Two-loop World-sheet Effective Action

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Abstract

We are studying quantum corrections in the earlier proposed string theory based on world-sheet action which measures the linear sizes of the surfaces. At classical level the string tension is equal to zero and as it was demonstrated in the previous studies one loop correction to the classical world-sheet action generates Nambu-Goto area term, that is nonzero string tension. We extend this analysis computing the world-sheet effective action in the second order of the loop expansion.

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1 Introduction

In [1, 2] authors suggested a model of string theory based on world-sheet action which measures the linear sizes of the surfaces, so-called gonihedric model. The world-sheet action of the theory is defined by the integral:

\[ S = m \cdot L = m \int_{\Sigma} d^2 \zeta \sqrt{g} \sqrt{K^{ia} K^{ib}}, \]  

(1)

where \( L \) measures the linear size of the surface \( \Sigma \), \( m \) has dimension of mass and \( K^{ia} \) is a second fundamental form (extrinsic curvature)\(^3\).

At the classical level string tension is equal to zero \( T_{\text{classical}} = 0 \), because the action (1) is equal to the perimeter of the flat Wilson loop \( S \rightarrow m(R + T) \). It was demonstrated in [1, 7] that quantum fluctuations generate dynamically the Nambu-Goto area term \( A_{NG} \) in the world-sheet effective action

\[ S_{\text{eff}} = S + T_q A_{NG} + \ldots, \]  

(2)

with non-zero string tension \( T_q = mM \). Dynamical generation of string tension \( T_q \) has been found in the discrete formulation of the theory when the action (1) is written for the triangulated surfaces [1]. In [7] the authors have computed one-loop effective lagrangian \( L_{\text{eff}} = L_0 + L_1 + L_2 \) and found that similar generation of non-zero string tension takes place in a continuum formulation of the theory (the \( L_{\text{eff}} \) is a functional of the Lagrange multiplier \( \lambda_{ab} = \lambda_{\eta_{ab}} \)).

Here we shall extend this analysis computing the world-sheet effective Lagrangian in the second order of the loop expansion \( L_{\text{eff}} = L_0 + L_1 + L_2 \). We get the following result for the world-sheet effective action up to two-loop order which depends on \( \lambda \):

\[ S_{\text{eff}} = m \left[ 1 + a \frac{\lambda}{m} \left( \ln \left( \frac{\lambda}{M} \right) - 1 \right) + b \left( \frac{\lambda}{m} \right)^2 \ln^2 \left( \frac{\lambda}{M} \right) \right] \int d^2 \zeta \sqrt{g} \sqrt{K^{ia} K^{ib}} \]  

(3)

together with the term which is proportional to the Polyakov-Kleinert action

\[ S_{PK} = \frac{1}{4\pi e_{\text{eff}}^2} \int d^2 \zeta \sqrt{g} K^{ia} K^{ib}, \quad \frac{1}{e_{\text{eff}}^2} = a \frac{\lambda}{m} \ln^2 \left( \frac{\lambda}{M} \right) \]

where \( a = \frac{(D-3)}{4\pi}, \quad b = \frac{-3(D-3)(D-5)}{32\pi^2} \). Quantum fluctuations generate high powers of the extrinsic curvature tensor \( K^{ia}_{ab} \) in the form of Polyakov-Kleinert action with the induced coupling constant \( e_{\text{eff}}^2 \).

From these results we conclude that the extremum of the effective action with respect to \( \lambda \) defines a non-trivial saddle point solution and exhibits the condensation of the Lagrange multiplier at the value \( < \lambda > = M \) (this extremum is equivalent to imposing the constraint \( g_{ab} = \partial_a X_\mu \partial_b X_\mu \)). One can also see that the coupling constant \( 1/e_{\text{eff}}^2 \) is zero at the saddle point and high derivative term \( S_{PK} \) actually is not present.

\(^3\)It differs from the models considered in the previous studies [3, 4, 5, 6], because the action has dimension of length \( L \propto \text{length} \), it is proportional to the linear size of the surface similar to the path integral action. This is in contrast with the previous proposals where the extrinsic curvature term is a dimensionless functional \( S(\text{extrinsic curvature}) \propto 1 \).
2 Perturbation of World-sheet Sigma Model

