} = -\frac{2\pi}{e} n_W.
\]

In other words, each vortex carries an integer multiple of the *flux quantum* \( \Phi_0 = 2\pi/e \).

This can be seen by placing a superconductor in an external magnetic field. As observed earlier, the superconductor repels the magnetic field, but in many cases, if the field is strong enough, it will penetrate the superconductor and form Abrikosov flux tubes, which are exactly what we have
called gauged vortices. Superconductors for which this happens are said to be of Type II. In Type I superconductors, the magnetic field forms one thick vortex that carries the whole magnetic flux. If \( \lambda < e^2 \), the Abelian Higgs model behaves like a Type I superconductor. Vortices with multiple windings are stable and two vortices of equal sign attract each other. In the opposite case with \( \lambda > e^2 \), two equal-sign vortices repel each other, and the theory resembles a Type II superconductor.

The simplest example of a gauge theory with monopoles is the \textit{Georgi-Glashow model}. It consists of a real three-component scalar field \( \Phi^a \), where \( a \in \{1, 2, 3\} \), and a three-component gauge field \( A^a \). The gauge field makes the theory invariant under local (i.e. position-dependent) rotations of \( \Phi^a \) in the three-dimensional internal space. This symmetry is spontaneously broken by a vacuum expectation value \( \Phi^a \Phi^a = v^2 \). However, it is still possible to do rotations with \( \Phi^a \) as an axis without changing it, and therefore there is a residual U(1) symmetry. We have seen before that a local U(1) symmetry gives rise to the electromagnetic field, and this happens in this case, too.

In normal electrodynamics, the field lines of the magnetic field cannot end, which means that there cannot be magnetic charges. The same is true in the broken phase of the Georgi-Glashow model as long as \( \Phi^a \neq 0 \). However, \( \Phi^a \) vanishes at the centre of the monopole solution in Eq. (7), and the corresponding solution in the Georgi-Glashow model does indeed have a non-zero magnetic charge. This solution, which is known as the \textit{'t Hooft-Polyakov monopole} \cite{16, 17}, is therefore a magnetic monopole. The total energy of a \textit{'t Hooft-Polyakov monopole} is finite, but it has a long-range \( 1/r \) contribution. Because the monopoles have magnetic charges, it is easy to understand this as the magnetic analogue of the Coulomb interaction.

Similar \textit{'t Hooft-Polyakov monopoles} exist in essentially all theories with a residual U(1) subgroup. In particular, this includes all GUTs, and therefore they generally predict the existence of magnetic monopoles. The masses of these monopoles are determined by the energy scale at which the U(1) group appears. In the SU(5) GUT, the monopoles would be extremely heavy, \( m_M \approx 10^{17} \) GeV, but for instance in SO(10), somewhat lighter monopoles with \( m_M \approx 10^{13} \) GeV would be possible (see Section 4).

6 Early universe

It was observed by Hubble in 1929 that the light of distant galaxies is redshifted, and that the amount of redshift is proportional to the distance. The simplest explanation to this is that the galaxies are moving away from us and the redshift is caused by the Doppler effect. Because the velocity is proportional to the distance from us, this does not imply that we occupy a special position at the “centre” of the universe, but that the universe is expanding with the same rate everywhere.

In order to study the evolution of the whole universe, we assume the \textit{cosmological principle}, which means that the universe looks the same at every point and in every direction, which seems to be a good approximation at very long distances. In technical terms, we assume that the universe is homogeneous and isotropic.

This assumption, together with the observation that the universe is spatially flat \cite{18} means that we can describe its expansion by two equations, the Friedmann equation and the conservation of energy,

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho, \quad (11)
\]

\[
\dot{\rho} = -3 \left( \rho + \frac{p}{c^2} \right) \frac{\dot{a}}{a}. \quad (12)
\]

Here \( a(t) \) is the \textit{scale factor} of the universe, the dot indicates a time derivative, \( \rho \) is energy density and \( p \) is pressure. The scale factor shows how the space stretches. If the distance between two objects at rest is at time \( t_1 \) would be by a factor of \( a(t_1)/a(t_0) \) longer than it was at time \( t_0 \). As the Hubble parameter \( H = \dot{a}/a \) is positive, the universe is expanding, and distant galaxies would indeed appear to be moving away from us.

Conservation of energy (12) shows that as the universe expands, the energy density decreases. Consequently, the temperature of the universe has been higher in the past. If we know the equation
of state, i.e. the pressure $p$ and a function of $\rho$ and $T$, we can use Eqs. (11) and (12) to follow the time evolution of the scale factor $a$ arbitrarily far back in time. If we assume that the equation of state does not change dramatically, this analysis shows that the scale factor started from zero about $13.7$ billion years [18] ago. We should not take this result too literally, because at that time, the energy density would have been infinite and we have no reason to believe that our assumptions were valid.

In any case, we can quite confidently extrapolate the evolution of the scale factor almost all the way back to this initial singularity, which we can choose to be at $t = 0$. By observing the cosmic microwave background (CMB) radiation, we can actually see all the way to $t \approx 380000$ years [18], when the universe became transparent. Although we cannot see beyond this “surface of last scattering”, we can use theoretical calculations to find out what happened before that.

