Chiral symmetry breaking solutions for QCD in the truncated Coulomb gauge

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In this paper we study the power-like confining potentials $r^\alpha$. The region of allowed $\alpha$’s is identified, the mass-gap equation is constructed for an arbitrary $\alpha$ and solved for several values of the latter, and the vacuum energy and the chiral condensate are calculated. The question of replica solutions to the mass-gap equation for such potentials is addressed and it is demonstrated that the number of replicas is infinite for any $\alpha$, as a consequence of the peculiar behaviour of the quark self-energy in the infrared domain.


I. INTRODUCTION

The problem of the spontaneous breaking of chiral symmetry (SBCS) and its relation to confinement is one of the QCD cornerstones. Although the basic ideas of SBCS are already a textbook topic, this problem still lies at the crossroad of many studies and approaches to QCD. In this paper we exploit the potential model for QCD, whose origins can be traced back to QCD in the truncated Coulomb gauge and which is proven to be successful in studies of the low-energy phenomena in QCD (see, for example, [1]). This class of models can be indicated as Nambu-Jona-Lasinio-(NJL-)type models [2] with the current-current quark interaction and the corresponding formfactor coming from the bilocal gluonic correlator.
The standard approximation in such type of models is to neglect the retardation and to approximate the gluonic correlator by a confining potential of a certain form. The power-like potentials, which are the most natural candidates for the role of the confining force, are the subject of the present investigation. In the course of this paper we

- revisit the problem of SBCS for the for power-like confining potentials $V(r) = K_0^{1+\alpha} r^\alpha$ with $\alpha \geq 0$;
- restrict the range of allowed $\alpha$’s and, for several values of the latter, find numerically solutions to the corresponding mass-gap equation, as well as the vacuum energy density, and the chiral condensate for the chirally noninvariant vacuum of the theory;
- study in detail the problem of existence of the replica solutions to the mass-gap equation for power-like confining potentials.

The problem of instability of the chirally invariant vacuum for power-like confining potentials was studied in detail in the middle of 80’s by the Orsay group [3] and such an instability was proved for the range $0 \leq \alpha < 3$. For numerical studies the harmonic oscillator type potential, $\alpha = 2$, was chosen by these authors, as well as by the Lisbon group [4], and a set of results for hadronic properties were obtained in the framework of the given model. In this paper we study the mass-gap equation for an arbitrary value of $\alpha$ ranging from 0 to 2, with special attention payed to $\alpha$’s close to unity, since the linearly rising potential is known to be preferred by phenomenology as the most successful candidate for the confining force giving the correct Regge trajectories behaviour, possessing a clear connection to the QCD string, and so on (see [5] and references therein). It is also claimed to be singled out by the lattice calculations. We exclude the region $\alpha > 2$ since the corresponding mass-gap equation diverges for such $\alpha$’s. On the other hand, it would be hard to justify the use of such a strong confining force in phenomenological models for QCD. We solve the mass-gap equation explicitly for several values of $\alpha$ from the allowed region and demonstrate that the chiral angle, the vacuum energy density, and the chiral condensate are smooth slow functions of the form of the confining potential, so that the results obtained for the potential of a given form — the linear confinement being the most justified and phenomenologically successful choice — have a universal nature for any quark-quark kernel of such a type.

Following the set of recent publications devoted to possible multiple solutions for the chirally noninvariant vacuum in QCD [6, 7] (see also [8] where a similar conclusion was made
in a different approach), we address the question of replicas existence for various power laws $r^\alpha$. We find that for a whole range of allowed powers, $0 \leq \alpha \leq 2$, replica solutions do exist, similarly to the case of $\alpha = 2$ studied in detail in [3, 4]. We give the profiles of several replicas for the linear confinement and argue that the number of such solutions is infinite for any power $\alpha$, including the weakest, logarithmic, potential which corresponds to $\alpha = 0$.

We argue that the source of replicas is the infrared behaviour of the single-quark self-energy — the dressed quark dispersive law $E(p)$ — which, for small values of the quark momentum $p$, becomes a negative sharp function of $p$ thus enabling fast oscillations of the chiral angle with the frequency increasing with vanishing momentum $p$. Since this property of the quark dispersive law is expected to be an integral part of any confining interaction, then we confirm the conclusion made in [7] that “... across all these different quark kernels, the existence of vacuum replicas should constitute the rule rather than the exception.” We argue that, in real QCD, with the confining interaction flattening at large distances due to the effect of the QCD string breaking, the number of replicas becomes finite. We find the parameter of the SBCS given by the replica solutions to decrease fast with the number of nodes of the chiral angle so that one has a well defined perturbative series in replicas and, therefore, the account only for the first replica may be sufficient in many phenomenological applications (the details of the formalism which allows one to incorporate replicas into quark models can be found in ref. [7]).