One can consider the action (1) as an example of nonlinear higher derivative world-sheet sigma model and our aim is to develop perturbation expansion of this system around some classical solution and to compute two-loop effective action [8, 9, 10, 11, 12, 13]. The action can be represented in the form

\[ S = m \int d^2 \zeta \sqrt{\bar{g}} \sqrt{\left( \Delta(g) X_\mu \right)^2}, \]

here \( g_{ab} = \partial_a X_\mu \partial_b X_\mu \) is induced metric, \( \Delta(g) = 1/\sqrt{\bar{g}} \partial_a \sqrt{\bar{g}} g^{ab} \partial_b \) is a Laplace operator and \( K_a^a K_b^b = (\Delta(g) X_\mu)^2 \). The second fundamental form \( K \) is defined through the relations:

\[
K_{\mu}^{ab} n^i = \partial_a \partial_b X_\mu - \Gamma_\mu^{ab}_{\delta c} \partial_c X_\mu = \nabla_a \partial_b X_\mu, \tag{5}
\]

\[
n^i n^j = \delta_{ij}, \quad n^i \partial_a X_\mu = 0, \tag{6}
\]

where \( n^i \) are \( D-2 \) normals and \( a, b = 1, 2; \quad \mu = 0, 1, 2, ..., D-1; \quad i, j = 1, 2, ..., D-2 \).

In order to consider \( g \) and \( X \) as independent field variables we should introduce standard Lagrange multipliers \( \lambda^{ab} \) and add the corresponding term to the action [3]

\[ S_\lambda = -m \int d^2 \zeta \lambda^{ab} (\partial_a X^\mu \partial_b X_\mu - g_{ab}) \tag{7} \]

The action \( S + S_\lambda \) is invariant under two-dimensional general coordinate transformations and we can fix the conformal gauge

\[ g_{ab} = \rho \eta_{ab} \] \tag{8}
and should add the corresponding Faddeev-Popov action \( S_{FP} \). The gauge fixed action to be studied is \( S + S_\lambda + S_{FP} \). We split all fields into a sum of classical solution plus a quantum fluctuations consider:

\[
X^\mu = X_0^\mu + X_1^\mu \tag{9}
\]

\[
\rho = \rho_0 + \rho_1 \tag{10}
\]

\[
\lambda^{ab} = \lambda_0^{ab} + \lambda_1^{ab}. \tag{11}
\]

In order to investigate the saddle point solution for the Lagrangian multiplier we shall consider the Ansatz of the form

\[
\lambda_0^{ab} = \lambda \sqrt{\bar{g}} g^{ab} = \lambda \eta^{ab} \tag{12}
\]

where \( \lambda \) is a constant field.

Since we are interested in two-loop approximation of the effective action we have to expand the action \( S \) up to cubic and quartic interaction in the quantum fields. We easily get

\[
S + S_\lambda = m \int d^2 \zeta \sqrt{n^2} + \tag{13}
\]

\[
+ m \int d^2 \zeta \left[ \frac{1}{2 \sqrt{n^2}} X_1^\mu \partial^4 (\eta_{\mu\nu} - \frac{\bar{n}_\mu \bar{n}_\nu}{n^2}) X_1^\nu \right] - \lambda \partial^a X_1^\mu \partial_a X_1 \mu - 2 \lambda_1^{ab} \partial_a \partial_b X_1^\mu \lambda_1 \rho_1 
\]

\[
- m \int d^2 \zeta \lambda_1^{ab} \partial_a X_1^\mu \partial_b X_1 \mu 
\]