Atomic and nuclear physics, and particle physics that has been tested and confirmed in accelerator experiments, can take us all the way back to $t \approx 10^{-10}$ s, which corresponds to the temperature of $100$ GeV. To reach ever earlier times, we have to make assumptions about physics that is out of the reach of any experiments. On the other hand, we can use the universe as the ultimate accelerator experiment, and calculate what kind of observational predictions follow from a given theory. This gives us a way of reaching much higher energies than particle accelerators ever can, albeit with the price that we cannot repeat the experiment.

As we will see in Section 7, symmetries that are broken at zero temperature are usually unbroken at high temperatures. If the temperature of the universe was initially high enough, the universe would have undergone a phase transition from the unbroken to the broken phase as it expanded and cooled down. We therefore expect that there were a number of symmetry breaking phase transitions in the early universe, and they may have led to defect formation.

On the other hand, this hot Big Bang scenario cannot have been valid all the way to the initial singularity, and perhaps the most compelling reason is the horizon problem [19]. Different directions on the sky look very similar, and if we actually measure the temperature of the cosmic microwave background, it is uniform up to one part in $100$ $000$ in different directions. Yet, this light was emitted only $380000$ years after the initial singularity, during which time any information could only have travelled a distance of about $1$ degree on the sky. Therefore, there is no physical mechanism that could have made different direction in thermal equilibrium with each other. There are also other problems with the Big Bang scenario, such as the absence of magnetic monopoles, to which we will return later.

In order to solve these problems, Alan Guth proposed the theory of inflation [19] (see also Refs. [20, 21]), whose basic idea is that the universe was dominated by vacuum energy at some early stage. This can be achieved with a real scalar field, which is known as the inflaton and which has a potential with the shape shown in Fig. 3. The energy density $\rho$ is mostly due to the kinetic energy $\frac{\dot{\phi}^2}{2}$ and the potential energy $V(\phi)$. If the field is initially placed at the local minimum at $\phi = \phi_0$, the time derivative $\dot{\phi}$ vanishes, and only the vacuum contribution $V(\phi_0)$ remains. Because $\phi$ does not change, the energy density $\rho$ is constant and Eq. (11) implies that the scale factor $a(t)$ grows exponentially.

Inflation not only explains why the universe is so homogeneous, but also where all the structure we see in it today came from. The exponential expansion amplifies quantum fluctuations, and they act as seeds for structure formation and can be seen as fluctuations in the CMB. Recent measurements of these fluctuations [18] agree remarkably well with the predictions.

In order to explain the observed homogeneity over the whole sky, inflation must have expanded the universe by at least a factor of around $10^{20}$, but after that, it must have come to an end in order to match with the ordinary Big Bang cosmology. Because energy is conserved, the energy density that was driving inflation, was released as radiation during this process. The dynamics of this reheating depends on the details of the inflationary model, but eventually the universe reached thermal equilibrium at the reheat temperature $T_{rh}$. Its value is not known, but the asymmetry between matter and antimatter requires it to be at least $100$ GeV.

In Guth’s original inflationary model did not describe reheating in a satisfactory way, but there are a vast number of other inflationary models that have been proposed to address this and other unattractive features of the original proposal. Most of them are based on the assumption of slow roll. In an expanding universe, the equation of motion (1) is modified to

$$\ddot{\phi} + 3H \dot{\phi} - \nabla^2 \phi = -V'(\phi),$$

(13)
where $H = \dot{a}/a$ is the Hubble parameter. The extra term in this equation behaves as a friction term and slows down the motion of the field in an expanding universe. Just like an object falling in the atmosphere reaches a terminal velocity due to the air drag, the scalar field reaches a slow roll solution

\[
\dot{\phi} = -\frac{V'(\phi)}{3H}.
\]  

(14)

If the slope of the potential is small compared with the Hubble rate, the value of the potential changes only very slowly. The energy density $\rho$ is, again, almost constant, and we find an exponentially growing, inflationary, solution.

In this setup, inflation can end smoothly if the potential starts to fall steeply at some value of $\phi$, as shown in Fig. 4. Perhaps the most elegant way of achieving that is by adding another scalar field $\chi$ with the potential

\[
V(\phi, \chi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \chi^2 + \frac{1}{4} (\chi^2 - v^2)^2.
\]  

(15)

The shape of this potential is shown in Fig. 4, and it is symmetric under $\chi \rightarrow -\chi$. This model is known as hybrid inflation [22].

We can see that if $\phi$ has initially a large value and $\chi$ is zero, the fields roll down slowly along the valley towards the origin and the universe inflates. However, when $\phi$ reaches $\phi_0 = m/g$, the “waterfall” field $\chi$ has to choose between the positive and negative branches, and this breaks the symmetry spontaneously. In either branch, the potential is much steeper, and therefore the slow roll assumption fails and inflation ends. The interactions between the inflaton and other fields reheat the universe as the inflaton rolls quickly down the hill to the minimum.

There are many variants of this hybrid inflationary model. One attractive feature of the models in this class is that potentials of the form of Eq. (15) arise relatively naturally in many particle physics models. As we have seen, inflation is ended by a symmetry breaking phase transition in these models, and it is therefore important to understand what happens at this transition and, in particular, whether topological defects are formed.
Figure 4: a) A typical potential for slow roll inflation. During inflation (A), the field is in a part of the potential that is almost flat, and therefore it is moving very slowly. This gives rise to almost exponential expansion. Inflation ends when the potential becomes steeper (B). The universe reheats, and the inflation field ends up in the minimum of the potential (C). b) The potential for hybrid inflation. At any time, the waterfall field $\chi$ is at its minimum, and therefore the potential is effectively of the type shown in (a). At the end of inflation, the $\chi \leftrightarrow -\chi$ symmetry is broken spontaneously at a phase transition.

