Strings in an electric field, and the Milne Universe

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Abstract: Arguably the simplest model of a cosmological singularity in string theory, the Lorentzian orbifold \( R^{1,1}/\text{boost} \) is known to lead to severe divergences in scattering amplitudes of untwisted states, indicating a large backreaction toward the singularity. In this work we take a first step in investigating whether condensation of twisted states may remedy this problem and resolve the spacelike singularity. By using the formal analogy with charged open strings in an electric field, we argue that, contrary to earlier claims, twisted sectors do contain physical scattering states, which can be viewed as charged particles in an electric field. Correlated pairs of twisted states will therefore be produced, by the ordinary Schwinger mechanism. For open strings in an electric field, on-shell wave functions for the zero-modes are determined, and shown to analytically continue to non-normalizable modes of the usual Landau harmonic oscillator in Euclidean space. Closed strings scattering states of the Milne orbifold continue to non-normalizable modes in an unusual Euclidean orbifold of \( \mathbb{R}^2 \) by a rotation by an irrational angle. Irrespective of the formal analogy with the Milne Universe, open strings in a constant electric field, or colliding D-branes, may also serve as a useful laboratory to study time-dependence in string theory.

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1. Introduction

One of the most fundamental problems in quantum gravity is the nature of the Big Bang singularity, or more general space-like singularities. This is interesting both from a theoretical viewpoint, as it is likely to cause a breakdown of effective field theory, and from a practical viewpoint, as our Universe may have started in such a singularity. In the context of string theory, this issue is also tied with the problem of time dependence, and might play a role in solving the still mysterious vacuum selection problem.

A possible avenue into the study of gravitational singularities is to investigate exact two-dimensional conformal field theories which realize them in target space. A particularly simple example is the orbifold of the two-dimensional Minkowski space $\mathbb{R}^{1,1}$ by a discrete boost $X^\pm \equiv e^{\pm \beta} X^\pm$, tensored with a $\hat{c} = 8$ superconformal field theory such as flat Euclidean space $\mathbb{R}^8$ [1, 2]. Part of the geometry described by this orbifold CFT corresponds to the Milne Universe, i.e. a circle shrinking linearly in time till a Big Crunch, or growing linearly in time after a Big Bang (see Figure 1). It also includes two disconnected regions with closed time-like curves (CTC) attached to the cosmological singularity, whose existence may yet be a fatal flaw or a bounty. Despite being a very non-generic case of a Kasner singularity, the same geometry arises locally in many other examples based on gauged Wess-Zumino models [3, 4, 5, 6]. Other time-dependent orbifolds have been discussed recently in [9, 10, 11, 12].

Although the Lorentzian orbifold CFT may seem to be well defined using ordinary orbifold techniques, tree-level scattering amplitudes of untwisted string states are divergent, casting doubt on the validity of perturbation theory [1, 8] (see [2, 3] for related work in the case of null singularities). The divergences may be traced back to large graviton exchange near the singularity [8], where incoming particles become infinitely blue shifted. A non-perturbative instability toward large black hole creation has also been argued to result from the same mechanism [14]. On the other hand, the dynamics in the twisted sector is far less understood, and leaves open the possibility that a condensation of twisted states would remedy the afore-mentioned difficulties. This expectation is supported by the often noticed formal similarity, at the level of the CFT, of this closed string background with open strings in an electric field (see [15] for further time-dependent open/closed string analogies). Indeed, the structure of excited modes of first quantized closed strings in the twisted sector of the Milne orbifold essentially amounts to two copies of that of charged
open strings in a constant electric field, while the zero-modes are identical between the two cases without doubling, just as in flat space. As we shall see, in both the electric and Milne case, the zero-mode degrees of freedom describe the motion of a charged particle (the string center of motion) in an electric field.

To be more precise, consider two parallel D1-branes with anti-parallel Abelian electric fields $F^{(0)} = edx^+dx^-$, $F^{(1)} = -edx^+dx^-$ (see Figure 1). Open strings stretching between the two carry a net electric charge under the electric field $F^{(0)} - F^{(1)}$, and are analogous to closed strings in the $w$-twisted sector of the orbifold with boost parameter $\beta$, upon identifying $2\pi e \sim -w\beta$ for small $e$. Open strings in the 0-0 or 1-1 sectors on the other hand have no total charge, and as we will see, behave analogously to closed strings in the untwisted sector. In contrast to the physics of closed strings near a cosmological singularity, the dynamics of charged open strings in an electric field is fairly well understood [16, 17, 18, 19]: by the usual Schwinger mechanism [20], pairs of charged open strings are created from the vacuum, and cause the electric field to decay in a sequence of plasma oscillations, as they move off to infinity and discharge the condenser plates that created the electric field in the first place [21, 22, 23, 24]. At late times, one therefore recovers a situation with zero electric field.

As a way to make contact with the Milne Universe, one may consider the T-dual of the electric field configuration, namely two D-branes moving with opposite velocities $\pm e$, respectively: for $e \neq 0$, the distance between the two D-branes decreases linearly with time until they collide, just like the radius of the circle in the Milne universe before the Big Crunch. Just as in the electric field case, stretched open strings are pair-produced as the D-branes move away from (resp. toward) each other, and decelerate (resp. accelerate) the motion [25, 26]. In contrast to the electric field case, the final outcome of the D-brane collision is however less clear, as it depends on the ratio between production and recombination rate of open strings, as well as on the details of the bound state formation process.

By analogy with these open string processes, one may therefore expect that the cre-
ation of pairs of twisted closed strings will slow down the contraction rate of the Milne Universe, and possibly prevent the compact circle from reaching zero-size. Indeed, once produced, the twisted closed strings contribute a tensive energy that grows linearly with the radius of the Milne universe, thus mimicking the effect of a two-dimensional positive cosmological constant\(^1\). Whether the circle will further re-expand indefinitely or reach a constant radius cannot be decided on the basis of this analogy alone, as the D-brane collision process above illustrates. In contrast to the resolution of the usual time-like orbifold singularities by condensation of a coherent state of the twisted sector moduli field, this process involves the condensation of correlated multi-particle states, which go beyond standard string perturbation techniques. We hope to return to this problem in a future publication.

More modestly, our aim in this paper will be to understand the kinematics of twisted sector states, for which purpose the analogy with open strings in an electric field will turn out to be quite illuminating. In particular, the implicit assumption we made above that twisted sectors contain physical states appears to contradict a recent investigation of the one-loop amplitude in the Lorentzian orbifold \(^2\). As the analogy with open strings in electric fields will make it clear, upon appropriate quantization of the string zero-modes there does in fact exist a continuum of delta-normalizable physical scattering states, describing the unbounded trajectories of charged particle in an electric field. Such a spectrum in fact consistent with the one-loop amplitude computed in \(^2\), as the latter may be viewed either as the contribution of discrete Euclidean or continuous Lorentzian states.

The physical origin of this continuous spectrum of physical states is clear: in either the electric field and Milne universe cases, these states arise from quantizing the charged Klein-Gordon equation in two dimensions. In static coordinates, they are the scattering states of the inverted harmonic oscillator. In light-cone coordinates, they are eigenmodes of the scaling operator on phase space \(^2\) \(H = pq\). In Rindler coordinates, best suited for the Lorentzian orbifold case where one needs to project on states with integer boost momentum \(J\), they correspond to eigenstates of the Schrödinger equation with Liouville-like potential \(V(y) = M^2e^{2y} - \left(J + \frac{1}{2}e^{2y}\right)^2\). In all these cases, the potential is unbounded from below, so that there is no vacuum in the first quantized approach. The second quantized theory however is well defined and has a variety of in and out states, due to Schwinger pair creation (and, in the Rindler case, Hawking particle production). The latter can be understood very simply as the tunneling under the barrier in these unbounded potentials \(^2\).

Having found that the charged (resp. twisted) sectors of the open (resp. closed) string contain physical states, an intermediate goal before taking their back-reaction into account is to determine their scattering amplitudes, and check whether they turn out to be less singular than those of untwisted states. For this one needs to perform an analytic Wick rotation both on the world-sheet and in target space, an often perilous task in generic time-dependent backgrounds. Despite the lack of a global time-like Killing vector, we show that the Lorentzian orbifold does admit a well defined analytic continuation, upon simultaneously Wick rotating the target space light-cone coordinates \(X^\pm\) into a pair of

\(^1\)We thank B.Craps for a discussion on this point.

\(^2\)The latter quantization scheme was in fact suggested in a footnote of \(^2\).
complex conjugate coordinates $Z, \bar{Z}$, as well as continuing the real boost parameter $\beta$ to
an imaginary value. In the open string case, this prescription is precisely what allows one
to convert an electric field in Minkowski space to a magnetic field in Euclidean space. In
the closed string case, one obtains instead an orbifold of the Euclidean plane by a rotation
of angle $\beta$. Instead of reducing to the usual rotation orbifold $\mathbb{C}/\mathbb{Z}_N$ for rational values
of $\beta$, as often assumed in other instances of conical singularities \cite{33, 34}, we find that
it behaves continuously with respect to the rotation angle $\beta$, thanks to the inclusion of
string states that wind several times around the origin. This is of course necessary if the
analytic continuation back to the Lorentzian orbifold is to make sense. In fact, the same
rotation orbifold construction arises in the context of closed strings in plane gravitational
waves supported by a Neveu-Schwarz magnetic flux \cite{35, 36}, for which usual current algebra
techniques allow one to compute the scattering amplitudes \cite{37}.

Although this analytic continuation of the electric field (resp. Lorentzian orbifold)
gives the same result as the continuation of a magnetic field (resp. Euclidean rotation
orbifold) with respect to a transverse time coordinate, it is important to realize that the
electric/Milne physical states continue to an altogether different set of observables. While
the states usually considered in a magnetic field are the usual normalizable Landau states
with positive energy, the analytic continuation of the scattering states in an electric field
involves instead negative energy non-normalizable states of the harmonic oscillator; these
states blow up in the Euclidean direction where the particle comes from. Similarly,
the states of interest in the rotation orbifold case are not the usual localized twisted states,
but non-normalizable states which blow up either at the origin of the plane or at infinity.
It would be interesting to adapt the techniques in \cite{37} for such states, although we will not
attempt this in this paper.

The outline of this paper is as follows. In Section 2, we review some of the literature on
the first quantization of charged open strings in a constant electric field, and of closed strings
in the twisted sector of the Milne orbifold. In Section 3, we propose a quantization scheme
for the zero-mode sector which leads to physical states in the twisted sectors. We explain
its physical origin in static coordinates, which are most convenient to discuss Schwinger
pair creation in an electric field, and illustrate the prescription for excited states up to
level 1. In Section 4 we turn to Milne space, which requires quantizing the zero-mode wave
functions from the point of view of an accelerated observer, i.e. in Rindler coordinates. We
evaluate semi-classically the distribution of produced pairs, noting that it can be obtained
by projecting the usual homogeneous pair production in an electric field to boost invariant
states. Section 5 is devoted to analyzing the continuation to Euclidean space, both in the
electric field and Lorentzian orbifold cases. Our conclusions are presented in Section 6,
together with comments on open issues. Appendices contain further material on light-cone
quantization, Wick rotation and a review of parabolic cylinder and Whittaker functions.

2. First quantization, reviewed

In this section, we first review elementary aspects of charged open strings in an electric
field, following \cite{13} up to small changes in notations. We then turn to twisted closed strings
in the Lorentzian orbifold, following [28], and discuss how the two problems are related.

2.1 Charged particle in an electric field

Before turning to the case of open strings in a constant electric field, it is useful to recall some basic features of charged particles in an electric field.