\[
- m \int d^2 \zeta \frac{1}{2 \sqrt{n^2}} \left[ (\partial^2 X_1)^2 - (\frac{\partial^2 X_1 \cdot \bar{n}}{\bar{n}^2})^2 \right] \left[ \frac{\bar{n} \cdot \partial^2 X_1}{\bar{n}^2} \right] + \frac{(\partial^2 X_1)^2}{4 \bar{n}^2} - \frac{5(\partial^2 X_1 \cdot \bar{n})^2}{4 \bar{n}^2} \right],
\]
where we have introduced convenient notations: 
\[ \lambda_1 = \eta_{ab} \lambda_1^{ab}, \]
\[ \bar{n}^\mu = \partial^2 X_0^\mu, \quad \bar{e}^a_\mu = \partial^a X_{0\mu}, \]
and used the relation \((\partial_a X_0)^2 = 2 \rho_0\). Here we consider the quantities \(\bar{n}^\mu, \bar{e}^a_\mu, \rho_0\) to be \(\zeta\)-independent in contrast to the fast quantities. Like in [7] we shall make an expansion of \(X_1^\mu\) fields into tangential and normal fields \(\phi^a, \xi^i:\)
\[ X_{1\mu} = \phi^a \bar{e}_a_\mu + \xi^i \bar{n}^i_\mu \]
where the last relation, together with the following ones \(\bar{\tilde{\eta}}_\mu \bar{n}_\mu = g \tilde{K}^a_\alpha \tilde{K}^b_\beta \equiv g \tilde{K}^2, \) \(\bar{n}_\mu \bar{n}^i = \sqrt{g} \tilde{K}^{ai}_\alpha \equiv \bar{n}^i,\) can be easily seen in conformal gauge. After the substitution of them into (13) we find that the relevant terms for our calculation can be collected in the form
\[
\hspace{1cm}
\]
\[ S_1 = \frac{m}{2} \int d^2 \zeta \left\{ \xi^i \left[ \frac{\Pi^{ij}}{\sqrt{n^2}} \partial^4 + 2 \lambda \partial^2 \delta^{ij} \right] \xi^j + \phi^a \rho_0 \left[ \frac{1}{\sqrt{n^2}} \partial^4 + 2 \lambda \partial^2 \right] \phi^a - 4 \rho_0 \lambda^{ab} \delta^{ab} \phi^a \right\} + m \int d^2 \zeta \lambda \rho_1 - m \int d^2 \zeta \left\{ \lambda^{ab} \partial^a \phi^i \delta^{ij} \phi^j + \frac{1}{2(\bar{n}^2)^{3/2}} \left[ \bar{n}^i \Pi^{im} \partial^2 \xi^i \partial^2 \xi^m \right] \right\} - \frac{m}{8(\bar{n}^2)^{3/2}} \int d^2 \zeta \Pi^{ij} \partial^2 \xi^i \partial^2 \xi^j \Pi^{lm} \partial^2 \xi^l \partial^2 \xi^m. \tag{14} \]
All contractions in (14) have been performed using the tensor \(\eta_{ab}\). Moreover we have denoted by
\[ \Pi^{ij} = \delta^{ij} - \frac{\bar{n}^i \bar{n}^j}{\bar{n}^2}, \quad \Pi^{lm} = \delta^{lm} - \frac{\bar{n}^l \bar{n}^m}{\bar{n}^2}, \quad tr \Pi_1 = D - 7. \tag{15} \]
The operator \(\Pi\) is a projector, with the following properties
\[ \Pi^2 = \Pi, \quad \Pi \bar{n}^i = 0, \quad tr \Pi = D - 3 \tag{16} \]
and
\[ (\Pi_1 \Pi)^{ij} = (\Pi \Pi_1)^{ij} = \Pi^{ij}. \tag{17} \]
Let us find now the propagators for all these fields. The propagators for the \(\xi\) fields are easily determined by inverting the \(\xi \xi\) part in (14):
\[ \langle \xi^i(p) \xi^j(-p) \rangle = \frac{i}{m} \sqrt{\bar{n}^2} \left( \frac{\delta^{ij}}{p^4 - 2p^2 \lambda \sqrt{\bar{n}^2}} - \frac{1}{2 \lambda \sqrt{\bar{n}^2}} \frac{\bar{n}^i \bar{n}^j}{\bar{n}^2} \right) \tag{18} \]
where \(p\) is momentum. In order to find the propagator for \(\phi, \lambda_1\) and \(\rho_1\) we use in the momentum space the decomposition of \(\lambda_1\) proposed in [3]:
\[ \lambda_1^{ab}(p) = \omega(p)(\eta^{ab} - \frac{p^a p^b}{p^2}) + (p^a f^b + p^b f^a - (p \cdot f) \eta^{ab}), \quad tr \lambda_1 = \omega. \tag{19} \]
The quadratic part of the action for the fields \(\phi, \lambda_1\) is found to be
\[ \int \frac{d^2 p}{4\pi^2} \left\{ \frac{m}{2 \sqrt{\bar{n}^2}} \rho_0 \phi^a(-p) p^4 \phi^a(p) - 2 i m \rho_0 p^2 \phi^a(-p) f^a(p) - m \lambda \rho_0 p^2 \phi^a(-p) \phi^a(p) \right\}, \tag{20} \]
consequently we get the following nontrivial propagators:

\[
< \phi^a(-p) f^b(p) > = \frac{\eta^{ab}}{2m\rho_0 p^2} \tag{21}
\]

\[
< f^a(-p) f^b(p) > = -i \frac{\eta^{ab}}{4m\rho_0} (\frac{1}{\sqrt{\tilde{n}^2}} - \frac{2\lambda}{p^2}) \tag{22}
\]

It is worth to note here that the propagators for \( \phi \) fields are zero, meaning that longitudinal components of \( X_1 \) fluctuations are not propagating. For this reason we disregarded in (14) all interaction terms depending on \( \phi \). Finally we have to consider the term

\[
m \int \frac{d^2 p}{4\pi^2} \lambda_1(-p)\rho_1(p) = m \int \frac{d^2 p}{4\pi^2} \omega(-p)\rho_1(p)
= \int \frac{d^2 p}{4\pi^2} [\frac{m}{2} a_+(-p)a_+(p) - \frac{m}{2} a_-( -p)a_-(p)],
\]

where we have defined the fields

\[
a_+ = \omega + \rho_1 \quad a_- = \omega - \rho_1.
\]

In this way we get the contractions:

\[
< a_+(-p)a_+(p) >= -\frac{i}{m} \tag{23}
\]

\[
< a_-( -p)a_-(p) >= +\frac{i}{m}. \tag{24}
\]