7 Thermal phase transitions

At zero temperature, the equilibrium states of the system correspond to the minima of the potential. However, it is very well known from many everyday systems that a symmetry that is spontaneously broken at low temperatures can be restored at higher temperatures. One example of this is water: The crystal structure of ice breaks the rotation invariance, but when the temperature is above 0°C, ice melts and the rotation invariance is restored.

In field theories, one can see this by defining the effective potential $V_{\text{eff}}(\phi)$, which is essentially the free energy of a system in which the spatial average of the scalar field $\phi(x)$ is constrained to be $\phi$. Because the free energy is minimized in thermal systems, the equilibrium state corresponds to the minimum of the effective potential.

The effective potential is typically calculated using perturbation theory [23], and the leading contribution comes from one-loop Feynman diagrams. In the scalar theory, the main effect is to give a temperature-dependent quadratic contribution to the potential, $\Delta V(\phi) \propto \lambda T^2 \phi^2$ [24, 25]. As shown in Fig. 5a, this restores the symmetry when the temperature is above the critical temperature $T_c \approx v$.

The minimum of the effective potential gives the mean value $\langle \phi \rangle$, but the field fluctuates around this value. Therefore, even if the symmetry is restored and the mean value $\langle \phi \rangle$ vanishes, the actual value of the field at any given point is typically non-zero. In fact, the field only vanishes in a set
Figure 5: a) The effective potential in a continuous phase transition. Above the critical temperature $T_c$, the origin $\phi = 0$ is the only minimum, and the symmetry is restored. Two degenerate symmetry-breaking minima appear at $T = T_c$ and the origin becomes unstable. b) The effective potential in a first-order phase transition. At very high temperatures, the origin $\phi = 0$, is the only minimum, but when the temperature goes down, two metastable symmetry-breaking minima appear. At the critical temperature $T_c$, these minima are degenerate with the symmetric one. Below $T_c$, the symmetric minimum becomes metastable and eventually decays through bubble nucleation.

To leading approximation, the thermal fluctuations have a Gaussian distribution and therefore their properties are specified by the two-point function, which typically has an exponentially decaying form. The decay rate is characterized by the correlation length $\xi$. In practice, it means that fluctuations at two points separated by a distance greater than $\xi$ are independent.

When the transition point is approached, the correlation length diverges with a critical exponent $\nu$,

$$\xi(T) \sim \left(\frac{T - T_c}{T_c}\right)^{-\nu},$$

which depends on the universality class of the system. At the same time, the dynamics of the system becomes slower, and this critical slowing down can be expressed in terms of the relaxation time $\tau(T)$, which diverges with a (different) critical exponent $\mu$,

$$\tau(T) \sim \left(\frac{T - T_c}{T_c}\right)^{-\mu}.$$

For a real scalar field $\nu \approx 0.630$ [26], and for a complex field $\nu \approx 0.672$ [27]. In most systems, $\mu = 1$.

Unfortunately, it is often not possible to use the simple description in terms of the effective potential in gauge field theories. Thermal fluctuations can easily change the field configuration to any of its gauge transforms, because that does not cost any energy. This means that if we average over these fluctuations, the mean value $\langle \phi \rangle$ is always zero. This simple observation means that a local gauge symmetry cannot strictly speaking be broken spontaneously [29], and that one should only consider manifestly gauge-invariant quantities.

Nevertheless, it is possible to calculate the effective potential, if one first fixes the gauge. That means imposing a constraint, which breaks the gauge invariance, but does not change any gauge-invariant quantity. Then, one can calculate the effective potential and use it to study the phase transition. This may give non-zero values to non-gauge-invariant quantities such as $\langle \phi \rangle$, but one should keep in mind that this is just a non-physical consequence of the gauge fixing.

Even after the gauge fixing, the calculation of the effective potential is difficult in gauge field theories, and the reason is that the gauge bosons are massless in the symmetric phase. If the field in the one-loop Feynman diagram that contributes to the effective potential is massless, the diagram diverges and the calculation breaks down. This infrared problem can be partly solved by using a trick called resummation, but even that only works when the scalar coupling $\lambda$ is weak.
compared to the gauge coupling $g$ [30]. In that case, there is an extra cubic contribution $g^3 T \phi^3$ to the effective potential. There is, therefore, a range of temperatures in which the potential has two minima, one at zero and one at a non-zero value (see Fig. 5b). Furthermore, the values of a gauge-invariant quantities such as $\langle |\phi|^2 \rangle$ are different in these two minima, and therefore they actually correspond to physically different states.

When the minimum that corresponds to non-zero $\langle \phi \rangle$ becomes the global minimum, we have exactly the same situation as in Guth’s original inflationary model. The symmetric vacuum becomes metastable, and bubbles of the true, broken vacuum start to nucleate, in perfect analogy with bubbles of vapour in boiling water. A transition like this is known as a first-order phase transition. This does not always lead to inflation, though, because in many cases the energy of the thermal fluctuations dominates over the potential energy.

On the other hand, when the scalar coupling is strong relative to the gauge coupling, the perturbative calculation of the effective potential is unreliable. Generally, one will then have to use numerical lattice Monte Carlo simulations. It is known that for the Abelian Higgs model there is a phase transition, and the two phases can be characterized by whether the magnetic field has long-range correlations or, in other words, whether there is Meissner effect or not.