The paper is organised as follows: In the second section we give the necessary details of the formalism and derive the mass-gap equation for the power-like confining potential, which is studied in detail in the third section, first, qualitatively, then quantitatively, and, finally, numerically. The mass-gap equation is solved numerically for several values of $\alpha$ and the chiral condensate and the excess of the vacuum energy density over the trivial solution are calculated for the found solutions. In the fourth section, devoted to replicas, we demonstrate how an infinite number of solutions to the mass-gap equation appears and build explicitly a couple of replicas for the linear confinement. Our conclusions are the subject of the last, fifth, section.
II. THE MASS-GAP EQUATION

The chiral model which we use for our studies is given by the Hamiltonian with the current-current interaction parametrised by the bilocal correlator $K_{\mu\nu}^{ab}$,

$$H = \int d^3x \bar{\psi}(\vec{x}, t) \left(-i\vec{\gamma} \cdot \vec{\nabla}\right) \psi(\vec{x}, t) + \frac{1}{2} \int d^3xd^3y \, J^a_\mu(\vec{x}, t)K_{\mu\nu}^{ab}(\vec{x} - \vec{y})J^b_\nu(\vec{y}, t),$$ (1)

where the quark current is $J^a_\mu(\vec{x}, t) = \bar{\psi}(\vec{x}, t)\gamma^a \psi(\vec{x}, t)$, and the gluonic correlator is approximated by a potential,

$$K_{\mu\nu}^{ab}(\vec{x} - \vec{y}) = g^\mu_0g^\nu_0 \delta^{ab} V_0(|\vec{x} - \vec{y}|),$$ (2)

with

$$V_0(|\vec{x}|) = K_0^{\alpha+1}|\vec{x}|^\alpha.$$ (3)

In order to include the logarithmic potential into consideration the obvious modification of the potential is needed:

$$V_0(|\vec{x}|) \rightarrow \tilde{V}_0(|\vec{x}|) = K_0 \left(\frac{(K_0|\vec{x}|)^\alpha - 1}{\alpha}\right)_{|\alpha\rightarrow 0} = K_0 \ln(K_0|\vec{x}|).$$ (4)

The model contains the only dimensional parameter — the strength of the confining force $K_0$. For further convenience we shall consider a modified version of the potential (3),

$$V_0(|\vec{x}|) = K_0^{\alpha+1}|\vec{x}|^\alpha e^{-m|\vec{x}|},$$ (5)

where $m$ plays the role of the regulator for the infrared behaviour of the interaction. The limit $m \rightarrow 0$ is understood.

The standard technique used in such models is the Bogoliubov-Valatin transformation from bare to dressed quarks parametrised by the chiral angle — the main entity defining the chiral symmetry breaking, the structure of the BCS vacuum of the theory, as well as the properties of the hadronic states built over this vacuum [3, 4]. For the application of this technique to the two-dimensional QCD see the papers [10]. We choose the following parameterisation:

$$\psi(\vec{x}, t) = \sum_{\xi=1,\downarrow} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} [b_\xi(\vec{p}, t)u_\xi(\vec{p}) + d^\dagger_\xi(-\vec{p}, t)v_\xi(-\vec{p})],$$ (6)

$$u(\vec{p}) = \frac{1}{\sqrt{2}} \left[ \sqrt{1 + \sin \varphi_p} + \sqrt{1 - \sin \varphi_p} \right] u(0),$$

$$v(-\vec{p}) = \frac{1}{\sqrt{2}} \left[ \sqrt{1 + \sin \varphi_p} - \sqrt{1 - \sin \varphi_p} \right] v(0),$$ (7)
where $E_p$ (the shorthand notation for $E(p)$) stands for the dispersive law of the dressed quarks, the chiral angle $\varphi(p)$ (we also use the shorthand notation $\varphi_p$ for it) varies in the range $-\pi/2 < \varphi_p \leq \pi/2$ with the boundary conditions $\varphi(0) = \pi/2, \varphi(p \to \infty) \to 0$.

The Hamiltonian (8) normally arranged in the basis (8) splits into the vacuum energy, the quadratic and the quartic parts in terms of the quark creation/annihilation operators.

For the vacuum energy density one has

$$E_{\text{vac}}[\varphi] = \frac{1}{V} \langle 0 | TH[\varphi] | 0 \rangle = -\frac{g}{2} \int \frac{d^3p}{(2\pi)^3} \left( A(p) \sin \varphi_p + [B(p) + p] \cos \varphi_p \right),$$

where $V$ is the three-dimensional volume; the degeneracy factor $g$ counts the number of independent quark degrees of freedom,

$$g = (2s + 1)N_C N_f,$$

with $s = \frac{1}{2}$ being the quark spin; the number of colours, $N_C$, is put to three, and the number of light flavours, $N_f$, is two. Thus we find that $g = 12$. The auxiliary functions $A(p)$ and $B(p)$ are defined as

$$A(p) = \frac{1}{2} C_F \int \frac{d^3k}{(2\pi)^3} V_0(p) \sin \varphi_k,$$

$$B(p) = p + \frac{1}{2} C_F \int \frac{d^3k}{(2\pi)^3} (\hat{p} \cdot \hat{k}) V_0(p) \cos \varphi_k,$$

where $C_F = \frac{4}{3}$ is the $SU(3)_C$ Casimir operator in the fundamental representation. The actual form of the chiral angle is such that the quadratic part of the normally ordered Hamiltonian diagonalises, or alternatively, the vacuum energy takes its minimal value. The corresponding equation