In order to regularize divergent integrals we shall use the dimensional regularization considering our world-sheet action in \( 2 - 2\epsilon \) complex dimensions and shall use the so-called \( \overline{MS} \) scheme in order to renormalize the theory. All quantities in the above expressions, fields and coupling constants, are bare quantities. In the two-loop approximation the effective action shows up divergences as poles in \( \epsilon \) up to the order \( 1/\epsilon^2 \). To have better control of divergences we shall use explicit loop expansion of any bare quantity, for example:

\[
\lambda = \lambda_R + \hbar \delta \lambda_1 + \hbar^2 \delta \lambda_2 . \tag{25}
\]

The dimensionality of the world-sheet fields is as follows:

\[
[\xi] = -1 \quad [\phi^a] = 0 \quad [\sqrt{\tilde{n}^2}] = -1
\]

\[
[\rho_1] = -2 \quad [\rho_0] = -2 \quad [m] = 1
\]

\[
[\lambda] = 1 \quad [\lambda_1] = 1 \quad [p] = 0.
\]

### 3 Tree and one-loop diagrams

In tree approximation the effective Lagrangian is

\[
\mathcal{L}_0 = (m_R + \hbar \delta m_1 + \hbar^2 \delta m_2)\sqrt{\tilde{n}^2} . \tag{26}
\]
In the following the subscript indicating the renormalized quantity will be omitted for brevity of notation. In the first order, according to [7] we have
\[ L_1 = i \frac{\hbar}{2} \ln \det[\Pi_{ij} \partial_4 + 2(\lambda + \hbar \delta \lambda) \sqrt{\bar{n}^2} \delta^{ij} \partial_4] \]
\[ = i \frac{\hbar}{2}(D - 3) \int \frac{d^2k}{(2\pi)^2} \ln(k^2 - 2(\lambda + \hbar \delta \lambda) \sqrt{\bar{n}^2}) . \]
The integration is regularized in \( n = 2 - 2\varepsilon \) complex dimensions and we get
\[ L_1(\varepsilon) = -\frac{\hbar}{2}(D - 3) \lambda \sqrt{\bar{n}^2} \left\{ \frac{1}{\varepsilon} - \ln \left( \frac{\lambda}{M} \right) - 1 \right\} \]
(27)
We can expand (27) in \( \varepsilon \) and \( \hbar \) obtaining the following expressions
\[ L_1(\varepsilon) = -\frac{\hbar}{4\pi}(D - 3) \lambda \sqrt{\bar{n}^2} \left\{ \frac{1}{\varepsilon} - \ln \left( \frac{\lambda}{M} \right) + \frac{\varepsilon}{2} \left[ \ln^2 \left( \frac{\lambda}{M} \right) + \frac{\pi^2}{6} \right] \right\} , \]
(28)
\( \text{together with the counterterm } \delta \lambda \text{ of order } \hbar^2 \)
\[ -\frac{\hbar^2}{4\pi}(D - 3) \sqrt{\bar{n}^2} \delta \lambda \left\{ \frac{1}{\varepsilon} - \ln \left( \frac{\lambda}{M} \right) + \frac{\varepsilon}{2} \left[ \ln^2 \left( \frac{\lambda}{M} \right) + \frac{\pi^2}{6} \right] \right\} , \]
(29)
where \( M = \frac{2\pi e^{-\gamma}}{\sqrt{\bar{n}^2}} \)
and \( \gamma \) is the Euler’s constant. The divergence of order \( \hbar \) is cancelled by requiring:
\[ \delta m_1 = \frac{(D - 3) \lambda}{4\pi} \frac{\varepsilon}{\varepsilon} . \]
(30)
Thus we reproduce the one loop result of [7]
\[ L_1 = \frac{(D - 3)}{4\pi} \lambda \sqrt{\bar{n}^2} \left[ \ln \left( \frac{\lambda}{M} \right) - 1 \right] . \]
(31)
The terms of order \( \hbar^2 \) in (26) and (29) will provide the counterterms for two-loop diagrams. In the next section we shall proceed with calculation of two-loop effective Lagrangian.