These correlations can be seen in the thermal fluctuations of the magnetic field, which are actually nothing but thermal radiation, which we will approximate by blackbody radiation. We can think of the fluctuations as a superposition of plane waves with different wave numbers $k$, each of which has a Gaussian probability distribution with variance $G(k)$. In the symmetric phase, this is simply equal to the energy of the degree of freedom, and because of classical equipartition of energy, we have

$$ G(k) = T. \tag{18} $$

In the broken phase, the Meissner effect starts to suppress long-wavelength fluctuations, and we find

$$ G(k) = \frac{T}{1 + m^2_\gamma/k^2}, \tag{19} $$

where $m_\gamma$ is the inverse photon correlation length, which is also known as the penetration depth in superconductivity. Near the transition in the broken phase, the value of $m_\gamma$ is approximately given by

$$ m^2_\gamma \approx e^2 (T_c^2 - T^2). \tag{20} $$

In non-Abelian theories such as the electroweak theory, the Georgi-Glashow model or GUTs, the transition actually disappears and becomes a smooth crossover when the scalar self-interactions dominate over the gauge coupling [31]. This may sound odd, because one might have thought about using the value of $\langle \phi \rangle$ to distinguish between the two phases, but this quantity is not gauge-invariant and does not therefore have a physical interpretation. In fact, because the gauge symmetry cannot be broken, there is no qualitative difference between the two phases, and in this sense the disappearance of the transition should not be surprising. After all, exactly the same happens to the water-vapour transition at a high enough pressure.

8 Global symmetries – the Kibble-Zurek mechanism

Let us first consider what happens in theories with global symmetries if a symmetry that was initially present gets spontaneously broken. The equilibrium state of the system after the transition should in principle correspond to a constant, non-zero value of $\phi$ with some small thermal fluctuations around it. However, the choice of $\phi$ is not unique, because it could equally well be any point on the vacuum manifold. It was pointed out by Tom Kibble in 1976 that this leads to defect formation [2].

If we think of two points that are far enough from each other, there is no reason to expect that they would make the same choice. In the cosmological context, an obvious constraint is that any two points must make this choice independently of each other if they are so far from each

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2Because we are interested in long wavelengths, the fluctuations are described by the classical Rayleigh-Jeans spectrum (18). This spectrum cannot be valid at very short wavelengths where $k \gg T$, because it would lead to the famous ultraviolet catastrophe. This was the reason why Planck postulated that the energy of radiation is quantized, which eventually led to the development of quantum mechanics. However, because we are considering macroscopic effects, the Rayleigh-Jeans spectrum is sufficient.
Figure 6: Vortex formation by the Kibble mechanism. Because the system has only a finite
time available, it can only be ordered at distances less than a critical value $\xi$. Therefore, it forms
domains of these size, inside each of which the scalar field, depicted by the small arrows, is aligned.
The system tries to interpolate the field between the domains, but it can only do it if the field
vanishes somewhere between the domains. The vortex appears where the field vanishes.

other that even a light signal would not have had time to travel from one to the other. However,
it is equally obvious that this only gives the upper limit for the distance. For the moment, let us
just say that there is a critical length scale $\xi$, and at longer distances the choice of the vacuum is
uncorrelated.

In this case, the system cannot reach the true equilibrium state, because that would require
making the same choice at infinite distances. Instead, the best it can do to minimize the energy
is to make the field more or less constant at distances less than $\xi$. To get a picture of this, we
can think that it forms domain of radius $\xi$, inside each of which the field is constant and between
which it is random.

Of course, there would not really be well-defined domains, because the field would vary
smoothly from one domain to another. However, even that is not possible everywhere. Imag-
ine, for instance, that a $U(1)$ symmetry is broken in two dimensions. Fig. 6 shows a possible
configuration of three neighbouring domains, and one can interpolate the field from 1 to 2, from
2 to 3 and from 3 to 1, but if one does that, one finds that there is a vortex at the point where
these domains meet. This means that there is roughly one vortex per domain, and the number
density of vortices formed in the transition is

$$n \sim 1/\xi^2.$$  \hspace{1cm} (21)

The same estimate applies to strings in three dimensions, except that that number density should
then be interpreted as the number density per cross-sectional area. In other words, the transition
produces a string network, and the number of strings that cross any two-dimensional plane is given
by Eq. (21).

It is easy to generalize this argument to monopoles, and that yields a number density per unit
volume of

$$n \sim 1/\xi^3.$$ \hspace{1cm} (22)

Because global monopoles have very strong interactions due to their linear interaction potential,
this expression is only valid immediately after the transition. Later on, the interaction starts
to pull monopoles and antimonopoles together, and whenever they meet they annihilate. The
number density of monopoles would therefore drop very quickly from its initial value in Eq. (22).
To a somewhat lesser extent, the same is also true for global vortices, which have a logarithmic
interaction potential.
Figure 7: Snapshot of a liquid crystal after the phase transition. Vortices are the places where the dark regions form a cross. (From Digal et al. [38])

The predictions in Eqs. (21) and (22) are not very useful unless we know how to calculate $\hat{\xi}$. It was argued by Zurek in 1985 [3] that it depends mostly on the critical exponents $\mu$ and $\nu$ defined in Eqs. (16) and (17). For simplicity, we assume that the temperature is decreasing linearly,

$$T(t) = \left(1 - \frac{t}{\tau_Q}\right) T_c.$$  \hspace{1cm} (23)