$$\frac{\delta E_{\text{vac}}[\varphi]}{\delta \varphi_p} = 0,$$

known as the mass-gap equation, reads

$$A(p) \cos \varphi_p - B(p) \sin \varphi_p = 0.$$

For the generalised power-like potential (5) one can find

$$A(p) = -C_F \Gamma(\alpha + 1) K_0^{\alpha+1} \int_{-\infty}^{\infty} \frac{dk \ k \ cos[(\alpha + 1) \ arctan \frac{k-p}{m}]}{2\pi \ [m^2 + (k - p)^2]^{\alpha+1}} \sin \varphi_k,$$
\[ B(p) = p - C_F \Gamma(\alpha) \frac{K_0^{\alpha+1}}{p^2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ \frac{\alpha pk \cos[(\alpha + 1) \arctan \frac{k-p}{m}]}{[m^2 + (k-p)^2]^\frac{\alpha+1}{2}} - \frac{\cos[(\alpha - 1) \arctan \frac{k-p}{m}]}{(\alpha - 1)[m^2 + (k-p)^2]^\frac{\alpha-1}{2}} \right] \cos \varphi_k, \]

where, for the sake of convenience, we continued the integral to the negative values of \( k \) assuming \( \cos \varphi_{-k} = -\cos \varphi_k, \sin \varphi_{-k} = \sin \varphi_k \) (the most natural realisation of these conditions can be achieved in terms of some even function \( m_p \), such that \( \sin \varphi_p = m_p \sqrt{p^2 + m_p^2}, \cos \varphi_p = \frac{p}{\sqrt{p^2 + m_p^2}} \), which plays the role of the effective mass of the quark). Consequently, the mass-gap equation (14) takes the form:

\[ p^3 \sin \varphi_p = C_F K_0^{\alpha+1} \Gamma(\alpha) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \frac{\alpha pk \cos[(\alpha + 1) \arctan \frac{k-p}{m}]}{[m^2 + (k-p)^2]^\frac{\alpha+1}{2}} \sin[\varphi_p - \varphi_k] \right. \]

\[ + \left. \left( \frac{\cos[(\alpha - 1) \arctan \frac{k-p}{m}]}{(\alpha - 1)[m^2 + (k-p)^2]^\frac{\alpha-1}{2}} + \frac{(k-p) \sin[\alpha \arctan \frac{k-p}{m}]}{[m^2 + (k-p)^2]^\frac{\alpha}{2}} \right) \cos \varphi_k \sin \varphi_p \right\}, \]

and it is the main object of our studies.

III. INVESTIGATION OF THE GENERAL FORMULA

A. Qualitative analysis

As the first step in studies of the general formula (17), we perform its simple qualitative analysis. Using the techniques described in [6], we assume that a solution \( \varphi^{(0)}(p) \) to this equation exists, and the vacuum energy is minimal on this solution. If the function \( \varphi^{(0)}(p/A) \) with an arbitrary scale parameter \( 0 \leq A < \infty \) is substituted into the vacuum energy (9), then the function \( \mathcal{E}_{\text{vac}}(A) \) must reveal a minimum for \( A = 1 \). Moreover, the corresponding minimum should lie lower than the one for the trivial solution \( \varphi(p) \equiv 0 \), when there is no dressing of quarks and the chiral symmetry is unbroken. If the regulator \( m \) is removed from the vacuum energy functional, then the strength of the potential remains the only dimensional parameter in the theory, so that, after a proper rescaling of the integration variables, one arrives at the simple formula, for an arbitrary number of spatial dimensions \( d \),

\[ \mathcal{E}_{\text{vac}}(A) = C_1 A^{d+1} + C_2 K_0^{\alpha+1} A^{d-\alpha}, \]
where $C_1$ and $C_2$ are two constants independent of $A$ which are interrelated by the constraint $\frac{\partial E_{\text{vac}}(A)}{\partial A} |_{A=1} = 0$. The first term in Eq. (18) comes from the kinetic energy, the second term is due to the interaction. The following four situations are possible: i) $0 < \alpha < d$, ii) $\alpha > d$, and two boundary cases, iii) $\alpha = d$, and iv) $\alpha = 0$. In the first case the vacuum energy has a double-well form with two minima: trivial for $A = 0$ and nontrivial for $A = 1$. The difference $E_{\text{vac}}(A = 1) - E_{\text{vac}}(A = 0)$ is negative, so that the chirally nonsymmetric nontrivial solution is energetically preferable. For the second case one has the interaction term in (18) containing negative powers of $A$ and, as a result, the trivial solution, with unbroken chiral symmetry and which corresponds to $A = 0$, possesses an infinite energy and therefore it does not exist. In the meantime, a nontrivial solution with $A = 1$ still may be present. The boundary case of $\alpha = d$ leads to the logarithmic dependence of the vacuum energy on the parameter $A$, 

$$E_{\text{vac}}(A) = C_1 A^{d+1} + C_2 K_0^{d+1} \ln \frac{A}{K_0},$$

(19)

so that qualitatively the same conclusion holds — the theory possesses only chirally nonsymmetric phase. The two-dimensional QCD [9] is an example of the theory with such a logarithmic dependence (see [6] for the details).