2.1.1 Classical trajectories and conserved charges

The Lagrangian for a particle of mass $m$ and unit charge in a constant electric field $F = e \, dx^+ \wedge dx^-$ reads

$$L = \frac{1}{2} m \left(-2 \partial_\tau X^+ \partial_\tau X^- + (\partial_\tau X^i)^2\right) + \frac{e}{2} \left(X^+ \partial_\tau X^- - X^- \partial_\tau X^+\right)$$

(2.1)

where we defined the light-cone coordinates $X^\pm = (X^0 \pm X^1)/\sqrt{2}$, and work with the mostly plus metric $\text{d}s^2 = -2dX^+dX^- + (dX^i)^2$. The equations of motion are easily integrated to

$$X^\pm = x^\pm_0 \pm \frac{a^\pm_0}{e} e^{\pm \sigma / m}, \quad X^i = x^i_0 + p^i / m$$

(2.2)

hence classical trajectories are hyperbolas centered at an arbitrary point $(x^+_0, x^-_0)$ and asymptoting to the light-cone (see Figure 2). The canonical momenta

$$\pi^\pm = m \partial_\tau X^\pm \mp \frac{e}{2} x^\pm = \mp x^\pm_0 + \frac{1}{2} a^\pm_0 e^{\pm \sigma / m}, \quad \pi^i = m \partial_\tau X^i = p^i$$

(2.3)

satisfy the usual equal-time commutation rules

$$[\pi^+, X^-] = [\pi^-, X^+] = i, \quad [\pi^i, X^j] = i \delta_{ij}$$

(2.4)

The world-line Hamiltonian derived from (2.1) reads

$$H = \frac{1}{2m} \left[2(\pi^+ + \frac{e}{2} X^+)(\pi^- - \frac{e}{2} X^-) - p^2_i \right] = \frac{2a^+_0 a^-_0 - p^2}{2m}$$

(2.5)

and should equal $m/2$ by the mass-shell condition. The equation of the trajectory in the light-cone directions may thus be written

$$(X^+ - x^+_0)(X^- - x^-_0) + \frac{M^2}{2e^2} = 0$$

(2.6)

where we denoted by $M^2 = m^2 + p^2_i$ the two-dimensional mass. By convention, we will call particles (or electrons) those following the right branch of this hyperbola (i.e. $a^+_0 > 0$), and anti-particles (or positrons) those following the left branch (i.e. $a^-_0 < 0$). The coordinates of the center of the hyperbola are conserved charges, equal to the generators of translations in target space

$$P^\pm = m \partial_\tau X^\pm \mp e x^\pm_0$$

(2.7)

The commutation relations (2.4) imply that the two positions $x^\pm_0$ and velocities $a^\pm_0$ cannot be measured simultaneously, but rather

$$[x^+_0, x^-_0] = -\frac{\hat{i}}{e}, \quad [a^+_0, a^-_0] = -i e$$

(2.8)
Finally, one should recall that a constant electric field is invariant under Lorentz boosts $X^\pm \rightarrow e^{\pm \beta} X^\pm$. The infinitesimal generator for this symmetry, commuting with the world-line Hamiltonian $H$, is

$$j = X^+ \pi^- - X^- \pi^+ = e x_0^+ x_0^- + \frac{a_0^+ a_0^-}{e}$$

where we recall that $a_0^+ a_0^- = M^2/2$. In particular, it follows from (2.6) that, just as for neutral particles, trajectories of charged particles with $j = 0$ go through the origin $(X^+, X^-)$.

![Figure 2: Charged particle in an electric field](image)

**Figure 2:** Charged particle in an electric field. The left (right) branch of the hyperbola represents an positron (electron). The hyperbola is centered at $(x_0^+, x_0^-)$. The right branch intersects the light cone at $x^+ = j/(e x_0^-)$ and $x^- = j/(e x_0^+)$.  

### 2.1.2 Vacuum energy and Schwinger pair production

We now consider the one-loop vacuum free energy associated to a particle of mass $m$ and spin $s$ in a constant electric field. Using the Schwinger proper time representation of the propagator, we get

$$\mathcal{F} = - \frac{1}{2(2\pi)^{D-2}} \int_0^\infty \frac{dt}{t^{D/2}} \left[ \frac{e \sinh[(2s + 1)et]}{\sinh^2 et} - \frac{2s + 1}{t} \right] e^{-i(m^2 - i0^+)^2}$$

As is well known, the free energy density has a non-zero imaginary part, which can be identified as the production rate of charged pairs [20]. The imaginary part can be computed

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by residues, and reads
\[
\mathcal{W} = \Im \mathcal{F} = \frac{1}{2(2\pi)^{D-1}} (2s + 1) \sum_{k=1}^{\infty} (-1)^{F(k+1)} \left( \frac{e}{k} \right)^{D/2} \exp \left(-\pi k \frac{m^2}{|e|} \right) \tag{2.11}
\]
where the \(k\)-th term in the sum comes from the pole at \(et = ik\pi\), and \(F = 2s \mod 2\) denotes the statistics of the particle. As charged particles get created, they move off to infinity and discharge the condenser plates which created the electric field. The electric field therefore relaxes to zero, possibly in a sequence of plasma oscillations due to recombination phenomena \[21, 22\].

### 2.2 Charged string in an electric field

Let us now recall the basic features of charged open strings propagating in a constant electric field. We restrict to the bosonic string for simplicity, and follow \[19\] except for a few minor changes in the notations.

#### 2.2.1 Normal modes and world-sheet Hamiltonian

The world-sheet action of the bosonic open string in an electric field reads
\[
S_{bos} = -\frac{1}{4\pi \alpha'} \int d\sigma d\tau \left( \partial_a X^\mu \partial^a X_\mu + \frac{1}{2} \int d\tau \, F^{(0)}_{\mu\nu} X^\nu \partial_\tau X^\mu|_{\sigma=0} + \frac{1}{2} \int d\tau \, F^{(1)}_{\mu\nu} X^\nu \partial_\tau X^\mu|_{\sigma=\pi} \right) \tag{2.13}
\]
where \(F^{(a)}_{\pm} = e_a, a = 0, 1\) are the values of the electric field at the two ends of the string, which we may assume to be on different D-branes. The embedding coordinates are therefore harmonic functions on the world-sheet, satisfying the boundary conditions
\[
\begin{align*}
\partial_\sigma X^\pm &= \mp \pi e_0 \partial_\tau X^\pm \quad (\sigma = 0), \tag{2.14} \\
\partial_\sigma X^\pm &= \pm \pi e_1 \partial_\tau X^\pm \quad (\sigma = \pi). \tag{2.15}
\end{align*}
\]
A simple computation shows that the eigenmodes are integer spaced but with an imaginary part,
\[
\omega_n = n \pm i\nu, \quad \pi\nu = \text{arcth}(\pi e_0) + \text{arcth}(\pi e_1) \tag{2.16}
\]
except when the string starts and end on the same brane, whence \(e_0 + e_1 = 0\). The light-cone embedding coordinates may be expanded in orthonormal modes, (setting \(\alpha' = 1/2\))
\[
X^\pm = x^\pm_0 + i \sum_{n=-\infty}^{+\infty} (-)^n (n \pm i\nu)^{-1} a^\pm_n e^{-i(n\pm i\nu)\tau} \cos[(n \pm i\nu)\sigma \mp i \text{arcth}(\pi e_0)] \tag{2.17}
\]
where our choice of normalization for excited modes reduces to the standard one \[38\] as \(\nu \to 0\). Reality further demands that \((a^\pm_n)^* = a^{\mp}_{-n}\). In particular, the zero-modes \(a^\pm_0\) and \(x^\pm_0\) are hermitian operators. It is worth noting that while each of the points of the string becomes infinitely accelerated as \(\tau \to \infty\), the length of the string remains finite.
The contribution of the light-cone coordinates $X^\pm$ to the world-sheet Hamiltonian $L_0$ is now easily computed,

$$L_{0}^{l.c.} = -\frac{1}{\pi} \int_{0}^{\pi} (\partial_{\tau} X^{-} \partial_{\sigma} X^{+} + \partial_{\sigma} X^{+} \partial_{\tau} X^{-}) = -\frac{1}{2} \sum_{m=-\infty}^{\infty} (a_{-m}^{+}a_{m}^{-} + a_{m}^{-}a_{-m}^{+})$$  \hspace{1cm} (2.18)

which we kept Weyl-ordered at this stage, while their contribution to the higher Virasoro generators reads

$$L_{n}^{l.c.} = -\sum_{m=-\infty}^{\infty} a_{m+n}^{+}a_{m}^{-} + L_{n}^{int}$$ \hspace{1cm} (2.19)

We will often assume that the internal conformal field theory is just flat $\mathbb{R}^{24}$, or $\mathbb{R}^{8}$ in the superstring case. The canonical commutation relations may be computed easily,

$$[a_{m}^{+}, a_{n}^{-}] = -(m + i\nu)\delta_{m+n} , \hspace{0.5cm} [x_{0}^{\pm}, x_{0}^{\pm}] = -\frac{i}{\epsilon_{0} + \epsilon_{1}}$$ \hspace{1cm} (2.20)

and ensure that $L_0$ generates the time evolution on the world-sheet,

$$[L_0, x_{0}^{\pm}] = -(m \pm i\nu)a_{m}^{\pm}$$ \hspace{1cm} (2.21)

Note that the operators $x_{0}^{\pm}$ and $a_{0}^{\pm}$ satisfy the same algebra as for a charged particle (2.8), in the limit of small electric field, $\nu \sim \epsilon_{0} + \epsilon_{1}$. In fact, just as in the particle case $x_{0}^{\pm}$ is proportional to the generators of translations along the light-cone directions,

$$P^{\pm} = \frac{1}{\pi} \int_{0}^{\pi} \partial_{\tau} X^{\pm} - \epsilon_{0}X^{\pm}(0) - \epsilon_{1}X^{\pm}(\pi) = \mp(\epsilon_{0} + \epsilon_{1})x_{0}^{\pm}$$ \hspace{1cm} (2.22)

Given these commutation relations, it is natural to try and quantize the string by assuming the existence of a ground state annihilated by all strictly positive frequency modes $a_{n>0}^{+}, a_{n>0}^{-}$ and, say, by $a_{0}^{+}$. The world-sheet Hamiltonian, normal ordered with respect to this vacuum, takes the form

$$L_{0}^{l.c.} = -\sum_{m=0}^{\infty} a_{-m}^{+}a_{m}^{-} - \sum_{m=1}^{\infty} a_{-m}^{-}a_{m}^{+} + \frac{i\nu}{2}(1 - i\nu) - \frac{1}{12}$$ \hspace{1cm} (2.23)

As we shall see shortly, this quantization prescription is related by analytic continuation to the prescription for an Euclidean magnetic field. It is perfectly satisfactory for the purposes of computing the one-loop vacuum energy, however it does not yield any physical states, in contrast with classical expectations. The easily verified hermicitity of $L_0$ does not contradict the fact that states built on the vacuum by acting with creation operators $a_{-n}^{\pm}$ have complex energy, since these states have also zero-norm, just as in flat Minkowski space. In the case of the superstring, the fermion vacuum energy in the Ramond sector, $E_{R} = 1/8 - i\nu(1 - i\nu)/2$ completely cancels the boson contribution, while in the Neveu-Schwarz sector, $E_{NS} = -\nu^{2}/2$ only offsets the quadratic term.
2.2.2 One-loop amplitude and particle production

Using this quantization scheme, and freely Wick rotating to an Euclidean world-sheet, we may now compute the one-loop vacuum free energy,

\[ A_{\text{bos}} = \frac{i\pi V_{26}(e_0 + e_1)}{2} \int_0^\infty \frac{dt}{(4\pi^2 t)^{13}} \eta^{21}(it/2) \theta_1(tv/2; it/2) e^{-\pi v^2 t/2} \] (2.24)

where \( \theta_1 \) is the Jacobi theta function,

\[ \theta_1(v; \tau) = 2q^{1/8} \sin \pi v \prod_{n=1}^\infty (1 - e^{2\pi iv} q^n)(1 - q^n)(1 - e^{-2\pi iv} q^n), \quad q = e^{2\pi i\tau} \] (2.25)

Just as in (2.10), each of the poles at \( t = 2k/\nu \) contributes to the imaginary part, yielding the rate for charged string pair production,

\[ W = \frac{1}{2(2\pi)^{25}} \frac{(e_0 + e_1)}{\nu} \sum_{k=1}^\infty (-)^{k+1} \left( \frac{|\nu|}{k} \right)^{13} \sum_{N=-1}^\infty c_b(N) \exp \left( -2\pi k \frac{N}{|\nu|} - 2\pi k|\nu| \right) \] (2.26)

where we expanded \( \eta^{-24}(q) = \sum_{N=-1}^\infty c_b(N)q^N \). The result (2.26) can be viewed as the sum of the production rates (2.11) for each state in the string spectrum, of mass \( m^2 = 2N + \nu^2 \).