4 Contributions of two-loop diagrams

Now let us consider two-loop contribution to the effective action. We are looking for the connected, single-particle irreducible graphs of order \( \hbar^2 \) in the expression:
\[-i \hbar < 0 | T \exp \left\{ \frac{i}{\hbar} \int d^2 \zeta \left[ -\frac{m}{8(\bar{n}^2)^{3/2}} \Pi_{ij} \partial_2 \xi^i \partial_2 \xi^j \Pi_{lm} \partial_2 \xi^l \partial_2 \xi^m \right. \right. \]
\[ - \frac{m}{2(\bar{n}^2)^{3/2}} \left[ \bar{n}^i \Pi_{lm} \partial_2 \xi^i \partial_2 \xi^j \partial_2 \xi^m \right] \left[ \bar{n}^i \Pi_{lm} \partial_2 \xi^i \partial_2 \xi^j \partial_2 \xi^m \right] \right] \]
\[ + \frac{i}{\hbar} \int \frac{d^2p d^2q}{(2\pi)^4} m \left\{ (p^2 q \cdot f(p + q) + q^2 p \cdot f(p + q)) \xi^i(-p) \xi^i(-q) + \frac{1}{2} a_+(p + q) \xi^i(-p) \xi^i(-q) \left( \frac{(p \cdot q)^2 - p^2 q^2}{(p + q)^2} \right) + \frac{1}{2} a_-(p + q) \xi^i(-p) \xi^i(-q) \left( \frac{(p \cdot q)^2 - p^2 q^2}{(p + q)^2} \right) \right\} |0 > \]
Upon rescaling the fields in (32) like $\psi \rightarrow \frac{1}{\hbar^{1/2}}\psi$, expanding the exponential to the relevant order and applying Wick’s theorem, we are left with three integrals. Let’s consider the integral coming from the expansion of (32):

$$I_1 = -\frac{3m}{48(n^2)^{3/2}}(\Pi^{lm}\Pi^{ij}_i + \Pi^{jm}\Pi^{il}_i + \Pi^{mij}\Pi^{il}_i + \Pi^{imj}\Pi^{il}_i + \Pi^{ij}\Pi^{lm})$$

$$\int \frac{d^2pd^2q}{(2\pi)^4} \frac{-i\sqrt{n^2}}{m} \left( \frac{\delta^{ij}}{p^4 - 2p^2\lambda\sqrt{n^2}} - \frac{1}{2\lambda\sqrt{n^2}(p^2 - 2\lambda\sqrt{n^2})} \right)^2 \frac{\bar{n}^{i\bar{n}^j}}{m^2} =$$

$$= \frac{1}{8m\sqrt{n^2}} \int \frac{d^2pd^2q}{(2\pi)^4} \left[ (D - 3)\Pi^{ij}_i + \frac{2\lambda\sqrt{n^2}(D - 3)\Pi^{ij}_i}{p^2 - 2\lambda\sqrt{n^2}} \right]$$

$$+ (D + 1)\Pi^{ij}_i + \frac{4p^2}{2\lambda\sqrt{n^2}}\Pi^{ij}_i + \frac{2(D + 1)\lambda\sqrt{n^2}\Pi^{ij}_i}{p^2 - 2\lambda\sqrt{n^2}}$$

$$\left[ \frac{\delta^{ij}}{q^2 - 2\lambda\sqrt{n^2}} + \frac{2\lambda\sqrt{n^2}}{q^2 - 2\lambda\sqrt{n^2}} \delta^{ij} - \frac{q^4}{2\lambda\sqrt{n^2}(q^2 - 2\lambda\sqrt{n^2})} \right] \bar{n}^{i\bar{n}^j}.$$ 

Let us evaluate now this integral in $n = 2 - 2\varepsilon$ complex dimensions. Using the result that

$$\forall \alpha \geq 0 \quad \int d^np(p^2)^\alpha = 0 \quad (33)$$

we get

$$I_1 = \frac{1}{8m\sqrt{n^2}} \int \frac{d^2pd^2q}{(2\pi)^2n} \left[ \frac{4\bar{n}^2\lambda^2(D - 3)(D - 7) + 24\bar{n}^2\lambda^2(D - 3)}{(p^2 - 2\lambda\sqrt{n^2})(q^2 - 2\lambda\sqrt{n^2})} \right]$$

$$= \frac{\lambda^2\sqrt{n^2}}{2m}(D - 1)(D - 3) \int \frac{d^2pd^2q}{(2\pi)^2n} \frac{1}{(p^2 - 2\lambda\sqrt{n^2})(q^2 - 2\lambda\sqrt{n^2})}$$

and due to

$$\left[ \frac{d^np}{(2\pi)^n} \right]^2 = \frac{-\Gamma^2(\varepsilon)(4\pi)^\varepsilon(2\lambda\sqrt{n^2})^{-2\varepsilon}}{16\pi^2}$$

we obtain finally:

$$I_1 = -\frac{\lambda^2\sqrt{n^2}}{32m\pi^2}(D - 1)(D - 3)\Gamma^2(\varepsilon)(4\pi)^\varepsilon(2\lambda\sqrt{n^2})^{-2\varepsilon}. \quad (35)$$