Finally, for the case iv), that is, for the logarithmic potential (4) one has

$$E_{\text{vac}}(A) = C_1 A^{d+1} + C_2 K_0 A^d \ln \frac{A}{K_0},$$

(20)

where the logarithmic growth of the energy, when approaching the trivial solution limit $A = 0$, is cancelled by the power factor $A^d$, so that both, chirally symmetric and nonsymmetric, solutions coexist in this case, similarly to other potentials with $0 < \alpha < d$.

Thus we conclude that once the power of the potential reaches the critical value equal to the number of spatial dimensions the behaviour of the theory changes drastically, the chirally symmetric phase being swept off. Meanwhile the qualitative analysis performed above ignored the problem of convergence of the integrals in the expression for the vacuum energy and in the corresponding mass-gap equation. It also fails to answer the question as to how many solutions to the mass-gap equation exist. In what follows we turn to the quantitative and numerical analysis of these problems.
B. Quantitative analysis

Now we turn to the detailed analysis of the mass-gap equation (17), such as the problem of the convergence, the allowed region for $\alpha$’s, the dependence on the regulator $m$, and so on.

First of all, one can easily check that the case $\alpha = 0$ brings no difficulties — the r.h.s. vanishes if the limit $\alpha \to 0$ is taken naively, whereas to arrive at the mass-gap equation for the logarithmic potential one is to divide the r.h.s. by $\alpha$ (see Eq. (4)) which leads to a finite result after taking the limit $\alpha \to 0$.

For $\alpha = 1$ the divergent term proportional to $1/(\alpha - 1)$ vanishes on the r.h.s. of Eq. (17) since the cosine of the chiral angle is odd. An accurate expansion of this term for $\alpha \to 1$ brings about logarithmic terms.

Now let us check the largest value of $\alpha$ which does not lead to divergences in the mass-gap equation. When the regulator $m$ tends to zero, the first term in the curly brackets in Eq. (17), formally, is the most singular term for $k \sim p$, and it can be written as

$$\int_{-\infty}^{\infty} \frac{dk \, pk \cos[(\alpha + 1) \arctan \frac{k-p}{m}] \sin[\varphi_p - \varphi_k]}{2\pi \sqrt{m^2 + (k-p)^2}^{\alpha+1}}. \quad (21)$$

In the region $k \sim p$ the integrand admits an expansion in powers $(k-p)^n$:

$$\frac{\cos[(\alpha + 1) \arctan \frac{k-p}{m}]}{2\pi [m^2 + (k-p)^2]^{\alpha+1}} \left\{ p^2(k-p)\varphi_p' + (k-p)^2 \left[ p\varphi_p'' + \frac{1}{2}p^2\varphi_p'' \right] \ldots \right\}, \quad (22)$$

where the first term in the curly brackets is odd and, therefore, it vanishes in the integral in $k$ around $k = p$. The remaining expression, as well as other terms on the r.h.s. of Eq. (17), behaves as $|k-p|^{1-\alpha}$ and the corresponding contribution to the mass-gap equation converges for $\alpha < 2$, which is the upper limit of the range of valid powers $\alpha$. For such $\alpha$’s,

$$\arctan \frac{k-p}{m} \to \frac{\pi}{2} \text{sign}(k-p) + O(m), \quad (23)$$

and the regulator $m$ can be removed from the mass-gap equation. For the case of $\alpha = 2$ the entire r.h.s. of Eq. (17) vanishes after the substitution (23). To be more precise, one has to keep next terms in the expansion (23), with positive powers of the regulator $m$, reproducing the well-known representation of the delta function,

$$\delta(p-k) = \lim_{m \to 0} \frac{1}{\pi \frac{m}{m^2 + (k-p)^2}},$$
and its derivatives. We shall consider this special case separately.

Notice that the proof of the chirally symmetric vacuum instability given in [3] was based on the consideration of the self-energy functional

$$F(\mathbf{p}) = \int \frac{d^3 k}{(2\pi)^3} V_0(\mathbf{p} - \mathbf{k})(\mathbf{p} \cdot \mathbf{k}) \propto \frac{1}{p^\alpha},$$

which gives infrared divergent contribution to integrals in $d^3 p$ if $\alpha \geq 3$. In the meantime, as demonstrated above, the requirement of finiteness of the mass-gap equation imposes a stronger restriction on $\alpha$: $\alpha \leq 2$.

One encounters no more difficulties for $\alpha'$s within the interval $0 \leq \alpha < 2$ and, when the regulator $m$ is removed, arrives at the mass-gap equation in the ultimate form:

$$p^3 \sin \varphi_p = C_F K_0^{\alpha+1} \Gamma(\alpha + 1) \sin \frac{\pi \alpha}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \frac{\varphi_p}{|p - k|^{\alpha+1}} \right\}. \quad (24)$$

We introduced the term $1/\lambda^{\alpha-1}$, with an arbitrary mass parameter $\lambda$, in order to emphasise the convergence of the integral for $\alpha = 1$. This extra term does not contribute to the integral due to the parity of $\cos \varphi_k$.