Note that the poles originate from the \( \sin(\pi \nu t/2) \) term in (2.25), which represents the contribution of the zero-modes.

Similarly, the one-loop amplitude in the oriented superstring case reads

\[ A_{\text{ferm}} = \frac{i\pi V_{10}(e_0 + e_1)}{2} \int_0^\infty \frac{dt}{(4\pi^2 t)^{13}} \eta^{11}(it/2) \theta_1(tv/2; it/2) \theta_4(tv/4; it/2) \] (2.27)

The imaginary part can be computed as above, and yields

\[ W = \frac{1}{2(2\pi)^{9}} \frac{e_0 + e_1}{\nu} \sum_{k=1, k \text{ odd}}^\infty \left( \frac{|\nu|}{k} \right)^{5} \sum_{N=0}^\infty c_f(N) e^{-2\pi kN/|\nu|} \] (2.28)

where \( (\theta_4^3 - \theta_4^5)/\eta^{12}(q) = \theta_2^4(q)/\eta^{12}(q) = \sum_{N=0}^\infty c_f(N)q^N \) is the partition function in the Neveu-Schwarz sector, equal to that in the Ramond sector. This is in agreement with Schwinger’s formula (2.11), upon identifying \( m^2 = 2N \). In particular, there is no quadratic mass shift, despite the occurrence of such a term in the vacuum energy of \( L_0 \) in the Neveu-Schwarz sector: the contribution of the massless sector of the open superstring therefore is only power suppressed as a function of \( k \), and converges only when sufficiently many non-compact dimensions are included. Note also that the contributions with even Schwinger index \( k \) cancel between bosons and fermions, as a consequence of supersymmetry\(^3\).

2.2.3 Normal ordering and physical states

Let us now return to our normal ordered expression (2.23) for \( L_0 \). While it leads to a one-loop amplitude (2.24) with a satisfactory interpretation, it suffers from a major drawback:

\(^3\)In this sense, introducing an electric field breaks SUSY only softly.
the energy of any state constructed by applying creation operators on this vacuum has an imaginary part which is an odd multiple of $i\nu/2$, and therefore never satisfies the physical condition $L_0 = 0$. One may object that a uniform constant electric field is not physical anyway, since infinite energy may be transferred to charged particles; furthermore, a realistic electric field could only occur in a finite region of space bounded by condenser plates or other device that creates it. This answer is not satisfactory, as one may well scatter electrons and positrons in a finite region immersed in an electric field, of size much larger than the wavelength of the incoming particles. Clearly there is no problem in setting this up classically, and a quantum induced vacuum energy should not cause electrons to vanish into inconsistency.

Let us analyze the reason for this imaginary vacuum energy. The Fock space condition $a_0^+|0\rangle = 0$ is simply the analytic continuation of the $a_0^-|0\rangle = 0$ condition which picks out the normalizable eigenstates of the two-dimensional harmonic oscillator describing the Landau orbits of charged particles in a constant magnetic field: upon analytically continuing $B \rightarrow i\nu$ so as to obtain an electric field, these states become eigenstates of an inverted harmonic oscillator with imaginary energy $i\left(n + \frac{1}{2}\right)\nu$. Nevertheless, the inverted harmonic oscillator admits a perfectly well behaved (although unbounded) spectrum of delta-normalizable scattering states with real energy: those are the ones which describe the unbounded trajectories of the electron and positron in an electric field. Upon continuation back to the Euclidean magnetic case, these states would correspond to negative energy non-normalizable states.

Given that the inverted harmonic oscillator does not have a ground state, there is no reason to normal order in the zero-mode sector, and we therefore write

$$L_{0,c}^{L} = -\frac{1}{2} (a_0^+ a_0^- + a_0^- a_0^+) - \sum_{m=1}^{\infty} (a_0^+ a_m^- + a_m^- a_0^+) + \frac{1}{2} \nu^2 - \frac{1}{12}$$

This generator now has a real vacuum energy in the excited sectors. We follow the same ordering rule for the generator of boosts,

$$J = \frac{1}{\pi} \int_0^\pi \left( X^+ \partial_\tau X^- - X^- \partial_\tau X^+ \right) + \frac{e_0}{\pi} X^+ X^-(0) + \frac{e_1}{\pi} X^+ X^-(\pi)$$

$$= \frac{1}{2} (e_0 + e_1) (x_0^+ x_0^- + x_0^- x_0^+) + \frac{1}{2\nu} (a_0^+ a_0^- + a_0^- a_0^+) + i \sum_{m=1}^{\infty} \left( \frac{a_0^+ a_m^- + a_m^- a_0^+}{m + \nu} - \frac{a_0^+ a_m^- + a_m^- a_0^+}{m - \nu} \right)$$

where the zero-mode contribution reproduces the boost momentum $j$ of a charged particle (2.9). We will discuss in Section 3 the physical content of this ordering prescription.

2.3 Twisted closed strings in the Milne Universe

Let us now turn to the case of closed strings propagating in flat two-dimensional Minkowski space orbifolded by a discrete Lorentz boost $X^\pm = e^{\pm \beta X^\pm}$. We recall that this space can be interpreted as a cosmological universe with a circle contracting linearly with time until a Big Crunch singularity, followed by a linear Big Bang expansion:

$$ds^2 = -2dX^+dX^- = -dT^2 + T^2 d\theta^2$$

(2.31)
where the Milne coordinates $X^\pm = T e^{\pm \eta}/\sqrt{2}$ cover the forward ($T > 0$) and past ($T < 0$) light-cone of two-dimensional Minkowski space. The effect of the orbifold is to make the spatial coordinate compact, $\theta \equiv \theta + \beta$. In addition, the left and right quadrants of the covering space descend two regions attached to the singularity at $T = 0$, with metric

$$ds^2 = -2dX^+ dX^- = -r^2 d\eta^2 + dr^2$$ (2.32)

where $X^\pm = \pm re^{\pm \eta}/\sqrt{2}$ parameterize Rindler space. The orbifold action identifies the Rindler time under $\eta \equiv \eta + \beta$ hence generates CTC’s. It will be also useful to define $r = \pm e^y, T = \pm e^\tau$ (with appropriate signs in each patch) so that the metric becomes conformal to a cylinder,

$$ds^2 = e^{2y} (-d\eta^2 + dy^2) = e^{2\tau} (-d\tau^2 + d\theta^2)$$ (2.33)

with the singularity being pushed at $y = -\infty$ or $\tau = -\infty$. Finally, the light-cone $X^+ X^- = 0$ on the covering space descends to a non-Hausdorff null space lying arbitrarily close to the cosmological singularity.

While the Lorentzian orbifold exhibits a cosmological singularity and CTC’s, it has been suggested to resolve these problems by combining the boost with a translation on an extra spatial direction $[29, 39, 40]$. It is unclear however whether this deformation survives quantum dynamics, or relaxes dynamically to the singular configuration. Upon reduction along the direction where the translation is performed, this “electric Melvin universe” can be interpreted as an electric background for the Kaluza-Klein gauge field, on a curved geometry induced by the electric energy. Closed time-like curves still exist but are shielded behind a Cauchy horizon. Kaluza-Klein modes along the compact circle are charged under the electric field, and so may be pair-produced $[27]$. Our concern in this paper however is in the kinematics of twisted strings, which are neutral under this gauge field, and insensitive to the deformation.

2.3.1 Normal mode expansion

As in any orbifold, closed strings fall into different twisted sectors depending how many times they wind around the space-like circle $S^1_\theta$ in the past and forward patches, or the time-like circle $S^1_\eta$ in the left and right patches. Strings in the untwisted sector have the standard free mode expansion, but for the condition that they should have integer total momentum along $\theta$, or energy along $\eta$. Closed strings in the $w$-th twisted sector on the other hand satisfy the twisted periodicity condition

$$X^\pm (\sigma + 2\pi, \tau) = e^{\pm w/\beta} X^\pm (\sigma, \tau)$$ (2.34)

and hence have complex proper frequencies $\omega_n = n \pm i\nu$. This is analogous to the spectrum of open strings upon identifying the product of the boost parameter by the winding number with the electric field, according to

$$w/\beta = -2\pi \nu = -2\text{arcth}(\pi e_0), \quad e_1 = 0.$$ (2.35)

Upon continuing $\beta$ to $i\beta$, one thus obtains the identification appropriate for thermal Rindler space.
The string embedding coordinates may be expanded in left and right moving eigenmodes
\[ X^\pm_{closed} = X^\pm_R(\tau - \sigma) + X^\pm_L(\tau + \sigma) \] (2.36)
where, in analogy with (2.17),
\[ X^\pm_R(\tau - \sigma) = \frac{i}{2} \sum_{n=-\infty}^{\infty} (n \pm i\nu)^{-1} a^\pm_n e^{-i(n\pm i\nu)(\tau - \sigma)} \] (2.37)
\[ X^\pm_L(\tau + \sigma) = \frac{i}{2} \sum_{n=-\infty}^{\infty} (n \mp i\nu)^{-1} \tilde{a}^\pm_n e^{-i(n\mp i\nu)(\tau + \sigma)} \] (2.38)
The closed string oscillators satisfy the following commutation relations and hermiticity properties
\[ [\alpha^+_n, \alpha^-_m] = -(m + i\nu)\delta_{m+n}, \quad [\tilde{\alpha}^+_m, \tilde{\alpha}^-_n] = -(m - i\nu)\delta_{m+n}, \] (2.39)
\[ (\alpha^-_n)^* = \alpha^+_n, \quad (\tilde{\alpha}^-_n)^* = \tilde{\alpha}^+_n \] (2.40)
The left and right-moving world-sheet Hamiltonians are easily found to be
\[ L^c_{0} = -\frac{1}{2}(\alpha_0^+ \alpha_0^- + \alpha_0^- \alpha_0^+) - \sum_{n=1}^{\infty} (\alpha^+_n \alpha^-_n + \alpha^-_n \alpha^+_n) + \frac{1}{2}\nu^2 - \frac{1}{12} \] (2.41)
\[ \tilde{L}^c_{0} = -\frac{1}{2}(\tilde{\alpha}_0^+ \tilde{\alpha}_0^- + \tilde{\alpha}_0^- \tilde{\alpha}_0^+) - \sum_{n=1}^{\infty} (\tilde{\alpha}^+_n \tilde{\alpha}^-_n + \tilde{\alpha}^-_n \tilde{\alpha}^+_n) + \frac{1}{2}\nu^2 - \frac{1}{12} \] (2.42)
where we followed the same normal ordering prescription as in the open string case. Again, this prescription is needed to obtain non trivial physical states, and will be discussed at length in Section 3. We follow the same rule in displaying the generator of boosts,
\[ J = \frac{1}{\pi} \int_{0}^{2\pi} (X^+\partial_\tau X^- - X^-\partial_\tau X^+) = \frac{1}{2\nu} (\alpha_0^+ \alpha_0^- + \alpha_0^- \alpha_0^+) - \frac{1}{2\nu} (\tilde{\alpha}_0^+ \tilde{\alpha}_0^- + \tilde{\alpha}_0^- \tilde{\alpha}_0^+) \] 
\[ + \frac{1}{2\nu} \sum_{m=1}^{\infty} \left( \frac{\alpha^-_m \alpha^+_m - \tilde{\alpha}^-_m \tilde{\alpha}^+_m}{m + i\nu} + \frac{\tilde{\alpha}^-_m \tilde{\alpha}^+_m - \alpha^-_m \alpha^+_m}{m - i\nu} \right) \] (2.43)
Physical states satisfy \( L_0 = \tilde{L}_0 = 0 \). In contrast to the open string case, the orbifold projection demands that the total boost momentum \( J \) be an integer multiple of \( 2\pi/\beta \). In particular, transverse massless states necessarily have \( J = 0 \), which as we shall see in Section 4.2 localizes particle production to the light-cone.