We shall perform the $\varepsilon$ expansion of $I_1$ later. Now let us consider the second integral coming from the expansion of (32):

$$I_2 = \frac{6}{8(n^2)^{3/2}} \int \frac{d^2pd^2q}{(2\pi)^4} \left[ \frac{\bar{n}^{i\bar{n}^m}\Pi^{il}_i + \bar{n}^{m\bar{n}^i}\Pi^{il}_i + \bar{n}^{i\bar{n}^m}\Pi^{lm}}{3} \right] \left( \frac{\bar{n}^{i\bar{n}^m}\Pi^{il}_i + \bar{n}^{m\bar{n}^i}\Pi^{il}_i + \bar{n}^{i\bar{n}^m}\Pi^{lm}}{3} \right)$$

$$\left[ \frac{-p^2}{2\lambda\sqrt{n^2}} \delta^{ij} + \Pi^{ij}_i \left( 1 + \frac{2\lambda\sqrt{n^2}}{p^2 - 2\lambda\sqrt{n^2}} + \frac{p^2}{2\lambda\sqrt{n^2}} \right) \right]$$

$$\left[ \frac{-q^2}{2\lambda\sqrt{n^2}} \delta^{ij} + \Pi^{ij}_i \left( 1 + \frac{2\lambda\sqrt{n^2}}{q^2 - 2\lambda\sqrt{n^2}} + \frac{q^2}{2\lambda\sqrt{n^2}} \right) \right]$$

$$\left[ \frac{(p + q)^2}{2\lambda\sqrt{n^2}} \delta^{ij} + \Pi^{ij}_i \left( 1 + \frac{2\lambda\sqrt{n^2}}{(p + q)^2 - 2\lambda\sqrt{n^2}} + \frac{(p + q)^2}{2\lambda\sqrt{n^2}} \right) \right].$$
If we define
\[
g(p^2) = 1 + \frac{2\lambda \sqrt{n^2}}{p^2 - 2\lambda \sqrt{n^2}} + \frac{p^2}{2\lambda \sqrt{n^2}},
\]
this diagram can be written as:
\[
\frac{I_2}{\hbar^2} = -\frac{(D-3)}{12(n^2)^{\frac{D}{2}} m} \int d^2 p d^2 q \left[ \frac{3}{8} \frac{p^2 q^2 (p + q)^2}{\lambda^3 \sqrt{n^2}} - \frac{p^2 g(q^2)(p + q)^2}{2\lambda} \right.
- \frac{q^2 g(p^2)(p + q)^2}{2\lambda} + \frac{g(p^2)g(q^2)(p + q)^2 n^2}{2\lambda \sqrt{n^2}}
\]
\[
+ \frac{p^2 g(q^2)g((p + q)^2)\sqrt{n^2}}{2\lambda} + \frac{q^2 g(p^2)g((p + q)^2)\sqrt{n^2}}{2\lambda} \right].
\]

Using the property (33) of dimensional regularization this diagram amounts to:
\[
\frac{I_2}{\hbar^2} = -\frac{(D-3)}{12\sqrt{n^2} m} \int d^2 p d^2 q \left[ \frac{8n^2 \lambda^2}{(p^2 - 2\lambda \sqrt{n^2})(q^2 - 2\lambda \sqrt{n^2})} \right.
+ \frac{p^2}{(p + q)^2 - 2\lambda \sqrt{n^2}} + \frac{q^2}{(p + q)^2 - 2\lambda \sqrt{n^2}}
\]
\[
+ \frac{2\lambda \sqrt{n^2}}{(q^2 - 2\lambda \sqrt{n^2})(p^2 - 2\lambda \sqrt{n^2})} \right].
\]

Due to the following general formulas:
\[
\int d^n k \frac{1}{(-k^2 - 2p \cdot k + C)^\alpha} = i \frac{\pi^{\frac{n}{2}}}{\Gamma(\alpha)} (C + p^2)^{\frac{n}{2} - \alpha} \Gamma(\alpha - \frac{n}{2})
\]
and
\[
\int d^n k \frac{k^a k^b}{(-k^2 - 2p \cdot k + C)^\alpha} = i \frac{\pi^{\frac{n}{2}}}{\Gamma(\alpha)} (C + p^2)^{\frac{n}{2} - \alpha} \Gamma(\alpha - 1 - \frac{n}{2}) (C + p^2)^{-\frac{n}{2}} \eta^{ab}
\]
and due to (33) we get:
\[
-\frac{(D-3)}{3m \sqrt{n^2}} \int d^2 p d^2 q \left[ \frac{2n^2 \lambda^2}{(p^2 - 2\lambda \sqrt{n^2})(q^2 - 2\lambda \sqrt{n^2})} \right.
+ \frac{p^2 \lambda \sqrt{n^2}}{(q^2 - 2\lambda \sqrt{n^2})(p^2 - 2\lambda \sqrt{n^2})} \right].
\]