In particular, the mass-gap equation for the logarithmic potential follows from (24) in the limit $\alpha \to 0$, if the proper modification of the potential, given in Eq. (4), is applied,

$$p^3 \sin \varphi_p = \frac{C_F K_0^2}{4} \int_{-\infty}^{\infty} \frac{dk}{|p - k|} \left[ \frac{\varphi_p}{|p - k|^{\alpha+1}} - \frac{1}{\lambda^{\alpha-1}} \right] \sin 2 \varphi_k \sin \varphi_p. \quad (25)$$

For the case of $\alpha = 1$, that is, for the linear confinement, the formula well known in the literature is readily reproduced (see, for example, [6]),

$$p^3 \sin \varphi_p = C_F K_0^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \frac{\varphi_p}{(p - k)^2} \sin [\varphi_k - \varphi_p] - \ln \frac{|p - k|}{\lambda} \cos \varphi_k \sin \varphi_p \right\}. \quad (26)$$

To complete our investigation let us consider the case of the harmonic oscillator potential, $\alpha = 2$. The Fourier transform of the potential $V(r) = K_0^3 r^2$ is the Laplacian of the three-dimensional delta function, so that the resulting mass-gap equation becomes differential

$$p^3 \sin \varphi_p = \frac{1}{2} C_F K_0^{\alpha+3} \left[ p^2 \varphi_p'' + 2p \varphi_p' + \sin 2 \varphi_p \right]. \quad (27)$$

Formally, Eqs. (17), (24) remain valid for $-1 < \alpha < 0$, and the ultraviolet divergence is encountered for $\alpha = -1$, that is, for the Coulomb potential. We disregard this region since
the resulting force fails to be confining. Thus the valid confining potentials, in momentum space, range as, symbolically, \((2\pi)^3 K_0 \delta^{(3)}(\vec{p}) < V(\vec{p}) \leq (2\pi)^3 K_0^3 \Delta \delta^{(3)}(\vec{p})\), that it, from constant to the harmonic oscillator potential, respectively.

For numerical investigation of the mass-gap equation (24) it is convenient to evaluate analytically the contribution of the strip \(|k - p| < \lambda\) into the integral on the r.h.s.,

\[
I_\lambda = \frac{C_F K_0^{\alpha+1}}{2\pi} \frac{\lambda^{2-\alpha}}{2-\alpha} \Gamma(\alpha + 1) \sin \frac{\pi \alpha}{2} \left[p^2 \varphi''_0 + 2p\varphi'_0 + \sin 2\varphi_0\right], \tag{28}
\]

where this integral admits an extra contribution if the width of the strip is chosen different from \(\lambda\). For \(\alpha = 2\) the dependence of the strip integral (28) on \(\lambda\) disappears and the full mass-gap equation (27) is readily reproduced as a consequence of the delta functional form of the Fourier transform of the harmonic oscillator potential.

C. Numerical analysis

In this subsection we present the results of the numerical studies of the mass-gap equation (24) with \(0 \leq \alpha \leq 2\). The equation for the harmonic oscillator potential (27) is studied in detail in the literature, so the interested reader can find the details, for example, in refs. [3, 4]. In Fig. 1(a) we plot the profile of the nontrivial solutions \(\varphi_0(p)\) to the mass-gap equation (24) for several values of \(\alpha\). The solution for \(\alpha = 2\), found in refs. [3, 4], is also depicted for the sake of completeness. In Fig. 1(b) we present the results for the chiral condensate \(\langle \bar{q}q \rangle = -\frac{3}{\pi^2} \int_0^\infty dp \ p^2 \sin \varphi_0(p) \equiv -\Lambda_\chi^3\) and for the excess of the vacuum energy \(\Delta \mathcal{E}_{\text{vac}} = \mathcal{E}_{\text{vac}}[\varphi_0] - \mathcal{E}_{\text{vac}}[\varphi \equiv 0] \equiv -\Lambda_\varepsilon^4\) over the trivial vacuum as functions of \(\alpha\). From Fig. 1(b) one can see that \(\Lambda_\chi \approx \Lambda_\varepsilon\) in the whole range of allowed \(\alpha\)'s and that their dependence on the form of the potential is smooth, solutions for \(\alpha\) around unity being quite close to one another (see Fig. 1(a)). Thus we conclude that the concrete form of the confining potential does not play a crucial role for the physics of chiral symmetry breaking, resulting only in minor numerical changes. In particular one can see that the qualitative behaviour of the solution is very stable against deviations of the potential from the purely linear form, usually adopted in phenomenological models and declared to be confirmed by lattice calculations. Therefore, at least SBCS and the spectrum of low-lying hadrons will not be strongly affected in case the behaviour of the confining potential is slightly changed, deviating from linearity, as suggested in [11].
FIG. 1: The solutions to the mass-gap equation (24) for \( \alpha = 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 1.9, \) and 2.0 (plot (a)); the curves localised closer to the origin correspond to smaller \( \alpha \)'s, and the mass parameters defining the chiral condensate, as \( \langle \bar{q}q \rangle = -\Lambda_\chi^3 \), and the excess of the vacuum energy density over the trivial vacuum, for two quark flavours and three colours, as \( \Delta E_{\text{vac}} = -\Lambda_\xi^4 \), (plot (b)). All dimensional quantities are given in the units of \( K_0 \).