### 2.3.2 Open vs closed strings

As a matter of fact, the open string mode expansion (2.17) may be obtained from (2.37) by identifying the left and right-moving oscillators as
\[ \alpha^\pm_n = (-)^n a^\pm_n e^{\pm \text{arctanh}(\pi e_0)} = -\tilde{\alpha}^\pm_{-n} \] (2.44)
and introducing an extra zero-mode \( x^\pm_0 \), so that
\[ X^\pm_{open} = x^\pm_0 + X^\pm_R(\tau - \sigma) + X^\pm_L(-\tau - \sigma) \] (2.45)
Despite the fact that the twisted closed string has twice as many excited modes as the charged open string, it is important to note that the zero-mode structures are in fact identical. The closed string possesses two pairs of level-zero oscillators \( \alpha_0^\pm \) and \( \tilde{\alpha}_0^\pm \), while the open string has two pairs of oscillators \( a_0^\pm \) and \( x_0^\pm \). The two satisfy the same algebra (2.8) upon identifying
\[
\alpha_0^\pm = a_0^\pm , \quad \tilde{\alpha}_0^\pm = \pm \sqrt{\nu(e_0 + e_1)} x_0^\pm .
\] (2.46)
They differ in the way they appear in the world-sheet Hamiltonian \( L_0 + \tilde{L}_0 \) however. Vertex operators at the zero-mode level can nevertheless be treated identically in the open and closed string case.

2.3.3 One-loop amplitude

The one-loop amplitude for the Milne orbifold has been computed in [28, 29], using an Euclidean world-sheet but a Lorentzian target space. In the bosonic string case, the result can be written as an integral over the fundamental domain of the modular group
\[
A_{bos} = \int_F \sum_{l,w=0}^\infty \frac{d\tau d\bar{\tau}}{(2\pi^2 \tau_2)^{13}} \left| \frac{\eta_{21}(\tau)}{\eta_{21}(i\beta(l + w\tau); \tau)} \right|^2 e^{-2\pi \beta^2 w^2 \tau_2}
\] (2.47)
whereas for the fermionic string,
\[
A_{ferm} = \int_F \sum_{l,w=0}^\infty \frac{d\tau d\bar{\tau}}{(2\pi^2 \tau_2)^{5}} \left| \frac{\theta_1^4(i\beta(l + w\tau)/2; \tau)}{\eta_9(\tau) \theta_1(i\beta(l + w\tau); \tau)} \right|^2
\] (2.48)
In either expression, the integers \( l, w \) denote the winding numbers along the two cycles of the torus. The partition functions (2.47) and (2.48) are indeed modular invariant, and agree with the standard quantization prescription based on a Fock vacuum annihilated by the oscillators \( \alpha_{m>0}^\pm, \tilde{\alpha}_{m>0}^\pm, a_0^0 \) and \( \tilde{\alpha}_0^+ \). In contrast to open string case however, the physical interpretation of the integrated amplitude is obscured by the existence of poles in the bulk of the fundamental domain. Indeed, the result (2.47) is very similar [29] to the one-loop amplitude for the Euclidean BTZ black hole [41]. In the BTZ case, the divergences were interpreted as contributions from continuous representations of affine \( \text{SL}(2)_k \), corresponding to long strings extending to the boundary of \( AdS_3 \). Whether long strings play a rôle in the Milne orbifold as well will be left to future work.

3. First quantization, revisited

As we have seen, both the charged open string and the twisted closed string in the Milne Universe behave like a charged massive particle in a constant electric field, from the point of view of their zero-mode degrees of freedom. In Section 3.1, we represent the zero-modes on the space of wave functions of the center of motion. In Section 3.2 we study in detail the wave functions of the charged Klein-Gordon equation, which provide the first approximation to the string vertex operators. We illustrate in Section 3.3 our prescription on the spectrum of low-lying physical states in the framework of old covariant quantization, and reconcile in Section 3.4 our quantization prescription with the one-loop amplitude.
3.1 Quantizing the zero-modes

3.1.1 Open strings in constant electric field

As we have seen, the charged open strings and twisted closed strings have isomorphic zero-modes structures. It is therefore important to have a complete physical understanding of these zero-modes. Let us therefore go back to the open string case

\[ [x_0^+, x_0^-] = -\frac{i}{e_0 + e_1}, \quad [a_0^+, a_0^-] = -i\nu \]  

(3.1)

with the two pairs of oscillators commuting. The relation of these operators to more familiar position and momenta may be obtained by looking at the limit of small electric field \( \nu \sim (e_0 + e_1) \to 0 \). The mode expansion is singular in this limit,

\[ X^\pm \rightarrow \left( x_0^\pm \pm \frac{a_0^\pm}{\nu} \right) + a_0^\pm \tau + \text{osc.} \]  

(3.2)

One should therefore identify the position and velocity operators as

\[ x^\pm = x_0^\pm \pm \frac{a_0^\pm}{\nu}, \quad p^\pm = a_0^\pm \]  

(3.3)

The commutation relations \((3.1)\) guarantee that they satisfy the correct commutation relation \([x^\mu, p^\nu] = i\eta^{\mu\nu}\) in the limit (where \(\eta^{+-} = -1\)). The velocity operators \(a_0^\pm = p^\pm\) may therefore be represented in the space of functions of the target space coordinates \(x^+, x^-\) as \(p^\pm \sim i\partial/\partial x^\pm\), up to corrections that vanish as \(\nu \to 0\). It turns out that \(a_0^\pm\) can be identified with the covariant derivatives in a constant electric field,

\[ a_0^\pm = p^\pm = i\partial_\mp \pm \frac{\nu}{2} x^\pm, \quad x_0^\pm = \mp \frac{1}{\sqrt{\nu(e_0 + e_1)}} \left( i\partial_\mp \mp \frac{\nu}{2} x^\pm \right) \]  

(3.4)

These operators are hermitian with respect to the \(L_2\) norm on target space \(\int d\bar{x}^+ d\bar{x}^- |f|^2\). The mass-shell condition \(L_0 = 0\) now becomes the Klein-Gordon equation for a charged particle in a constant electric field \(\nu\),

\[ L_0 = -\frac{1}{2} \left[ \left( i\partial_+ + \frac{\nu}{2} x^- \right) \left( i\partial_- - \frac{\nu}{2} x^+ \right) + \left( i\partial_- - \frac{\nu}{2} x^+ \right) \left( i\partial_+ + \frac{\nu}{2} x^- \right) \right] + \frac{1}{2} M^2 \equiv 0 \]  

(3.5)

where

\[ M^2 := a_0^+ a_0^- + a_0^- a_0^+ = -2 \sum_{m=1}^{\infty} (a_m^+ a_m^- + a_m^- a_m^+) + \nu^2 - \frac{1}{6} + 2L_0^{int} \]  

(3.6)

denotes the two-dimensional mass squared. The zero-modes \((a_0^\pm, x_0^\pm)\) therefore simply represent the degrees of freedom of the center of mass of the charged string. In this representation, the generator of boosts takes the simple form

\[ J = -\frac{i}{2} (x_+ \partial_+ - x_- \partial_-) + i \sum_{m=1}^{\infty} \left( \frac{a_m^- a_m^+}{m + \nu} - \frac{a_m^+ a_m^-}{m - \nu} \right) \]  

(3.7)
3.1.2 Closed strings in the Milne Universe

Similarly, the closed string zero-modes are described by a pair of hermitian canonical conjugate variables, satisfying

\[
\left[ \alpha^+_0, \alpha^-_0 \right] = -i \nu, \quad \left[ \tilde{\alpha}^+_0, \tilde{\alpha}^-_0 \right] = i \nu.
\] (3.8)

Their meaning may be understood in the flat space limit

\[
\nu \to 0, \quad x^+ \to -\frac{R}{\nu} + y^+, \quad x^- \to -\frac{R}{\nu} + y^-, \quad y^\pm \text{ finite}
\] (3.9)

where the Milne geometry reduces to a finite circle of constant radius \( R \), times the time direction. The string embedding coordinates reduce to

\[
X^\pm \to \pm \frac{\alpha^\pm_0 + \tilde{\alpha}^\pm_0}{\nu} \pm (\alpha^\pm_0 - \tilde{\alpha}^\pm_0) \tau - (\alpha^\pm_0 + \tilde{\alpha}^\pm_0) \sigma + \text{osc.}
\] (3.10)

so that \( \alpha^\pm_0, \tilde{\alpha}^\pm_0 \) are related to the closed string zero-modes in flat space \( y^\pm_0, p^\pm \) by the singular field redefinition,

\[
y^\pm_0 = \frac{1}{\nu} (\alpha^\pm_0 + \tilde{\alpha}^\pm_0 \pm R), \quad p^\pm = \pm (\alpha^\pm_0 - \tilde{\alpha}^\pm_0)
\] (3.11)

One may easily verify that the commutation relations (3.8) contract to the usual relations for the zero-modes on \( \mathbb{R} \times S^1 \), in the limit \( \nu \to 0 \) with fixed \( R \).

Thanks to the isomorphism (2.46), or on the basis of this identification, the same representation (3.4) in terms of differential operators acting on the space of wave functions \( \phi(x^+, x^-) \) can be used to describe the zero-mode degrees of freedom of the twisted closed string:

\[
\alpha^\pm_0 = i \partial_x \pm \frac{\nu}{2} x^\pm, \quad \tilde{\alpha}^\pm_0 = i \partial_x \mp \frac{\nu}{2} x^\pm
\] (3.12)

The zero-mode piece of \( L_0 \) and \( \tilde{L}_0 \)

\[
\mathcal{M}^2 := \alpha^+_0 \alpha^-_0 + \alpha^-_0 \alpha^+_0, \quad \tilde{\mathcal{M}}^2 := \tilde{\alpha}^+_0 \tilde{\alpha}^-_0 + \tilde{\alpha}^-_0 \tilde{\alpha}^+_0
\] (3.13)

are therefore the Klein-Gordon equation of a particle of mass squared and charge \( (\mathcal{M}^2, \nu) \) and \( (\tilde{\mathcal{M}}^2, -\nu) \), respectively, where

\[
\mathcal{M}^2 = -\sum_{n=1}^{\infty} (\alpha^+_n \alpha^-_n + \alpha^-_n \alpha^+_n) + \frac{1}{2} \nu^2 - \frac{1}{12} + L_{\text{int}}
\] (3.14)

\[
\tilde{\mathcal{M}}^2 = -\sum_{n=1}^{\infty} (\tilde{\alpha}^+_n \tilde{\alpha}^-_n + \tilde{\alpha}^-_n \tilde{\alpha}^+_n) + \frac{1}{2} \nu^2 - \frac{1}{12} + \tilde{L}_{\text{int}}
\] (3.15)

Their difference is equal to the zero-mode Rindler boost momentum \( j \),

\[
\mathcal{M}^2 - \tilde{\mathcal{M}}^2 = 2i \nu \left( x^+ \partial_+ - x^- \partial_- \right) = 2 \nu j
\] (3.16)
The level matching condition $L_0 - \tilde{L}_0 = 0$ relates the product of the boost momentum $J$ by the winding $w$ to the difference of occupation numbers of the excited levels,

$$\tilde{L}_0 - L_0 = \nu J + \sum_{m=1}^{\infty} \frac{m}{m + i\nu} (\alpha_m^+ \alpha^-_m - \tilde{\alpha}_m^+ \tilde{\alpha}^-_m) + \sum_{m=1}^{\infty} \frac{m}{m - i\nu} (\alpha^+_m \alpha^-_m - \tilde{\alpha}_m^+ \tilde{\alpha}_m^-)$$  \hspace{1cm} (3.17)

Notice that the right-hand side is integer valued as it should. The orbifold projection on the other hand requires that the total boost momentum $J$ be integer,

$$J = -i(x^+ \partial_+ - x^- \partial_-) + \nu \sum_{m=1}^{\infty} \left( \frac{\alpha_m^+ \alpha^-_m - \tilde{\alpha}_m^+ \tilde{\alpha}^-_m}{m + i\nu} + \frac{\tilde{\alpha}_m^+ \tilde{\alpha}_m^- - \alpha^+_m \alpha^-_m}{m - i\nu} \right) \in \frac{2\pi}{\beta} \mathbb{Z}$$  \hspace{1cm} (3.18)

We have therefore obtained our main insight into the dynamics of twisted closed strings in the Milne universe: from the point of view of their center of mass, they behave like particles of charge $w$ in a constant electric field $e$ related to the boost parameter $\beta$ by Eq. (2.35), with a further restriction on the boost momentum $J$ as specified by the matching relation.

### 3.2 Zero-mode wave functions in static coordinates

In this section, we quantize the charged Klein-Gordon equation (3.5) governing the zero-mode degree of freedom from the point of view of a static observer, who measures scattering processes with respect to the global time $X^0 = t$.