The quantity $\tau_Q$ characterizes the cooling rate, and is known as the quench timescale. When the temperature decreases towards $T_c$, the correlation length $\xi$ grows as $\xi(t) \sim (|t|/\tau_Q)^{-\nu}$, but at the same time the dynamics of the system becomes slower, $\tau(t) \sim (|t|/\tau_Q)^{-\mu}$. Eventually, at some time $\hat{t}$, the relaxation time $\tau$ becomes equal to the time that is left before reaching $T_c$, i.e., $\tau(\hat{t}) = |\hat{t}|$. After this time, the system cannot adjust to the change of the temperature fast enough, and falls out of equilibrium. This freeze-out time depends on $\tau_Q$ as $\hat{t} \propto -\tau_Q^{\mu/(1+\mu)}$. The correlation length cannot grow significantly after this time, and therefore we can, to a good approximation, say that it freezes to whatever its value was at time $\hat{t}$, and that determines the freeze-out scale $\hat{\xi}$ [3],

$$\hat{\xi} = \xi(\hat{t}) \propto \tau_Q^{\nu/(1+\mu)}.$$  \hspace{1cm} (24)

A similar, but much simpler, picture can be used for first-order transitions that take place through bubble nucleation. In that case, the domains would be replaced by bubbles, and the scale $\hat{\xi}$ would be the typical bubble radius at the time when the bubbles coalesce.

The predictions Eq. (21) and (24) have been tested experimentally in a variety of systems, ranging from superfluid $^4$He [32, 33] and $^3$He [34, 35] and liquid crystals (see Fig. 7) [36, 37, 38] to nonlinear optical systems [39] and convective fluids [40]. The results agree mostly with the predictions, although systematic tests have not been possible in many cases, because it is typically quite difficult to vary the quench time scale $\tau_Q$ and to determine the critical exponents $\mu$ and $\nu$.

Therefore, it is perhaps better to concentrate on how the defects and antidefects are distributed in space [38], because that is generally insensitive to details like these. Let us, for simplicity, discuss a two-dimensional system with vortices and consider the winding number $N_W(R)$ around a circle of radius $R$. On average, this is of course zero, $\langle N_W(R) \rangle = 0$, because there are as many vortices as antivortices. However, the variance $\sigma_W(R) \equiv \langle N_W(R)^2 \rangle$ is generally non-zero, and its behaviour as a function of $R$ reflects the spatial distribution of vortices.
If we assume that the locations of the vortices are more or less uniformly distributed, the number of vortices plus antivortices inside a circle of radius $R$ is $N_{\text{vort}} = n\pi R^2$, where $n$ is the number density.

If each of the $N_{\text{vort}}$ were just as likely to be positive as negative, $N_W(R)$ would be just a result of a random walk of $N_{\text{vort}}$ steps of unit length. The variance of that is well known to be

$$\sigma_W(R) = N_{\text{vort}} \propto R^2.$$  \hspace{1cm} (Random) \hspace{1cm} (25)

However, if the vortices were produced by the Kibble mechanism, we would expect something different. The arc of the circle would cross roughly $N_{\text{dom}} \approx R/\xi$ correlated domains. Each time we move from one domain to another, the phase angle changes by an essentially random amount, and around the whole circle, there are $N_{\text{dom}}$ of these changes. Therefore, we have a random walk of $N_{\text{dom}}$ steps, and consequently,

$$\sigma_W(R) \propto N_{\text{dom}} \propto R.$$ \hspace{1cm} (Kibble) \hspace{1cm} (26)

More generally, we can follow Ref. [38] and characterize the distribution of the vortices by writing

$$\sigma(R) \propto R^{4\nu_\sigma}.$$ \hspace{1cm} (27)

The above arguments show that a random distribution would give $\nu_\sigma = 1/2$, whereas the Kibble mechanism would predict $\nu_\sigma = 1/4$. These predictions do not depend on any quantities that would have to be measured or calculated separately such as the critical exponents.

The exponent $\nu_\sigma$ was measured experimentally for a phase transition in liquid crystals by Digal et al. [38], who found $\nu_\sigma = 0.26 \pm 0.11$, in agreement with what the Kibble mechanism predicts.

9 Gauge symmetries – trapping of thermal fluctuations

If the symmetry broken in a phase transition is a local gauge symmetry, it is not enough to consider the behaviour of the scalar field [41]. The gauge field has a dynamics of its own, and as Eq. (9) shows in the case of the Abelian Higgs model, the Higgs field does not even try to be aligned if $A$ is non-zero.

Let us start by discussing the Abelian Higgs model. If we imagine having initially a uniform magnetic field $B = B_0$ in the system, we know very well from superconductor experiments what happens when we cool it into the broken (superconducting) phase [11]: The magnetic field forms a lattice of vortices. Because all the flux must go into the vortices, the number density would be $n = (e/2\pi)|B_0|$. This happens no matter how slowly we cool the system, and therefore the formation of these defects is clearly not described by the Kibble mechanism.

Imagine now that the magnetic field is not uniform, but that there is a magnetic plane wave of some wavelength $\lambda$. In the absence of magnetic monopoles, the magnetic field lines cannot end, and the only way in which a field line can disappear is therefore by shrinking to a point. However, that cannot happen in less time than what it takes for light to travel from one end of the field line to the other. This time is simply the wavelength in units of the speed of light, and therefore the lifetime of a fluctuation cannot be shorter than the wavelength. It is actually more convenient to use the wave number $k = 1/\lambda$ than the wavelength $\lambda$. If we define the decay rate $\gamma(k)$ as the inverse of the time it takes for the amplitude of a fluctuation with wavenumber $k$ to halve, this result gives a causality bound $\gamma(k) < k$.