IV. THE REPLICA

As argued in a sequence of recent papers [6, 7], it is possible that “... the same ultraviolet behaviour, for instance for the quark propagator, bifurcates to different solutions when we go to the low-energy domain...” in QCD, and such a replica was discovered for a phenomenology inspired potential. Besides that, the whole infinite tower of excited solutions for the mass-gap equation (27) for the harmonic oscillator potential was found in ref. [3] and also confirmed in [6]. With the general form of the mass-gap equation (17) and (24) we are in the position to investigate the problem of replicas existence for various power-like confining potentials. We find that any power-like potential \( r^\alpha \), with the power \( 0 \leq \alpha \leq 2 \), maintains replicas. In the paper [3] a detailed analysis was performed for the harmonic oscillator potential mass-gap equation (27) and the existence of an infinite tower of solutions was proved analytically. We failed to repeat this analysis for the general form of the mass-gap equation (24) since, in contrast to Eq. (27), Eq. (24) is integral with the coefficients tuned to provide the overall convergence of the integral, but leaving no hope to use any expansions under the integral.
Instead let us use an approximate method in order to demonstrate how replica solutions occur for the mass-gap equation (24). To this end we use the parametrisation of the chiral angle through the effective quark mass and introduce a new function $\psi_p$:

$$
\sin \varphi_p = m_p D_p, \quad \cos \varphi_p = p D_p, \quad \psi_p = p \sin \varphi_p, \quad D_p^{-1} \equiv \sqrt{p^2 + m_p^2}.
$$

(29)

It is also convenient to use the following integral:

$$
\int_{-\infty}^{\infty} \frac{dk}{2\pi} \psi_k - \psi_p = \frac{1}{2\Gamma(\alpha + 1) \sin \frac{\pi \alpha}{2}} \int_{-\infty}^{\infty} dx |x|^\alpha \psi_x e^{ipx}.
$$

(30)

Then the mass-gap equation (24) can be rewritten in a simple physically transparent form:

$$
\left[ 2E(\hat{p}) + C_F K_0^{\alpha+1} |x|^{\alpha} \right] \psi_x = 0,
$$

(31)

where the operator $E(\hat{p})$ has the meaning of the quark self-energy and it is given by the following expression:

$$
E(p) = \frac{1}{D_p} - \frac{C_F K_0^{\alpha+1}}{p^2 D_p} \Gamma(\alpha + 1) \sin \frac{\pi \alpha}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ \frac{p^2 D_p - k^2 D_k}{|p - k|^\alpha} + \frac{pk D_p D_k}{(\alpha - 1)|p - k|^\alpha - 1} \right].
$$

(32)

The even function $m(p)$ takes its maximal value at $p = 0$ and then decreases rapidly. Thus, for momenta larger than some $p_0 \sim m(0)$, Eq. (31) can be linearised by putting $D_p^{-1} \approx |p|$ in the expression (32). Then the integral on the r.h.s. of Eq. (32) is easily evaluated and the self-energy takes the form:

$$
E(p) \approx |p| - \frac{2C_F K_0^{\alpha+1} \Gamma(\alpha + 1) \sin \frac{\pi \alpha}{2}}{\pi \alpha (2 - \alpha) |p|^{\alpha}}.
$$

(33)

Notice that for any confining potential, $\alpha \geq 0$, the second term on the r.h.s. of the expression (33) dominates in the low-momentum region bringing a large negative contribution to the quark self-energy (the corresponding term becomes logarithmic, $\sim K_0 \ln \frac{K_0}{|p|}$, for the potential (41)). This general feature of the quark self-energy in the confining potential has been discussed in literature not once (see, for example, [3, 4], or the papers [10] where the case of QCD$_2$ is discussed in detail) and it is known to play a crucial role for the properties of the theory. Besides that, replicas exist in the theory also due to this feature of the quark self-energy $E(p)$. In order to demonstrate this let us write the linearised mass-gap equation (31) in the form of a Schrödinger-type equation in momentum space,

$$
\left[ C_F K_0^{\alpha+1} |\hat{x}|^\alpha + 2|p| - \frac{4C_F K_0^{\alpha+1} \Gamma(\alpha) \sin \frac{\pi \alpha}{2}}{\pi (2 - \alpha) |p|^{\alpha}} \right] \psi_{\hat{p}} = \varepsilon \psi_{\hat{p}},
$$

(34)
and notice that the linearisation suppresses the first, positive, term on the r.h.s. of Eq. (32) and enhances the second, negative, term. As a result, we are interested in the odd (see the definition (29)) eigenstates of the linear equation (34) with negative eigenvalues. Each such state indicates a solution of the full nonlinear mass-gap equation (24) and can be used as the starting anzatz for the iterative numerical method to solve the latter. Therefore, in order to find the number of solutions to the mass-gap equation (24), it is sufficient to count the odd eigenstates, with negative eigenvalues, of the linear equation (34). If the Bohr-Sommerfeld quantisation procedure is applied directly to Eq. (34), then the quasiclassical integral, \( I_{KB} = \int_{p_{min}}^{p_{max}} x(p) dp \), diverges logarithmically at \( p = 0 \), where the chiral angle is not small anymore and the approximation \( D^{-1}_p \approx |p| \) obviously fails. In this region, the sharp behaviour of the self-energy is smeared by the effective quark mass, \( \mu \sim m_p (p \rightarrow 0) \), which plays the role of the effective regulator, so that we modify Eq. (34) accordingly:

\[
\frac{C_F \alpha}{\Gamma \left( \frac{1}{2} \right)} \Gamma \left( \frac{1}{1+\alpha} \right) \left( \frac{\pi}{2} \right) \left[ \frac{\sqrt{\pi \Gamma \left( \frac{4-\alpha}{2} \right)}}{2^\alpha \Gamma \left( \frac{1+\alpha}{2} \right)} \right] \left[ 2^{\alpha-1} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right)}{\Gamma (\alpha) \Gamma \left( \frac{\alpha}{2} \right)} \right] \right] \psi_p = \varepsilon \psi_p. \tag{35}
\]

The spectrum of eigenstates of Eq. (35) starts, for \( n = 0 \), at the bottom of the deep well described by the effective potential \( V(p) = 2E(p) \), \( \varepsilon_0 \approx V(p = 0) \approx -K_0^{1+\alpha}/\mu^\alpha \), and, for some \( n_{max} \), reaches zero from below. Then the spectrum continues for positive eigenvalues up to infinity. Since the last negative eigenenergy \( \varepsilon_{n_{max}} \) is small, then we expand the quasiclassical integral accordingly and find:

\[
\varepsilon_n \approx K_0 \left( \frac{C_F \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1+\alpha}{2} \right)} \right) \left[ \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right)}{\Gamma (\alpha) \Gamma \left( \frac{\alpha}{2} \right)} \right] \left[ \frac{\sqrt{\pi \Gamma \left( \frac{4-\alpha}{2} \right)}}{2^\alpha \Gamma \left( \frac{1+\alpha}{2} \right)} \right] \left[ 2^{\alpha-1} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right)}{\Gamma (\alpha) \Gamma \left( \frac{\alpha}{2} \right)} \right] \left[ \frac{\Gamma \left( \frac{1+\alpha}{2} \right)}{\Gamma (\alpha) \Gamma \left( \frac{1+\alpha}{2} \right)} \right] \left[ \frac{\varepsilon \sigma_n}{\mu} \right] \right] \tag{36}
\]

where we expressed \( \sin \frac{\pi \alpha}{2} \) through the Euler Gamma functions as

\[
\sin \frac{\pi \alpha}{2} = 2^{\alpha-1} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right)}{\Gamma (\alpha) \Gamma \left( \frac{\alpha}{2} \right)} \tag{37}
\]

Notice that in actuality the regulator \( \mu \) itself is not a constant, but it rapidly decreases with \( n \), and it is such that all eigenvalues of Eq. (35) vanish for all \( n \)’s. Then the corresponding eigenfunctions \( \psi_n \) provide, according to the definition (29), the solutions to the exact mass-gap equation (24). From Eq. (36) one can easily see that

\[
\mu_n(\alpha) = K_0 \exp \left( -C_\alpha \pi n \right), \quad C_\alpha = \left[ \frac{\sqrt{\pi \Gamma \left( \frac{4-\alpha}{2} \right)}}{2^\alpha \Gamma \left( \frac{1+\alpha}{2} \right)} \right]^{1/\alpha}. \tag{38}
\]
Using a more accurate expansion in Eq. (36), one can find the correction $\delta_\alpha$ to the leading regime Eq. (38), $\mu_n(\alpha) = K_0 \exp(-C_\alpha(\pi n + \delta_\alpha))$, as it was done in [3]. Unlike the leading logarithmic term, this correction is sensitive to the concrete form of the regularization of the potential $V(p)$, and we do not give it here.

From the formula (38) we conclude that, for any $\alpha$, the mass-gap equation supports an infinite number of solutions which, in the low-momentum region ($p \ll K_0$), behave as

$$\varphi_n(p \sim 0) = \frac{\pi}{2} - \frac{p}{K_0} e^{C_\alpha(\pi n + \delta_\alpha)} + \ldots,$$

then oscillate $n$ times and, finally, approach zero for the infinite momentum. The constant $C_\alpha$ is monotonously decreasing function of $\alpha$ in the whole interval $0 \leq \alpha \leq 2$ and it varies from $C_0 = \frac{1}{2} \exp\left[\frac{1}{2}(\gamma - 1 - \psi\left(\frac{1}{2}\right))\right] \approx 1.08$ to $C_2 = \sqrt{2} \approx 0.71$. Numerically, the same behaviour of the solutions to the mass-gap equation is observed — for smaller $\alpha$’s and higher $n$’s the chiral angle becomes steeper at the origin, in accordance with the obvious identification $\varphi_n’|_{p=0} = -1/\mu_n \propto \exp(C_\alpha[\pi n + \delta_\alpha])$ (see Figs. 1(a), 2).

The formula is approximate since the dependence on $\mu$ in Eq. (35) reproduces only the gross features of the quark self-energy, whereas the exact dependence of the self-energy on the chiral angle is, in turn, a consequence of the mass-gap equation, and the problem becomes self-consistent. To estimate the accuracy of the formula we continue it to $\alpha = 2$ and compare our result with the result found in [3]:

$$C_2 = \frac{1}{\sqrt{2}} \approx 0.71, \quad C_2(\text{ref. [3]}) = \frac{2}{\sqrt{7}} \approx 0.76,$$

that is, the error is about $6 \div 7\%$.