Since
\[
\int d^n p d^n q \frac{1}{(2\pi)^{2n} (p^2)^{\nu_3} (q^2)^{\nu_3} ((p + q)^2)^{\nu_3}} = \frac{i^{2+2n} (-b^2)^{n-\nu_1-\nu_2-\nu_3}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)(4\pi)^n} \left[ \frac{\Gamma(\nu_1 + \nu_2 + \nu_3 - n)}{\Gamma(\nu_1 + \nu_2 + 2\nu_3 - n)} \right] \left[ \frac{\Gamma(\nu_1 + \nu_3 - \frac{n}{2})}{\Gamma(\nu_2 + \nu_3 - \frac{n}{2})} \right]
\]
and due to (37) we obtain the following result
\[
I_2 = \hbar^2 \frac{(D-3)\sqrt{n^2} \lambda^2}{8m \pi^2} \Gamma^2(\varepsilon)(4\pi)^{2\varepsilon} (2\lambda \sqrt{n^2})^{-2\varepsilon}.
\]
We can find now the sum of the first two integrals (35) and (40) of the two-loop effective Lagrangian in the form

\[ I_1 + I_2 = -h^2 \frac{\lambda^2 \sqrt{n^2} (D - 3)(D - 5)}{32m \pi^2} \Gamma^2(\varepsilon)(4\pi)^{2\varepsilon}(2\lambda \sqrt{n^2})^{-2\varepsilon}. \]  

(41)

Now let us consider the last non-trivial contribution of order \( h^2 \) coming from (32). It amounts to the integral

\[ I_3 = \frac{h^2}{4\pi^2} \int \frac{d^2 p d^2 q}{(2\pi)^4} \left[ -i \frac{\sqrt{n^2}}{m} \left( \frac{\delta^{ij}}{p^2} - \frac{1}{2\lambda \sqrt{n^2}} \frac{n^i n^j}{\sqrt{n^2}} \right) \frac{1}{p^2 - 2\lambda \sqrt{n^2}} \right] (-im) \delta^{il} (p q^2 + p q^2)
\]

\[ \left[ -i \frac{\sqrt{n^2}}{m} \left( \frac{\delta^{tr}}{q^2} - \frac{1}{2\lambda \sqrt{n^2}} \frac{n^t n^r}{\sqrt{n^2}} \right) \frac{1}{q^2 - 2\lambda \sqrt{n^2}} \right] i m \delta^{jr} (p^2 q_d + q^2 p_d) \left[ -i \frac{\gamma^0}{4m \rho_0} \left( \frac{1}{\sqrt{n^2} - 2\lambda} \right) \right] \]

\[ = -h^2 \frac{\sqrt{n^2}}{4m \rho_0} \int \frac{d^2 p d^2 q}{(2\pi)^4} \left( \frac{D - 2}{q^2} + \frac{D - 2}{p^2} - \frac{1}{2\lambda \sqrt{n^2}} - \frac{q^2}{2p^2 \lambda \sqrt{n^2}} - \frac{p^2}{2q^2 \lambda \sqrt{n^2}} + \frac{p^2 + q^2}{4n^2 \lambda^2} \right) \]

\[ + \frac{h^2}{2m \rho_0} \int \frac{d^2 p d^2 q}{(2\pi)^4} \left( \frac{D - 2}{p^2} - \frac{1}{2p^2 \lambda \sqrt{n^2}} - \frac{1}{2q^2 \lambda \sqrt{n^2}} + \frac{1}{4n^2 \lambda^2} \right) \]

\[ \left( 1 + \frac{2\lambda \sqrt{n^2}}{q^2 - 2\lambda \sqrt{n^2}} + \frac{2\lambda \sqrt{n^2}}{p^2 - 2\lambda \sqrt{n^2}} + \frac{4n^2 \lambda^2}{(p^2 - 2\lambda \sqrt{n^2})(q^2 - 2\lambda \sqrt{n^2})} \right). \]

Using the property (33) of dimensional regularization and after simple algebraic calculations this graph reduces to the following integral

\[ I_3 = -h^2 \frac{\lambda n^2 (D - 3)}{2m \rho_0} \int \frac{d^2 p d^2 q}{(2\pi)^4} \frac{1}{(p^2 - 2\lambda \sqrt{n^2})(q^2 - 2\lambda \sqrt{n^2})} \]

\[ = h^2 \frac{\lambda^2 n^2 (D - 3)}{32m \pi^2 \rho_0} \Gamma^2(\varepsilon)(4\pi)^{2\varepsilon}(2\lambda \sqrt{n^2})^{-2\varepsilon}. \]  

(42)