We can also calculate $\gamma(k)$ more precisely. The behaviour of the magnetic field is determined by Maxwell’s equations, and in vacuum it would simply oscillate with frequency $k$. However, in any medium, such as the hot elementary particle plasma that filled the early universe, there are electric currents. If they obey Ohm’s law with conductivity $\sigma$, long-wavelength modes ($k \ll \sigma$) have a decay rate

$$\gamma(k) = \frac{k^2}{\sigma},$$ \hspace{1cm} (28)

which is much slower than the causality bound.

If we start with the magnetic plane wave and cool the system into the superconducting phase very rapidly, the wave does not have time to decay. Because there can be no magnetic field in the broken phase, it forms vortices instead, as illustrated in Fig. 8. The vortices point in the
Figure 8: Vortex formation by flux trapping in the Abelian Higgs model. 

a) In the symmetric phase, there are Gaussian thermal fluctuations of all wavelengths. 

b) During a rapid quench, longest wavelengths $\lambda > 1/k_c$ are too slow to react and freeze out. Shorter wavelengths disappear, leading to a smooth magnetic field, which varies over the distance scale $1/k_c$. 

c) The broken phase is a superconductor, and the frozen-out magnetic field becomes trapped into vortices.
same direction as the initial magnetic field, and their number density is higher where the field was stronger. In fact, if we looked at the system with a low enough resolution so that we could not see individual vortices but could still measure the average magnetic field arising from the flux quanta carried by the vortex, we would still see the initial magnetic field. On the other hand, if the cooling is slow enough, then the initial wave decays and no vortices are formed.

So, we have seen that a non-zero initial magnetic field leads to vortex formation, but where could that initial magnetic field come from? The answer is thermal fluctuations. We already saw that arbitrarily long wavelengths are present in the Rayleigh-Jeans spectrum (18) of thermal radiation. Let us consider a single mode with wave number \( k \), i.e., a single plane-wave component of the thermal spectrum. If the corresponding decay rate \( \gamma(k) \) is slower than what it would have to be in order for the mode to stay in equilibrium, the mode freezes out and keeps its initial equilibrium amplitude, forming vortices as we saw above.

If we know how the temperature changes, we know from Eq. (19), what the equilibrium amplitude of any given mode is and therefore how quickly the mode would have to decay in order to adjust to this change. It is obvious that there are always going to be some very long wavelengths that are unable to do it. Simply from causality, we obtain the lower bound

\[
k_c > \left( \frac{e^{2T^2}}{\tau Q} \right)^{1/3},
\]

and if Ohm’s law is valid, we have

\[
k_c \approx \left( \frac{e^{2T^2}}{\tau Q} \sigma \right)^{1/4}.
\]

Now, we will have to find out what this means in terms of the number density of vortices. By definition, the modes with \( k \leq k_c \) freeze out, and those with \( k \geq k_c \) equilibrate. This magnetic field is squeezed into vortices, but let us ignore that for a moment, and assume that all modes with \( k \geq k_c \) simply disappear. This means that the magnetic field configuration is a superposition of waves with wavelength longer than \( 1/k_c \), and if we look at the system at distances shorter than that, we simply see a uniform magnetic field. Eventually, all this flux turns into vortices (see Fig. 8c). If there was initially a flux \( \Phi(R) \) through a circle of radius \( R \leq 1/k_c \), it will form \( \Phi(R)/\Phi_0 \) vortices, all with the same sign. The number density \( n \) can then be calculated by dividing this by the area \( \pi R^2 \) of the circle.

On the other hand, if \( R > 1/k_c \), the Fourier modes with \( k \geq k_c \) only affect the spatial distribution of the flux inside the curve, but do not contribute to the net flux. Consequently, the final flux \( \Phi(R) \) after the transition is the same as it was in the symmetric phase before the transition. This can be calculated from \( G(k) \) or estimated using physical arguments, and the result is [41]

\[
\Phi(R) \approx \sqrt{TR}.
\]

If we now choose \( R = 1/k_c \), we are on the borderline of these limiting cases. We can say that the flux is given by Eq. (31), and that the whole flux is turned into vortices of equal sign. Therefore, we know what \( \Phi(1/k_c) \) is, and we can calculate the number density of vortex lines,

\[
n \approx \left. \frac{\Phi(R)/\Phi_0}{\pi R^2} \right|_{R=1/k_c} \approx \sqrt{e^{2T^3}}.
\]

Using Eq. (30), we can write this as

\[
n \approx e^{7/4} T^{5/4} \left( \frac{\sigma}{\tau Q} \right)^{3/8}.
\]

The two-dimensional analogue of this result has been tested in numerical simulations [42], and seems to work well (see Fig. 9). There have also been attempts to test the result experimentally with superconductors (see Fig. 10) [43, 44, 45]. However, the actual experiments are usually done with two-dimensional films. This is not equivalent to the fully two-dimensional setup used in the simulations, because in the experimental setup, the magnetic field extends outside the film. Therefore, the magnetic field behaves as in three dimensions and the scalar field as in two
Figure 9: A snapshot from a numerical simulation of the Abelian Higgs model. It shows the magnetic field configuration after a quench into the broken phase. Black and white correspond to magnetic fields pointing up and down, respectively. The clustering of vortices is obvious. (From Stephens et al. [42]).

dimensions. This situation was discussed briefly in Ref. [41], and a more detailed analysis is given in Ref. [46].