#### 3.2.1 The inverted harmonic oscillator

In order to go to static coordinates, we define linear combinations

$$a_0^\pm = \frac{P \pm Q}{\sqrt{2}}, \quad x_0^\pm = \frac{\hat{\mathcal{P}} \pm \hat{\mathcal{Q}}}{\sqrt{2\nu(e_0 + e_1)}}$$  \hspace{1cm} (3.19)

satisfying the canonical commutation relations

$$[P, Q] = i\nu, \quad [\hat{\mathcal{P}}, \hat{\mathcal{Q}}] = -i\nu$$  \hspace{1cm} (3.20)

Equivalently, using the representation of $a_0^\pm, x_0^\pm$ as covariant derivatives,

$$P = i\partial_t + \frac{\nu}{2}x, \quad Q = -i\partial_x + \frac{\nu}{2}t, \quad \hat{\mathcal{P}} = i\partial_t - \frac{\nu}{2}x, \quad \hat{\mathcal{Q}} = -i\partial_x - \frac{\nu}{2}t$$  \hspace{1cm} (3.21)

The charged Klein-Gordon operator can now be rewritten as an inverted harmonic oscillator

$$-M^2 = P^2 - Q^2$$  \hspace{1cm} (3.22)

Since the generator of spatial translations $\hat{\mathcal{P}}$ commutes with $M^2$, we may diagonalize it and further compute the wave functions in the $P$ representation. Since $P = \hat{\mathcal{P}} + \nu x$, we expand

$$f (x^+, x^-) = \int d\tilde{p} \psi_{\tilde{p}}(u)e^{-i(\tilde{p} \cdot \frac{\nu}{2}x + \frac{\nu}{2}t)}$$  \hspace{1cm} (3.23)
where we defined $u = (\tilde{p} + \nu x)\sqrt{2/\nu}$. The Klein-Gordon equation now takes the form of an inverted harmonic oscillator in one variable,

$$
\left(-\frac{\partial^2}{\partial u^2} - \frac{1}{4} u^2 + \frac{M^2}{2\nu}\right) \psi_p(u) = 0
$$

The physical interpretation of the motion in the inverted harmonic potential is now clear: particles coming from $u = +\infty$ in the inverted harmonic potential are just electrons coming from $x = +\infty$ and being slowed down by the electric field. At the turning point $u = M\sqrt{2/\nu}$ or $x = (M - \bar{p})/\nu$, they bounce against the potential barrier and escape to $x = +\infty$ again. In real space, the trajectory consists of a branch of an hyperbola centered at $x = -\tilde{p}/\nu$. Similarly, particles coming from $u = -\infty$ are positrons which follow the other branch of the hyperbola.

Quantum mechanically, there therefore exists a continuum of delta-function normalizable states with real positive $M^2$, in stark contrast with expectations based on analytic continuation from the Euclidean magnetic problem. Furthermore, the quantum mechanical tunnelling through the barrier can be simply interpreted as induced Schwinger pair production $e^- \rightarrow (1 + \eta)e^- + \eta e^+$. We will compute the production rate $\eta$ in the next subsection. Notice that the fact that the potential is unbounded from below does not cause any instability, as the energy is fixed equal to $M^2/(2\nu)$. On the other hand, the Schwinger pair production will be responsible for a dynamical decay of the electric field, as the charged particles move off to infinity and screen the electric field.

### 3.2.2 Parabolic cylinder functions

Eigenmodes of the inverted harmonic oscillator are well known under the name of parabolic cylinder functions. They already arose in the context of the $c = 1$ string theory [42], from where we borrow the following results: even and odd solutions of (3.24) under the parity transformation $u \rightarrow -u$ are given by

$$
\psi^+(u) = \frac{2^{1/4}}{\sqrt{4\pi(1 + e^{\pi M^2/\nu})^{1/2}}} \left| \frac{\Gamma\left(\frac{1}{4} + i\frac{M^2}{4\nu}\right)}{\Gamma\left(\frac{1}{4} + i\frac{M^2}{4\nu}\right)} \right| \frac{1}{2} e^{-iu^2/4} \mathbf{F}_1(1, \frac{1}{4} - i\frac{M^2}{4\nu}, 1; \frac{u^2}{2})
$$

$$
\psi^-(u) = \frac{2^{3/4}}{\sqrt{4\pi(1 + e^{\pi M^2/\nu})^{1/2}}} \left| \frac{\Gamma\left(\frac{3}{4} + i\frac{M^2}{4\nu}\right)}{\Gamma\left(\frac{1}{4} + i\frac{M^2}{4\nu}\right)} \right| \frac{1}{2} u e^{-iu^2/4} \mathbf{F}_1(1, \frac{3}{4} - i\frac{M^2}{4\nu}, 3; \frac{u^2}{2})
$$

These form a delta-normalizable complete orthogonal basis of functions on $\mathbb{R}$, normalized such that

$$
\sum_{\epsilon = \pm} \int_{-\infty}^{\infty} \frac{dM^2}{2\nu} \psi^\epsilon(M^2, u_1)\psi^\epsilon(M^2, u_2) = \delta(u_1 - u_2)
$$

$$
\sum_{\epsilon = \pm} \int_{-\infty}^{\infty} du \psi^\epsilon(M_1^2, u)\psi^\epsilon(M_2^2, u) = 2\nu\delta(M_1^2 - M_2^2)
$$

For large positive $u \gg M/\sqrt{\nu}$, they satisfy the BKW asymptotics

$$
\psi^\pm(u) \sim \left(2\pi u \sqrt{1 + e^{\pi M^2/\nu}}\right)^{-1/2} \left[ \sqrt{k} \cos \phi(u) \pm \frac{1}{\sqrt{k}} \sin \phi(u) \right],
$$

where $k = \nu u^2$.
where
\[ \phi(u) = \frac{1}{4} u^2 - \frac{M^2}{\nu} \log |u| + \frac{\pi}{4} + \frac{1}{2} \arg \Gamma \left( \frac{1}{2} + i \frac{M^2}{2\nu} \right) \] (3.28)
and \( k = \sqrt{1 + e^{\pi M^2/\nu} - e^{\pi M^2/\nu}} \). In particular they fall off at infinity as \( 1/\sqrt{|u|} \).

3.2.3 In and out states

It is now possible to construct plane-wave combinations, such that only incoming or outgoing waves are present at \( u = \pm \infty \):
\[ \psi^\pm_R(u) = \frac{1}{2} \left( k^{1/2} \pm ik^{1/2} \right) \psi^+ + \frac{1}{2} \left( k^{-1/2} \mp ik^{-1/2} \right) \psi^- \] (3.29)
\[ \psi^\pm_L(u) = \frac{1}{2} \left( k^{1/2} \pm ik^{1/2} \right) \psi^+ - \frac{1}{2} \left( k^{-1/2} \mp ik^{-1/2} \right) \psi^- \] (3.30)
such that
\[ \psi^\pm_R(u) \sim \left( 2\pi u \sqrt{1 + e^{\pi M^2/\nu}} \right)^{1/2} e^{\pm i\phi(u)}, \quad u \to +\infty \] (3.31)
\[ \psi^\pm_L(u) \sim \left( -2\pi u \sqrt{1 + e^{\pi M^2/\nu}} \right)^{1/2} e^{\pm i\phi(u)}, \quad u \to -\infty \] (3.32)

These combinations can in fact be written directly in terms of parabolic cylinder functions \( D_{1/2 \pm iM^2/2\nu}(\pm e^{\pi M^2/\nu} u) \) (see Appendix C). Using these modes we may now construct solutions of the Klein-Gordon equation corresponding to the creation of an electron or the annihilation of a positron at \( t = -\infty \),
\[ \phi_e^{\text{in}} = \psi^-_R(u)e^{-i(\tilde{p} + \frac{1}{4}\nu x)t}, \quad \phi_p^{\text{in}} = \psi^+_L(u)e^{-i(\tilde{p} + \frac{1}{4}\nu x)t} \] (3.33)
or at \( t = +\infty \):
\[ \phi_e^{\text{out}} = \psi^+_R(u)e^{-i(\tilde{p} + \frac{1}{4}\nu x)t}, \quad \phi_p^{\text{out}} = \psi^-_L(u)e^{-i(\tilde{p} + \frac{1}{4}\nu x)t} \] (3.34)
The semi-classical behaviour of these wave functions is summarized on Figure 3. The reflection and transmission coefficients of the wave function amplitude are easily computed by looking at the semi-classical expansion in the region \( u \to -\infty \),
\[ \left( \frac{\psi^+_R}{\psi^-_R} \right)(u) \sim \frac{i}{2} \left( -2\pi u \sqrt{1 + e^{\pi M^2/\nu}} \right)^{1/2} \begin{pmatrix} k - \frac{1}{k} & k + \frac{1}{k} \\ -k - \frac{1}{k} & -k + \frac{1}{k} \end{pmatrix} \begin{pmatrix} e^{i\phi(u)} \\ e^{-i\phi(u)} \end{pmatrix} \] (3.35)
which yield
\[ R = \frac{k^{-1} + k}{k^{-1} - k} = \frac{1}{\sqrt{1 + e^{-\pi M^2/\nu}}}, \quad T = \frac{2}{k - k^{-1}} = \frac{1}{\sqrt{1 + e^{\pi M^2/\nu}}} \] (3.36)
The fact that the modulus of the reflection coefficient is greater than one is a manifestation of the Klein paradox, due to the unboundedness of the potential. As usual, the resolution of the paradox is that the charge density \( q = u|\psi(u)|^2 \) is proportional to the modulus square of the amplitude, and conserved during the stimulated pair production process.
\( e^- \rightarrow R^2 e^- + T^2 e^+ \) thanks to the relation \( R^2 = 1 + T^2 \). The stimulated pair creation rate \( \eta \) introduced at the end of Section 3.2.1 is therefore equal to \( \eta = R^2 - 1 \).

A general solution of the Klein-Gordon equation may therefore be expanded in two different ways,

\[
\phi = \int d\tilde{p} \left( a_{\tilde{p}}^{*} \phi^\text{in}_{\tilde{p}} + b_{\tilde{p}} \phi^{\text{p, in}}_{\tilde{p}} \right) = \int d\tilde{p} \left( a_{\tilde{p}}^{\text{out}*} \phi^\text{out}_{\tilde{p}} + b_{\tilde{p}}^{\text{out}} \phi^{\text{p, out}}_{\tilde{p}} \right) \tag{3.37}
\]

Here the operator \( a^{*} \) creates an electron, whereas \( b^{*} \) creates a positron. The commutation relations read, either in the \( \text{in} \) or \( \text{out} \) basis,

\[
[a_{\tilde{p}}, a_{\tilde{p}'}^{*}] = [b_{\tilde{p}}, b_{\tilde{p}'}^{*}] = \delta(\tilde{p} - \tilde{p}') \tag{3.38}
\]

The non-trivial Bogolioubov transformation between these two bases is a consequence of pair production in an electric field. Using the identities (C.5), we obtain

\[
|0, \text{in}\rangle = N e^{\gamma} \int d\tilde{p} a^{\text{out}*}_{\tilde{p}} b^{\text{out}*}_{\tilde{p}} |0, \text{out}\rangle \tag{3.39}
\]

where \( \gamma, \delta \) are the Bogolioubov coefficients,

\[
\gamma = \frac{\sqrt{2\pi}}{\Gamma \left( \frac{1}{2} + i \frac{M^2}{2\nu} \right)} e^{-\frac{\pi M^2}{4\nu} - i\frac{\pi}{4}} , \quad \delta = e^{-\frac{\pi M^2}{2\nu}} e^{i\frac{\pi}{4}} \tag{3.40}
\]

The vacuum persistence amplitude is therefore given by the overlap

\[
|\langle 0, \text{in} | 0, \text{out} \rangle|^2 = \exp \left( - \int d\tilde{p} \ \ln\left( 1 + e^{-\pi M^2 / \nu} \right) \right) \tag{3.41}
\]

This agrees with Schwinger creation rate in two dimensions (2.11), upon appropriately interpreting the volume divergence in the integral over \( \tilde{p} \).
3.3 Physical states at level 0 and 1

We now determine the low-lying physical states of the bosonic open string in an electric field, in the framework of old covariant quantization, which is sufficient for our purposes. We represent the excited modes in a Fock space built on a vacuum $|0_{ex}\rangle$ annihilated by the positive frequency modes $a^+_{m>0}$. Zero-modes in the $(x^+, x^-)$ directions are quantized using the inverted harmonic oscillator scattering modes described in the previous section. In the transverse directions we assume a plane wave with momentum $a^i_0 = k_i$.