As an indirect confirmation of the conclusion made above let us mention that no critical phenomena are observed in the mass-gap equation for any $\alpha$ as well as in the limit $\alpha \to 2$, so that all properties of the mass-gap equation, including the infinite number of solutions, may be continued from $\alpha = 2$ to smaller $\alpha$’s.

In Fig. 2 we give the profiles of the ground-state, as well as of the first two replica solutions for the linear confinement, $\alpha = 1$. Solutions with larger number of nodes are also available for numerical study, but the method has to be very precise since each new solution possesses oscillations squeezed to zero and seen only with the help of methods with a sufficiently high resolution. According to the formula, the behaviour of these solutions at the origin can
FIG. 2: The three first solutions to the mass-gap equation \( (26) \) corresponding to the ground-state BCS vacuum \( (n = 0) \) and to the first two replica states \( (n = 1, 2) \). The momentum \( p \) is given in the units of \( K_0 \).

be approximated as

\[
\varphi_n(p \sim 0) = \frac{\pi}{2} - \frac{p}{K_0} \exp \left[ \frac{\pi}{4} (\pi n + \delta_1) \right] + \ldots,
\]

so that \( \mu_{n+1}(1)/\mu_n(1) \approx \exp \left( \frac{\pi^2}{4} \right) \sim 10 \), that is, the parameter of SBCS decreases about ten times for each next replica.

In ref. [7] a method is proposed which allows one to take the vacuum replicas into account in quark models. The generalisation of this approach to the case of many replicas is straightforward and assumes the summation over contributions of all replicas. On the other hand, since the mass parameter of the solution, for example, \( \mu_n \), decreases fast (approximately ten times for the linear confinement) for each next solution, then one has a sort of perturbative series with the well defined convergence parameter. Therefore it may be sufficient to consider only the first replica neglecting the contribution of higher replicas, starting from the second one, which are hard to distinguish numerically from the trivial solution \( \varphi_p \equiv 0 \). To have a good phenomenological description of quarkonia one should supply the purely confining potential by the short-range Coulomb interaction and, possibly, by a constant term in order
to fit for the right value of the chiral condensate (an attempt to evaluate this constant from first principles was undertaken in [12]). Such a potential was considered in [6, 7] and the solutions for the ground-state BCS vacuum as well as for one vacuum replica were found numerically.

V. CONCLUSIONS

In this paper we complete the study of the power-like potentials $K_0^{\alpha+1}r^\alpha$ from the point of view of SBCS and the number of nontrivial solutions of the mass-gap equation. We use the constructive method to prove the chirally symmetric vacuum instability for such potentials, solving the mass-gap equation explicitly and calculating the vacuum energy for the corresponding solution. We establish the region of allowed powers of $\alpha$ for such confining potentials which lead to a convergent mass-gap equation and a finite excess of the vacuum energy density over the trivial solution with unbroken chiral symmetry. Thus, using simple qualitative arguments, we demonstrate that for $0 \leq \alpha < d$, $d$ being the number of spatial dimensions, at least two, chirally symmetric and nonsymmetric, solutions should exist, whereas for $\alpha \geq d$ the trivial solution possesses an infinite energy density and disappears. In the meantime, the restriction for the parameter $\alpha$ exists, $0 \leq \alpha \leq 2$, which comes from the fact that the corresponding mass-gap equation should be convergent. We find numerically solutions to the mass-gap equation for various values of $\alpha$ from the allowed region and, for the found solutions, evaluate the vacuum energy density and the chiral condensate which are given by the same scale, $\Lambda_\varepsilon \sim \Lambda_\chi$, and turn out to be slow functions of the parameter $\alpha$.

We address the question of existence of the second, third, and higher chirally nonsymmetric solutions of the mass-gap equation for power-like confining potentials and find that any potential with $0 \leq \alpha \leq 2$ supports such solutions — the replicas — which is rather the rule for confining potentials and comes from the peculiar behaviour of the quark self-energy in the infrared domain. We find that the number of such replicas is infinite for any $\alpha$ and estimate the slope of the solutions to the mass-gap equation at the origin.

Thus we dare predict the existence of replicas regardless of the explicit form of the confinement and of the details of the model used in calculations. In real QCD, with light quark flavours and the quark-quark potential flattening at large distances due to the effect of the QCD string breaking (such an effect can be taken into account in the interquark potential
through a coordinate dependence of the effective string tension \[13\]), the number of replicas is expected to be finite. Indeed, the distance at which the string starts to break, \(L\), will play the role of the infrared regulator in the formula similar to \[35\], instead of \(\mu\). Therefore, as follows from Eq. (36), \(n_{\text{max}} \sim \ln(K_0L) \sim 1\), where we consider \(L \sim K_0^{-1}\). In other words, confinement becomes less “binding” due to string breaking and the corresponding mass-gap equation supports fewer solutions — “bound states”. A more detailed analysis of the QCD inspired interaction, including the proper string dynamics, from the point of view of replicas is in progress now and will be subject for future publications.

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