However, just like the Kibble mechanism, this scenario can also give predictions that are independent of dynamical parameters such as the conductivity. As we saw, the flux $\Phi(R)$ through a circle with $R = 1/k_c$ turns into vortices of equal sign, and therefore, if it is greater than $\Phi_0$, we should see clusters of equal-sign vortices. Based on Eq. (32), we would expect roughly $N_{cl} \approx (e^2 T/k_c)^{1/2}$ vortices per cluster. Because we can make $k_c$ arbitrarily small by cooling the system slower, we should be able to make these clusters if we can control the cooling rate.

If $N_{cl}$ is large enough, it is much more likely that a randomly picked vortex is inside a cluster than near its boundary. Therefore, if we calculate the quantity $\sigma_W(R)$ defined in Section 8, all the vortices contributing to $N_W(R)$ are of the same sign as long as $R$ is less than the cluster size $1/k_c$. Therefore $N_W(R)$ is proportional to its area, and consequently,

$$\sigma_W(R) \propto R^4,$$

(34)

at short distances. In other words, $\nu_\sigma = 1$.

Furthermore, if we can measure both the number density $n$ and the cluster size $N_{cl}$, the combination $n N_{cl}^3$ should be independent of $k_c$ and can therefore be used to test the scenario. On the other hand, if one wants to learn more about the dynamics of the phase transition, one would measure that combination $n/N_{cl}$, which would give the freeze-out scale $k_c$.

The result in Eq. (32) is proportional to the square root of temperature and would therefore vanish as $T$ goes to zero. This is important, because the transition that ended hybrid inflation would have taken place at zero temperature. Would defect formation then have happened according to the Kibble mechanism, or would the gauge fields still have played some role? In the above discussion, we have only considered classical thermal fluctuations, and it might be possible that quantum fluctuations play a similar role at zero temperature. More work is needed to find out whether that is really the case.

I have argued recently in Ref. [47] that the formation of magnetic monopoles in the GUT phase transition may take place in a somewhat similar way. There are several extra complications, though. In particular, as was already mentioned, there may actually be no true phase transition
Figure 10: The distribution of magnetic flux in an array of superconducting rings after a rapid phase transition. The similarity with Fig. 9 is striking, and the authors report that their results show that thermal fluctuations play an important role. The result is still not a conclusive proof of the theory discussed in Section 9, because a ring array is in many ways different from a superconducting film or indeed a three-dimensional space. (From Kirtley et al. [44])

at all, just a smooth crossover from one phase to another. One may then ask whether monopoles are formed at all.

It is, however, easy to see that the theory cannot stay in equilibrium if it is cooled at a constant rate. There are no long-range correlations in the symmetric phase of the theory, and it is generally known that any correlation length has to be of the order of \((g^2T)^{-1}\) or shorter [48], where \(g\) is again the gauge coupling constant. On the other hand, at a very low temperature we should recover normal electrodynamics, which has a massless photon. Correspondingly, the magnetic field has an infinite correlation length.

Admittedly, it is not obvious what “magnetic field” means in the symmetric phase, but no matter how we define it, we have a correlation length \(\xi_B\) that starts from a microscopic value \((g^2T)^{-1}\) and grows to an infinite value. Analogously with Kibble’s argument, we can say that \(\xi_B\) cannot grow arbitrarily fast, because causality alone restricts its growth rate to be less than the speed of light. It is therefore inevitable that if we keep on cooling the system, it falls out of equilibrium at some point and the magnetic correlation length freezes to a finite value, which we denote by \(\xi_B\).

The next key observation is that the finiteness of the correlation length is due to screening. This is analogous to the Debye screening of electric field by electric charges [49], and therefore it implies a presence of magnetic charges. Thus, because the correlation length is necessarily finite due to causality, magnetic monopoles must have been formed.

These arguments can be made more quantitative by noting that one can define a magnetic charge that is conserved even in the symmetric phase. This charge behaves in a similar way to the magnetic field in the Abelian Higgs model. In the symmetric phase, there are long-wavelength thermal fluctuations, which should disappear in the broken phase, but again, causality implies that fluctuations whose wavelength exceeds some critical value do not have time to decay. They freeze out and form vortices. This critical wavelength turns out to be exactly the value \(\xi_B\) to
which the correlation length of the magnetic field freezes. One finds that the number density of
monopoles formed at the transition would be roughly \[ n_M \approx \sqrt{\frac{g^2 T}{\xi_B^5}} \] (35)

As for vortices, one also find that if \( \xi_B \) is long enough, clusters of \( \sqrt{g^2 T \xi_B} \) monopoles are formed.

So far, we have been discussing continuous (or smooth) phase transitions, but discontinuous
first-order transitions are possible both in Abelian and non-Abelian theories, and in general they
take place if the Higgs self coupling constant \( \lambda \) is less than the square of the gauge coupling
constant.

In a first-order transition, bubbles of the broken phase are nucleated, and they grow and
coaalesce. We can safely assume that there are no defects inside the bubbles. Because magnetic
field and charge are conserved in Abelian and non-Abelian cases, respectively, they will be trapped
in the gaps between the bubbles. Therefore, we would expect that monopoles are formed in well-
localized clusters. In the Abelian case, vortices with multiple windings are stable if there is a
first-order transition, and therefore we would expect thick and massive vortex lines, which carry
much higher fluxes than in the continuous case [50].