3.3.1 The tachyon

Let us start with the ground state of the bosonic open string, which for $\nu = 0$ corresponds to a tachyon:

$$|T\rangle = \phi(x^+, x^-)|0_{ex}, k\rangle$$ (3.42)

The only non-trivial Virasoro constraint is

$$L_0|T\rangle = \left[-\frac{1}{2} (a^+_0 a^-_0 + a^-_0 a^+_0 ) + \frac{1}{2} \nu^2 - 1 + \frac{1}{2} k_i^2 \right] |T\rangle = 0$$ (3.43)

The tachyon wave function is therefore an eigenmode of the two-dimensional charged Klein-Gordon equation, with two-dimensional mass

$$M^2 \phi := (a^+_0 a^-_0 + a^-_0 a^+_0 ) \phi = (k_i^2 + \nu^2 - 2) \phi$$ (3.44)

3.3.2 The gauge boson

Now we turn to the first excited level, which for $\nu = 0$ corresponds to a gauge boson. A general state in this level may be written as

$$|A\rangle = (- f^+ a^-_{-1} - f^- a^+_{-1} + f^i a^i_{-1}) |0_{ex}, k\rangle$$ (3.45)

where we understood the dependence of $f^{\pm,0}$ on the coordinates $x^+, x^-$. The mass shell condition $L_0 |A\rangle = 0$ requires

$$[M^2 - k_i^2 - \nu^2 \mp 2i\nu] f^\pm = 0, \quad [M^2 - k_i^2 - \nu^2] f^i = 0.$$ (3.46)

In addition the Virasoro constraint $L_1 |A\rangle = 0$ implies

$$-(1 + i\nu) a^-_0 f^+ - (1 - i\nu) a^+_0 f^- + a^0_0 f^i = 0$$ (3.47)

States should be furthermore identified under variation by a spurious state,

$$\delta |A\rangle = L_{-1} \phi |0_{ex}\rangle = (- a^+_{-1} a^-_0 - a^-_{-1} a^+_0 + a^i_{-1} a^i_0) \phi |0_{ex}, k\rangle$$ (3.48)

under the condition that the right-hand side still be a physical state. The $L_0$ and $L_1$ constraints on $\delta |A\rangle$ demands that

$$[-(1 + i\nu) a^-_0 a^+_0 - (1 - i\nu) a^+_0 a^-_0 + a^i_0 a^i_0] \phi = 0$$ (3.49)

$$[M^2 - k_i^2 - \nu^2] \phi = 0$$ (3.50)
These two equations are in fact equivalent, thanks to a fortunate conspiracy with the value of the normal ordering constant in (2.41). The spurious state is therefore indeed a physical state, hence we should therefore identify

\[(f^+, f^-, f^i) \equiv (f^+ + a^+_0 \phi, f^- + a^-_0 \phi, f^i + a^i_0 \phi)\]  

(3.51)

This gauge symmetry may be fixed by choosing

\[f^+ = a^+_0 \psi, \quad f^- = a^-_0 \psi \text{ with } [M^2 - k_i^2 - \nu^2] \psi = 0\]  

(3.52)

The \(L_1\) constraint allows one to express \(\psi\) in terms of the transverse degrees of freedom, through

\[-k_i^2 \psi + k_i f^i = 0\]  

(3.53)

Despite the gap \(\nu^2\) in the two-dimensional mass squared, the first excited level therefore has \(D - 2\) transverse degrees of freedom, as required for a massless gauge boson in \(D\) dimensions. This is satisfying as we do not expect the number of degrees of freedom to changed as an electric field background is turned on. In particular, we see that there are no ghosts up to level 1.

### 3.3.3 Closed string states

Transposing to closed strings in the Milne orbifold, the same construction can be applied on the left and right movers separately. The sole effect of the orbifold projection is to restrict the zero-mode wave function to have integer two-dimensional boost-momentum \(\beta J/(2\pi)\), determined by the level-matching condition \(\text{(3.17)}\). In order to enforce this condition it is convenient to go to a basis where \(J\) is diagonal, as we shall proceed to do in Section 4. For now, we simply notice that this construction implies the existence of physical states of the bosonic string in each twisted sector consisting of a scalar tachyon at level \((0,0)\) and a massless graviton with \((D-2)^2\) transverse degrees of freedom at level \((1,1)\), both restricted to having zero boost momentum \(j\). In addition, there are transverse gauge bosons at level \((0,1)\) and \((1,0)\), with boost momentum \(\pm 2\pi/(\omega \beta)\). In the type II string, the tachyon and gauge bosons are projected out, while the Ramond sectors contribute further states.

### 3.4 One-loop amplitude and Lorentzian physical states

We now return to Euclidean one-loop vacuum free energy Eq. (2.24). Usually, knowledge of the partition function determines the spectrum, and indeed this intuition led to the discrete imaginary spectrum discussed in section 2.2.1. Nevertheless, we now show that the same partition function is consistent with an altogether different spectrum in Minkowski space.

To simplify the discussion, we focus on the zero-mode sector, where the two prescrip-
tions differ. We may thus restrict our attention to the factor \(1/ \sin(\pi \nu t)\) in the Jacobi theta function in (2.23), which summarizes the contributions of the zero-modes, and should be obtainable purely in quantum field theory. Next we go to a Minkowski world-sheet, by rotating the Schwinger parameter \(t \rightarrow it\) in (2.24) (recall that the expression (2.24) assumed a Lorentzian target space). This converts the zero-mode contribution into \(1/ \sinh(\pi \nu \theta)\), which is the expression we now would like to explain.
As is well known (see e.g. [44], section 4-3-3), the one-loop amplitude of a charged scalar field in a constant electric field may be rewritten as the trace of the heat kernel for the Klein-Gordon operator,

$$A_{1-loop} = i \int \frac{ds}{s} e^{-is(m^2 - i\epsilon)} \text{Tr} e^{is\Delta_E}$$  \hspace{1cm} (3.54)

where $\Delta_E$ is the electric field Klein-Gordon operator. The main point of this subsection, and the reason why we have the freedom to change the quantization scheme in Minkowski space, is that the propagator of the charged scalar in an electric field has two equivalent representations - one as a sum over discrete states with an imaginary energy and the other as a sum over a continuum of Minkowski modes. Both are therefore consistent with the form of the Minkowski one loop partition function.

The expression as discrete sum over states with imaginary energy is inherited from the Euclidean continuation (as already previewed in section 2.2.3, and further discussed in section 5). We start with the propagator for the harmonic oscillator $\mathcal{H} = -\partial^2 u + u^2/4$ evaluated in terms of the discrete spectrum,

$$\langle u_1 | e^{-2isH} | u_2 \rangle = \sum_{n=0}^{\infty} e^{i(n+1/2)s} \psi_n^*(u_1) \psi_n(u_2) = \frac{1}{\sqrt{4\pi s \sin s}} \exp \left[ -\frac{1}{4} \left( \frac{u_1^2 + u_2^2}{\tan s} - \frac{2u_1u_2}{\sin s} \right) \right]$$  \hspace{1cm} (3.55)

The one-loop free energy in an Euclidean magnetic field is given by restricting to coinciding points $u_1 = u_2 = u$ and integrating over $u$. One obtains:

$$\langle x | e^{-\Delta_B s} | x \rangle = \int_{-\infty}^{\infty} d\tilde{p} (\sqrt{2/B} \tilde{p} + Bx) | e^{-sH_m} | \sqrt{2/B} (\tilde{p} + Bx) \rangle = \frac{1}{\sinh(\pi Bs)}$$  \hspace{1cm} (3.56)

To go to an electric field in Minkowski space we rotate both $B \to i\nu$ and $s \to it$, hence reproducing the above factor $1/\sinh(\pi \nu t)$.

However, instead of computing the propagator of the inverted harmonic oscillator by analytic continuation, the same result may be obtained directly from the continuous spectrum. Indeed, the continuous spectrum of the inverted harmonic oscillator satisfies the completeness relation (see e.g. [42], eq A 12):

$$\int_{-\infty}^{\infty} dM^2 \frac{e^{iM^2 s}}{2\nu} \psi_n^*(M^2, u_1) \psi_n(M^2, u_2) = \frac{1}{\sqrt{4\pi i \sinh s}} \exp \left[ i \left( \frac{u_1^2 + u_2^2}{\tanh s} - \frac{2u_1u_2}{\sinh s} \right) \right] = \langle u_1 | e^{-2isH} | u_2 \rangle$$  \hspace{1cm} (3.57)

valid for $-\pi < \Im(s) < 0$. The same computation as above therefore leads to the expression $1/\sinh(\pi \nu s)$, without the need to analytically either $s$ or $\nu$.

To put it otherwise, the contribution of the zero-modes to the one-loop amplitude may be interpreted either as a sum over the discrete Euclidean spectrum, or as an integral over the continuous Lorentzian spectrum,

$$\frac{1}{2i \sin(\nu t/2)} = \sum_{n=1}^{\infty} e^{-i(n+\frac{1}{2})\nu t} = \int dM^2 \rho(M^2) e^{-M^2 t/2}$$  \hspace{1cm} (3.58)
where the density of states of the continuous spectrum follows in the usual manner from the reflection phase shift (3.27),

$$\rho(M^2) = \frac{1}{\nu} \log \Lambda - \frac{1}{2\pi i} \frac{d}{dM^2} \log \frac{\Gamma\left(\frac{1}{2} + i \frac{M^2}{2\nu}\right)}{\Gamma\left(\frac{1}{2} - i \frac{M^2}{2\nu}\right)}$$ (3.60)

where $\Lambda$ is an infrared cut-off.

This argument confirms the validity of our prescription for the electric case. The one-loop closed string amplitude (2.47) in the Lorentzian orbifold case does not seem to be obtainable by a similar heat-kernel argument. The two systems, however, are markedly similar and we shall assume that the physical picture we developed for charged strings carries over to the twisted closed string case (leaving the physical interpretation of the poles encountered in the definition of (2.47) to future work).

4. Strings in Rindler space and the Milne orbifold

As we have seen, the quantization of the twisted closed string zero-modes in the Milne orbifold proceeds identically with that of the charged open string in a constant electric field, only with a further projection on states with integer boost momentum $\beta J/(2\pi)$. It is thus important to determine the wave functions in Rindler coordinates, where the zero-mode boost momentum $j$ is diagonalized and the orbifold projection most easy to enforce.

4.1 Open strings in Rindler space coordinates

Quantization of charged particles in Rindler space was discussed extensively in [45]. We start by working in the right Rindler wedge, where $X^\pm = \pm re^{\mp\eta}/\sqrt{2}$ is a good coordinate system. Since the zero-mode boost momentum (or Rindler energy) $j$ commutes with $M^2$, we will focus on states with a well-defined momentum $j$:

$$f_j(r,\eta) = e^{-ij\eta}f_j(r)$$ (4.1)

4.1.1 Radial dynamics

The charged Klein-Gordon equation now implies the following second order ODE for the radial wave function,

$$\left[-r\partial_r r\partial_r + M^2 r^2 - (j + \frac{1}{2}\nu r^2)^2\right]f_j(r) = 0$$ (4.2)

In terms of the coordinate $y = \ln r$, this can be viewed as the Schrödinger equation for a particle in the potential

$$V(y) = M^2 e^{2y} - (j + \frac{1}{2}\nu e^{2y})^2$$ (4.3)

---

5We became aware of this work after completing a significant part of the analysis. Early references include [13], and other unfortunate rediscoverers are [14].
which controls the motion along the radial direction as a function of Rindler time,

\[
\left(\frac{dr}{d\eta}\right)^2 + V(r) = 0 \quad \text{or} \quad \left(\frac{dy}{d\eta}\right)^2 + V(y) = 0 \tag{4.4}
\]

In the absence of an electric field, the potential (4.3) looks like a Liouville barrier, so that all particles coming from \(r = 0\) \((y = -\infty)\) bounce back at the turning point \(r = M/|j|\). This corresponds to the fact that the classical trajectories are time-like straight lines in the \((x,t)\) plane, and therefore reach a finite Rindler radius. For \(j = 0\) the trajectories go through the origin.