There are also two integrals in (32) of order \( h^2 \), which cancel each other. They are

\[ I_\pm = \pm \frac{i h^2}{2} \int \frac{d^2 p d^2 q}{(2\pi)^4} \left( -i \frac{\sqrt{n^2}}{m} \right) \left( \frac{\delta^{ij}}{p^4 - 2p^2 \lambda \sqrt{n^2}} - \frac{1}{2\lambda \sqrt{n^2} (p^2 - 2\lambda \sqrt{n^2})} \frac{n^i n^j}{\sqrt{n^2}} \right) \]

\[ m^2 (p \cdot q)^2 (p^2 + q^2)^2 \delta^{tr} \delta^{lm} \left( -i \frac{\sqrt{n^2}}{m} \right) \left( \frac{\delta^{lm}}{p^4 - 2p^2 \lambda \sqrt{n^2}} - \frac{1}{2\lambda \sqrt{n^2} (p^2 - 2\lambda \sqrt{n^2})} \frac{n^l n^m}{\sqrt{n^2}} \right). \]

Therefore the pure two-loop contribution to the effective Lagrangian is:

\[ \mathcal{L}_2(\varepsilon) = I_1 + I_2 + I_3 = -h^2 \left[ \frac{\lambda^2 \sqrt{n^2} (D - 3)(D - 5)}{32m \pi^2} - \frac{\lambda n^2 (D - 3)}{32m \pi^2} \right] \Gamma^2(\varepsilon)(4\pi)^{2\varepsilon}(2\lambda \sqrt{n^2})^{-2\varepsilon}. \]  

(43)

The Laurent expansion around \( \varepsilon = 0 \) is

\[ \mathcal{L}_2(\varepsilon) = -h^2 \frac{(D - 3)}{32 \pi^2} \frac{\lambda}{m} \sqrt{n^2} \left( (D - 5)\lambda - \sqrt{n^2} \right) \left\{ \frac{1}{\varepsilon^2} - \frac{2}{\varepsilon} \ln \left( \frac{\lambda}{\rho_0} \right) + 2 \ln^2 \left( \frac{\lambda}{M} \right) + \frac{\pi^2}{6} \right\}. \]  

(44)
Summing all contributions in $h^2$-order coming from (28) and (44) we can cancel the non-polynomial divergencies first by requiring

$$h^2(D - 3)\frac{\lambda}{4\pi} \delta \lambda \sqrt{n^2} \ln \left( \frac{\lambda}{M} \right) + h^2 \frac{(D - 5)\lambda - \sqrt{n^2}}{16\pi^2\varepsilon} \frac{\lambda}{m} \sqrt{n^2} \ln \left( \frac{\lambda}{M} \right) = 0,$$

that is

$$\delta \lambda = -\frac{1}{4\pi \varepsilon} \frac{\lambda}{m} \left[(D - 5)\lambda - \frac{\sqrt{n^2}}{\rho_0}\right]$$

and then the polynomial divergencies requiring that

$$\delta m_2 = -\frac{1}{\varepsilon^2} \frac{32\pi^2}{m} \left[(D - 5)\lambda - \frac{\sqrt{n^2}}{\rho_0}\right].$$

Finally substituting into $\mathcal{L}_{\text{eff}} = \mathcal{L}_0(\varepsilon) + \mathcal{L}_1(\varepsilon) + \mathcal{L}_2(\varepsilon)$ the values found for $\delta \lambda$ and $\delta m_2$ from the equations (46) and (47) we get the finite $\overline{MS}$ world-sheet effective Lagrangian in two-loop order:

$$\mathcal{L}_2 = -\frac{(D - 3)(D - 5)}{32\pi^2} \frac{\lambda^2}{m} \sqrt{n^2} \ln^2 \left( \frac{\lambda}{M} \right)$$

together with the term which we shall write separately, because, as it is easy to see, it is proportional to the Polyakov-Kleinert Lagrangian

$$\mathcal{L}_{2\ PK} = \left(\frac{D - 3}{16\pi^2}\right) \frac{\lambda}{m} \ln^2 \left( \frac{\lambda}{M} \right) \frac{n^2}{\rho_0} \propto \frac{(\partial^2 X^\mu)^2}{\rho} = \sqrt{\varepsilon} K_i^a K_j^b.$$

Quantum fluctuations generate high powers of the extrinsic curvature tensor $K_i^{ab}$ in the form of Polyakov-Kleinert action with the induced coupling constant

$$\frac{1}{e^2_{\text{eff}}} = \left(\frac{D - 3}{16\pi^2}\right) \frac{\lambda}{m} \ln^2 \left( \frac{\lambda}{M} \right).$$

From these results we conclude that the extremum of the effective action with respect to $\lambda$ defines a non-trivial saddle point solution and exhibits the condensation of the Lagrange multiplier at the value $<\lambda> = M$ (this extremum is equivalent to imposing the constraint (7) $g_{ab} = \partial_a X_\mu \partial_b X_\mu$). One can also see that the coupling constant $1/e^2_{\text{eff}}$ is zero at the saddle point and high derivative term $\mathcal{L}_{PK}$ actually is not present. The absence of high derivative terms in the effective action can also be seen in physical gauge calculations [7], there are two terms (see formulas (50) and (52) in [7]) proportional to $\mathcal{L}_{PK}$, but they cancel each other.

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