10 Defects in cosmology

There is no conclusive evidence for the existence of topological defects in the universe, but there
is strong evidence against certain types of topological defects. First of all, domain walls are ruled
out unless they, for some reason, have an extremely low energy density [4]. The reason for this
is that the mass of a domain wall is proportional to its area and is therefore enormous for a wall
that extends through the whole universe.

Global strings or monopoles could arise from a spontaneous breakdown of a continuous global
symmetry, but as we saw in Section 2, that would predict the existence of massless Goldstone
bosons. No such particles have been observed, but they may still exist if they only interact
extremely weakly with other forms of matter.

Otherwise, we are left to consider gauged defects. Cosmic strings have been studied intensively,
because it was thought they could explain the primordial density perturbations that gave rise to
the large-scale structure of the universe and the temperature fluctuations of the cosmic microwave
background (CMB) radiation. However, recent CMB measurements [18] show that this is not the
case, and that these perturbations probably originated in quantum fluctuations during inflation.
This observation does not by any means rule out the existence of cosmic strings, and people
are still looking for evidence for or against them in the CMB radiation and in other astronomical
observations. One way of finding strings is by gravitational lensing. Because strings are so massive,
they would bend the light coming from galaxies behind them and therefore we would see two copies
of the same galaxy. A possible such observation of a cosmic string has been reported recently [51].

In principle, magnetic monopoles would be a natural candidate for the missing dark matter,
which is known to make up about 85% of all the matter in the universe [18]. They have been
searched for in experiments, but with no success [52]. This seems to rule out the possibility of
superheavy magnetic monopoles with GUT scale masses being the main component of the dark
matter. There is also a more severe problem: The number of magnetic monopoles produced at
the GUT phase transition has been estimated to be so high that their mass would have caused the
universe to collapse very soon after the transition [53]. This is known as the monopole problem.

In the standard Big Bang scenario with no inflation, we would assume that the temperature
was initially around the Planck scale \( 10^{19} \) GeV. When the universe was expanding and cooling
down, it went though the GUT transition at around \( 10^{16} \) GeV. Because all GUTs predict the
existence of magnetic monopoles and they would have been formed at this transition, this scenario
suffers from the monopole problem.

The monopole problem is solved by inflation if the reheat temperature was low enough so that
the GUT symmetry was never restored. So far, there are very few observational constraints on the
reheat temperature, and more or less the only other constraint comes from the matter-antimatter
asymmetry. There is overwhelming observational evidence that there is practically no antimatter
but a significant density of matter in the universe. It seems impossible to generate this asymmetry after the electroweak phase transition [54], which took place at around 100 GeV, and therefore the reheat temperature should be high enough for the electroweak symmetry to be restored. This gives the rough bounds,

$$100 \text{ GeV} \lesssim T_{\text{rh}} \lesssim 10^{16} \text{ GeV}. \quad (36)$$

If there were other phase transitions, cosmic strings may have been produced provided that the reheat temperature was above the corresponding critical temperature.

There could, however, be extra complications. First of all, it is not at all clear, how well the scenarios that assume an equilibrium initial state can really describe phase transitions that take place in a rapidly expanding universe only a short time after the end of inflation. The assumption of thermal equilibrium seems particularly questionable at long distances, between points that have only barely come within each other’s sight.

In many inflationary models, reheating can be a very violent non-equilibrium process, in which case it is often called *preheating* [55]. It may then be possible to produce topological defects [56, 57, 58, 59] even if the reheat temperature is well below the critical temperature. In some cases, it is possible to describe this as a “non-thermal” symmetry restoration. It is not known how well the scenarios discussed in Sections 8 and 9 can describe defect formation from preheating. In any case, it seems that we may have to rethink the upper bound in Eq. (36). The same applies to the lower bound, because similar effects may also create the matter-antimatter asymmetry [60, 61, 62].

There is also a third possibility in hybrid inflation (15). Because inflation ends at a breakdown of a symmetry, there is always a phase transition at the end of inflation, and that would typically lead to defect formation [22, 63, 64]. If we want to avoid domain walls and Goldstone bosons, the waterfall field $\chi$ will have to be a multi-component field with a gauge symmetry. Because the universe is at zero temperature at the time of the phase transition, it is not yet clear if the scenario discussed in Section 9 is applicable.

### 11 Conclusions

Theoretically, it seems plausible that topological defects should have been formed in the early universe. If signs of them are observed, that will give an enormous boost to our understanding of the universe, but the failure to find them can also tell us many things. Most significantly, the absence of the high number of monopoles predicted by the theory led to the development of the theory of inflation, which has become the standard paradigm in cosmology because of other supporting observations. The apparent absence of defects can also help us constrain certain parameters of the inflationary models, which would otherwise be undetermined. This will become more important in the future, when we start to home in on a particular class of inflationary models using data from other observations. It also has significance in particle physics, because it constrains possible high-energy extensions of the Standard Model of particle physics.

For all this it is important to have a good understanding of how defects are formed. Currently, there are theories that describe this process in transitions that start from thermal equilibrium in either global or gauge field systems. In the global case, some of the predictions have been confirmed in condensed matter experiments, and it seems that the gauge theory predictions will be tested with superconductors in the future.

However, the transitions in the early universe may have taken place far away from thermal equilibrium, and much more work is still needed before we can really claim that we understand how possible cosmological defects would have been formed.

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### References