For non-vanishing \(\nu\), say \(\nu > 0\), the situation is more interesting (see Figure 4). At large radius, the potential is now unbounded from below, corresponding to the fact that states with \(M^2 \neq 0\) may now reach \(r = \infty\). If \(\nu j/M^2 > 1\), the potential decreases monotonically from \(V = -j^2\) at \(r = 0\) \((y = -\infty)\). If \(1 > \nu j/M^2 > 1/2\), there is a bump at \(r = (M/\nu)\sqrt{2(1 - \nu j/M^2)}\), but the energy of the particle is greater than the height of the barrier \(V_{\text{max}} = (M^4/\nu^2)(1 - 2\nu j/M^2)\). Finally, if \(1/2 > \nu j/M^2\), the particle bounces off the barrier, at the turning points

\[
r_{\pm} = \left\{ \begin{array}{ll}
\frac{M}{\nu} \left( 1 + \sqrt{1 - 2\nu j/M^2} \right) & , \ j > 0 \\
\frac{M}{\nu} \left( 1 - \sqrt{1 - 2\nu j/M^2} \right) & , \ j < 0 
\end{array} \right. \tag{4.5}
\]

These are precisely the extremal values of the Rindler radius \(r\) evaluated along the classical trajectory (2.2),

\[
r^2 = -2X^+ X^- = -2x_0^+ x_0^- + \frac{M^2}{\nu^2} - 2\frac{a_0^+ x_0^-}{\nu} e^{\nu\tau/m} + 2\frac{a_0^- x_0^+}{\nu} e^{-\nu\tau/m} \tag{4.6}
\]

where we recall that the boost momentum is related to the position of the center of the classical trajectory through (2.9). A quantum state of definite \(j\) will therefore involve a superposition of many classical trajectories with a fixed value of

\[
x_0^+ x_0^- = \frac{M^2}{\nu^2} \left[ \frac{\nu j}{M^2} - \frac{1}{2} \right] \tag{4.7}
\]

Their behaviour can be followed on Figure 4, where we plotted two such trajectories with opposite values of \((x_0^+, x_0^-)\) for representative values of the dimensionless parameter \(\nu j/M^2\):

- If \(\nu j/M^2 < 0\) (Figure 4, top), the electron comes from \(r = \infty\) in the R patch, reaches a minimal Rindler radius \(r_+\), and bounces towards \(r = \infty\) again. The positron on the other hand comes from the past region \(P\), crosses the past horizon \(r = 0\) of the \(R\) patch \(H_R^-\), reaches a maximal radius \(r = r_-\), and falls through the future horizon of the \(R\) patch \(H_R^+\) into the future region \(F\). From the point of view of the radial potential in the \(R\) region (4.3), they correspond to a particle coming from \(y = +\infty\) (resp. \(y = -\infty\)) and bouncing off the potential barrier back to \(y = +\infty\) (resp. \(y = -\infty\)). Quantum mechanically, the particle may tunnel under the barrier and jump from the electron to the positron branch: as in static coordinates, this corresponds to stimulated Schwinger emission.

\(^6\)More precisely, the wave function describes the time-reverse of this process.
Figure 4: Classical trajectories in real space and radial dynamics in the $R$ region. Here $j$ is measured in units of $M^2/\nu$.

- If $0 < \nu j/M^2 < 1/2$ (Figure 4, middle), the positron branch now lies entirely outside the $R$ region. The electron branch of the symmetric hyperbola centered at $-(x_0^+, x_0^-)$ however does cross the $R$ region, entering from the past horizon $H_R^-$ and falling back
into the future horizon \( H_R^+ \). It corresponds again to a particle coming from \( y = -\infty \) in the potential \( V(y) \), and bouncing off the barrier. Quantum mechanically, the particle may again tunnel under the barrier and jump from the electron branch of one hyperbola to the electron branch of the other: this teleportation process is the analog of usual Hawking-Unruh pair creation for neutral particles, and proceeds by nucleation of electron-positron pairs in the middle.

- Finally, if \( \nu j/M^2 > 1/2 \) (Figure 4, bottom), the center of the hyperbola lies inside the past or forward region. Consequently, all particles either enter from the past horizon \( H_R^- \) and attain future infinity \( I^+_R \), or enter from past infinity \( I_R^- \) and fall into the future horizon \( H_R^+ \). This corresponds to the fact that the energy of the particle in the radial potential \( V(y) \) is sufficient to cross the barrier. Even so, part of the wave function will be reflected quantum mechanically, so that an electron coming from \( H_R^- \) may jump to the other hyperbola and exit the \( R \) patch through the horizon \( H_R^+ \) instead of reaching \( I_R^+ \). This process is again the counterpart of Hawking-Unruh pair creation for neutral particles.

From this, we can infer the semi-classical behaviour of the wave functions in each of the Rindler quadrants, displayed in Figure 3. Notice that the sign of the charges on each of the asymptotic regions \( I^\pm_{L,R} \) is fixed, while the sign of the charges on the horizons \( H_{L,R} \) depends on the sign of \( j \).

4.1.2 Normal modes in Rindler wedges

We may now construct superpositions of wave functions which have only incoming or outgoing components on either the horizon or the asymptotic region, in analogy with the usual modes for a neutral field in Rindler space [48]. In the R quadrant, incoming modes from \( I_R^- \) can be written in terms of Whittaker functions as [45]

\[
V_{in,R}^j = e^{-ij\eta r}M_{-i(\frac{j}{2} - \frac{M^2}{2\nu}), -\frac{j}{2}}(i\nu r^2/2)
\]

while incoming modes from \( H_R^- \) are

\[
U_{in,R}^j = e^{-ij\eta r}W_{-i(\frac{j}{2} - \frac{M^2}{2\nu}), -\frac{j}{2}}(-i\nu r^2/2)
\]

The first wave function \( V_{in,R}^j \) represents an electron of charge 1 coming in from \( I_R^- \), being deflected by the electric field and exiting at \( I_R^+ \); due to pair production, the outgoing charge \( q_1 \) at \( I_R^+ \) is greater than the incoming one, and accordingly there is also an outgoing charge \( \text{sgn}(j)q_2 \) at the future horizon \( H_R^+ \). The transmission and reflection coefficients have been calculated in [45], and read\(^7\)

\[
q_1 = e^{-\pi j} \frac{\cosh \left[ \frac{\pi M^2}{2\nu} \right]}{\cosh \left[ \pi \left( j - \frac{M^2}{2\nu} \right) \right]}, \quad q_2 = e^{-\pi \frac{jM^2}{2\nu}} \frac{|\sinh |j|\pi|}{\cosh \left[ \frac{\pi}{2} \left( j - \frac{M^2}{2\nu} \right) \right]}
\]

\(^7\)An absolute value is missing in [45], eq (4.14).
Figure 5: Classical trajectories in the four quadrants of Minkowski space.

Note that these two quantities are positive, and satisfy $q_1 + \text{sgn}(j)q_2 = 1$ by charge conservation. In contrast, $U^j_{in,R}$ describes a particle of charge $\text{sgn}(j)$ coming in from the past horizon $H^-_R$, and disappearing into the future horizon $H^+_R$. Due to particle creation, the outgoing charge at $H^+_R$ is $\text{sgn}(j)q_1$, while there are also $q_2$ electrons coming out at $I^+_R$. Using these two sets of modes, one can therefore expand

$$f(x^+,x^-) = \int_{-\infty}^{\infty} dj \left[ a^{in}_{\nu_R^+}(j) \nu_R^+ + \theta(j) a^{in}_{\nu_R^-}(j) + \theta(-j) b^{in}_{\nu_R^-}(j) \right] U^j_R$$

where $\theta(j)$ is the Heaviside function, and the oscillators $a$ and $b$ have canonical commutation relations. Outgoing modes $U^j_{out,R}$ and $V^j_{out,R}$ may be defined similarly by requiring that $U^j_{out,R}$ has no component on $H^+_R$, and $V^j_{out,R}$ has no component on $I^+_R$. Equivalently, they may be obtained by charge conjugation,

$$U^j_{out,R}(r,\eta) = [V^j_{in,R}(r,-\eta)]^*, \quad V^j_{out,R}(r,\eta) = [U^j_{in,R}(r,-\eta)]^*$$

Their transmission and reflection coefficients are still given in absolute value by $(q_1, q_2)$, with signs determined by the semi-classical analysis above. The Bogolioubov transformation relating the incoming and outcoming classical basis can be found in [45]. We quote in particular
the overlap

$$|R(0, in|0, out)R|^2 = \exp \left[-\int_{j<0} dj \ln \left(\frac{1 + e^{-\pi M^2/\nu}}{1 + e^{-\pi M^2/\nu + 2\pi j}}\right)\right]$$  \hspace{1cm} (4.13)$$

By extracting carefully the volume dependence, the numerator can be seen to reproduce the standard Schwinger creation rate, while the denominator is a subtraction which scales as the area, and corresponds to Unruh particle production from the horizon.

The same construction can be carried out in the other wedges. The $L$ wedge is essentially identical to the $R$ wedge, with same reflection coefficients ($q_1, q_2$). For the $P$ and $F$ wedges however, the transmission and reflection coefficients are instead given by

$$q_3 = e^{\pi (j - \frac{M^2}{2\nu}) \cosh \left[\frac{\pi M^2}{2\nu}\right]} \left|\sinh \pi j\right|, \quad q_4 = e^{-\pi \frac{M^2}{2\nu} \cosh \left[\frac{\pi (j - M^2/2\nu)}{2\nu}\right]} = q_3 - 1$$  \hspace{1cm} (4.14)$$

In our conventions, $q_3$ and $q_4$ are positive, and occur with signs determined by the semiclassical analysis in Figure 6.

![Figure 6: Transmission and reflection coefficients in the four quadrants of Minkowski space. The lower (upper) sign corresponds to $j < 0$ ($j > 0$).](image)

4.1.3 Unruh modes

Having determined the normal modes on each of the quadrants, it is a simple matter to fit them into modes on the full Minkowski plane, without singularities at the horizons: it suffices to start from the modes in one of the Rindler patches, say $P$,

$$\Omega_{in,+}^j = \mathcal{V}_{in,P}^j = W_{-i\left(\frac{\pi M^2}{2\nu}\right)}^{j/2} \left(-i\nu X^+ X^\dagger\right)^{-ij/2}$$

$$\Omega_{in,-}^j = \mathcal{U}_{in,P}^j = M_{i\left(\frac{\pi M^2}{2\nu}\right)}^{j/2} \left(i\nu X^+ X^\dagger\right)^{-ij/2}$$  \hspace{1cm} (4.15)$$

and analytically continue across the horizons. The result can be decomposed in terms of the Rindler modes on each patch,

$$\Omega_{in,+}^j = \mathcal{V}_{in,P}^j + e^{-i\psi}\sqrt{q_3} \mathcal{U}_{in,R}^j - i\sqrt{q_1} \mathcal{V}_{in,L}^j - ie^{-i\psi}\sqrt{q_1} \mathcal{U}_{out,F}^j$$

$$\Omega_{in,-}^j = \mathcal{U}_{in,P}^j + i\sqrt{q_4} \mathcal{U}_{in,R}^j + e^{i\psi}\sqrt{q_3} \mathcal{V}_{in,L}^j - ie^{-i\psi}\sqrt{q_1} \mathcal{V}_{out,F}^j$$  \hspace{1cm} (4.16)$$

which create or destroy particles of charge $\pm$ in the past patch $P$. In addition, there exists modes that vanish identically in the past, but can be obtained by analytic continuation of

---

8The reflection coefficients displayed in [6], eq. (6.14) and (6.16) are incorrect. To ease the notation, the Rindler modes are tacitly assumed to be truncated to their respective quadrants.
modes in the future region,

\[
\omega_{in,-} = e^{i\psi \text{sgn}(j)} \left[ V_{in,R}^j + i \sqrt{q_2} e^{i\psi'} V_{in,F}^j \right]
\]
(4.19)

\[
\omega_{in,+} = e^{-i\psi \text{sgn}(j)} \left[ U_{in,L}^j - i \sqrt{q_2} e^{-i\psi'} U_{in,F}^j \right]
\]
(4.20)

These modes create or destroy particles of charge \(\pm\) in the left or right whisker (see Figure 7). In these expressions, the reflection phases are given by

\[
\psi = \text{arg} \left\{ \frac{\Gamma \left[ i j \right]}{\Gamma \left[ \frac{1}{2} + i \left( j + \frac{M^2}{2\nu} \right) \right]} \right\}, \quad \psi' = \text{arg} \left\{ \frac{\Gamma \left[ \frac{1}{2} + i \left( j - \frac{M^2}{2\nu} \right) \right]}{\Gamma \left[ \frac{1}{2} + i \frac{M^2}{2\nu} \right]} \right\},
\]
(4.21)

Figure 7: Incoming Unruh modes \(\Omega_{in,-}^j\) (left) and \(\omega_{in,-}^j\) (right). Red lines denote tunnelling events.

In the case of the \(\Omega_{in,\pm}\) modes, one may see that charge conservation implies that \(q_1\) particles are produced in the forward region, of the same type as the incoming particle. This process involves four tunnelling events. For the \(\omega_{in,\pm}\) modes, which only involves two tunnelling events, particle creation can only take place to the future of \(L\), and so \(\omega_{in,\pm}\) vanish on \(P\) and on the other side of the whisker. These are the analogue to the Unruh modes in the neutral case [49]. Particles of either sign are produced in the future patch \(F\), with charge \(\pm(q_2q_4, -q_2q_3)\).

4.2 Particle production in Milne space

As we have seen, zero mode wave functions for twisted closed strings on Milne space in the \(w\)-th twisted sector are identical to those of a charged particle in an electric field \(e_0\) such that \(w\beta = -2\text{arcth}(\pi e_0)\), upon restricting to integer boost momentum \(j\) as specified by the matching condition. Since Schwinger pair production is equivalent to tunneling under the potential barrier, it is clear that pair production will take place in the Milne orbifold as well.

In contrast to the electric field however, where pair production occurred homogeneously throughout space and time, the orbifold projection restricts the emitted pairs to have fixed
integer boost momentum, e.g. \( j = 0 \) for left-right symmetric states such as the tachyon or the graviton. As represented in Figure 8, classical trajectories of massive untwisted particles with \( j = 0 \) correspond to straight lines going from the past region to the future region through the origin; massless particles on the other hand come from the past to the whisker, or from the whisker to the future region. For twisted states on the other hand, classical trajectories with \( j = 0 \) are hyperbolae, one branch of which goes through the origin. One of the member of the pair therefore goes from the past to the future region, while the other remains purely in the whisker. Particle production involves jumping from one branch to the other, hence the particles produced in the whisker are correlated to those in the future region. From the point of view of an observer in the whisker (which seems to be favored in WZW implementations of the Milne Universe [4, 6]), pair production is thus described by a density matrix of charged particles of a given sign.

One may evaluate the distribution of produced pairs semi-classically, by assuming an homogeneous distribution in the \((x^+_0, x^-_0)\) phase space (as is the case in the electric field), and asking how many classical trajectories with a given boost momentum go through a given point \((X^+, X^-)\) in space-time. The total number of pairs produced at \((X^+, X^-)\) is therefore given by the Jacobian

\[
w(X^+, X^-) = \int dx^+_0 dx^-_0 
\]

\[
\delta[(X^+ - x^+_0)(X^- - x^-_0) + \frac{M^2}{2\nu^2}] \delta[\nu x^+_0 x^-_0 + \frac{M^2}{2\nu^2} - j]
\]

We find

\[
w(X^+, X^-) = \frac{1}{\sqrt{(j - \nu X^+ X^-)^2 + 2M^2 X^+ X^-}}
\]

For \( j = 0 \) and \( M^2 > 0 \), the production therefore diverges on the light-cone, as well as around a definite trajectory in the whiskers. For \( M^2 = 0 \), the production diverges at an higher rate on the light cone, leading to a logarithmically divergent total number of produced pairs.

While this copious pair production of twisted states at the singularity is a promising indication that they may resolve the cosmological singularity, this is difficult to demonstrate with the current string perturbation techniques, as this mechanism involves condensation of correlated multi-particle states. Before addressing this challenging backreaction issue, an intermediate goal may be to send twisted squeezed states from \(-\infty\), along with untwisted states, and see whether scattering amplitudes are better behaved. While this is still a daunting problem, a necessary step is to determine the vertex operators for these twisted states, to which we now turn.
5. Analytic continuation and vertex operators

We have constructed above the physical spectrum of open strings in an electric field, and closed strings in Milne space, in a Lorentz signature target space. In order to compute S-matrix elements with more than one incoming particle and one outgoing particle, or for any genus $>1$ computation, a necessary step is to analytically continue to Euclidean signature both on the world-sheet and in target space. The Euclidean formulation that we will propose consists of two parts:

(i) An Euclidean world-sheet/ Euclidean target space CFT. In the electric field case this is simply a constant magnetic field in Euclidean space. In the Milne case, the analytic continuation will turn out to be a rotation orbifold of two-dimensional Euclidean space, although of a somewhat unusual type: the rotation angle $\theta$ is equal to the boost parameter $\beta$ of the Milne universe, and in general irrational. For $\beta = \frac{2\pi p}{q} \in \mathbb{Z}$, our proposal differs from the usual rotation orbifold by the fact that twisted sectors with deficit angle $\theta$ and $\theta + 2\pi$ should not be identified.

(ii) A set of vertex operators to be used for S-Matrix computations in these CFTs. In the magnetic field case, these vertex operators are not the ones associated to the states of the Euclidean theory (which are the standard normalizable Landau modes of the magnetic field problem), but rather a continuum of non-normalizable states. In the Euclidean Milne space these are certain Wilson line observables on the world-sheet, corresponding to non-normalizable states in $\mathbb{R}^2$.

Since our derivation of the Euclidean formulation will parallel the usual analytic continuation from flat Minkowski to Euclidean space, we start by revisiting the latter, before discussing the electric field and Milne space in turn.

5.1 From flat Minkowski space to flat Euclidean space

We begin with a simple field theory analysis of a scalar field in flat Minkowski space $\mathbb{R}^{1,d}$. The path integral describing the quantum theory is

$$\langle \mathcal{O}(\phi, \phi^*) \rangle = \int [d\phi][d\phi^*] \mathcal{O}(\phi, \phi^*) e^{-i \int dt d^4x \frac{i}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}$$

(5.1)

To go to Euclidean target space we rotate the $t$ contour integral gradually along the family of contours $t = e^{-i\theta} \tau$ (where $\tau$ runs from $-\infty$ to $\infty$) from $\theta = 0$ (the Minkowski theory) to $\pi/2$ (the Euclidean theory).

The S-matrix elements of the Minkowskian theory are given by the residues at the poles in the expression

$$\int d^4x_1 \ldots d^4x_n \ e^{i p_0^i x_{0i}} \langle T[\phi(x_1)\ldots \phi(x_n)] \rangle$$

(5.2)

where $p_0^i > 0$ for outgoing particles, and $p_0^i < 0$ for incoming particles. We now rotate the contour integrals $\int dt_k$ inside the path integral without changing the external momenta. Since we are rotating dummy integration variables, we remain on the poles and are still
computing the S-matrix although now we are using an Euclidean path integral. Under this contour deformation, the vertex operator $\int dt e^{ipt}$ becomes a non-normalizable operator $\int d\tau e^{ipt}$ which diverges at $\tau \to \pm \infty$ in order to allow for the outgoing (incoming) excited state to propagate to Euclidean time $\tau = +\infty$ (from $\tau = -\infty$). Of course, these non-normalizable operators need to be regularized, which can be done by introducing an infrared cut-off and expanding them on the basis of normalizable states.

Following the same procedure outlined above, the S-matrix for string theory on the same flat Minkowski space $\mathbb{R}^{1,d}$ can be written as the correlation function in the Euclidean theory,

$$\langle e^{E_1 \tau(z_1,\bar{z}_1)} e^{i\vec{p}_1 \vec{x}(z_1,\bar{z}_1)} \cdots e^{E_n \tau(z_n,\bar{z}_n)} e^{i\vec{p}_n \vec{x}(z_n,\bar{z}_n)} \rangle_E,$$

integrated over the moduli of the punctured Riemann surface. Note that the operator $e^{E \tau}$, for real $E$, does not appear in the state/operator correspondence of the Euclidean CFT – the latter are instead $e^{iE \tau}$ for real $E$. In flat space this does not cause any problem because one can analytically continue the energies $E_i$ so as to use the the local operators $e^{iqt}$ of the theory. In the electric field the analytic continuation of the Minkowskian wave functions to Euclidean space is again a class of non-normalizable vertex operators, but now markedly different than the states of the Euclidean theory, with no obvious way of continuing from one class to the other. In appendix B we outline how to compute the flat space correlators of $e^{E \tau}$ without analytically continuing $E$.

### 5.2 The electric field case

We are now ready to tackle the electric field case. Our discussion will be restricted to the field theory limit, since most of the subtleties are in the zero mode sector. The Lorentzian path integral in an electric background now is

$$\int [d\phi][d\phi^*] e^{-i \int d^d x dt \left\{ -|\partial_t - \frac{i}{2} \vec{b} \cdot \vec{x} |\phi|^2 + |\partial_x + \frac{i}{2} \tau |\phi|^2 \right\}}$$

We analytically continue $t$ as before. In order to obtain a manifestly positive definite expression we will also analytically continue $\nu$ counter to the rotation of $t$, i.e.,

$$t = e^{-i\theta} \tau, \quad \nu = e^{i\theta} b$$

where $\theta$ is varied from 0 to $\pi/2$ while keeping $\tau$ and $b$ fixed. This rotation multiplies the kinetic term by a phase and does not introduce relative phases between the derivative and connection terms. The result is, as one might have expected, a magnetic field $b$ in Euclidean space, described by the Euclidean path integral

$$\int [d\phi][d\phi^*] e^{- \int d^d x d\tau \left\{ |\partial_x - \frac{i}{2} b \tau |\phi|^2 + |\partial_x + \frac{i}{2} \tau |\phi|^2 \right\}}$$

The same Euclidean theory would have been obtained by analytic continuation of a magnetic field in Minkowski space. The difference between the two cases lies in the choice of vertex operators to be used in order to compute S-matrix elements in either theories: to compute the S-matrix in Minkowski space with a magnetic background one would use vertex operators which are non-normalizable in the Euclidean time direction transverse to the
magnetic field, whereas for the electric case at hand one should use an altogether different set, which are the analytic continuation of $\phi_{in}^\pm$ and $\phi_{out}^\pm$ from Section 3.2. For instance, the operator $\phi_{p}^{in,e}$ creating an electron coming from $t = -\infty$

$$\phi_{e}^{in} = D_{-\frac{1}{2} + i \frac{M^2}{2b}} \left[ e^{\frac{\pi i}{4}(\tilde{p} + \nu x)\sqrt{2/\nu}} e^{-i(\tilde{p} + \frac{1}{2} \nu x)t} \right] \tag{5.7}$$

rotates to

$$\phi_{e}^{E,in} = D_{-\frac{1}{2} + i \frac{M^2}{2b}} \left[ i(\tilde{p} + bx)\sqrt{2/b} \right] e^{-i(\tilde{p} + \frac{1}{2} bx)\tau} \tag{5.8}$$

where we also rotated the conserved momentum $\tilde{p} \rightarrow i\tilde{p}$. This indeed is a non-normalizable eigenmode of the upward harmonic oscillator with negative energy $E = -\frac{M^2}{2b}$. Using formula (C.5) for the asymptotic behavior of the parabolic cylinder function $D_{-1-p}(ix)$, we see that this indeed blows up at $x \rightarrow +\infty$, where the electron comes from. The same holds for all the other modes.

It is also interesting to analyze the analytic continuation of the eigenmodes in Rindler coordinates. Upon rotating $t = -i\tau$, the lightcone coordinates $X^\pm$ become a pair of complex coordinates in the two-dimensional plane, $X^+ = (x - i\tau)/\sqrt{2} := \bar{Z}, X^- = Z$. The boost momentum $j$ becomes the angular momentum in the $(Z, \bar{Z})$ plane, and the equation for the radial motion in Rindler space just becomes the Schrödinger equation for a two-dimensional harmonic oscillator in radial coordinates. The Rindler eigenmodes $V_{j,in,R}$ and $V_{j,in,R}$ once again become non-normalizable states of the two-dimensional harmonic oscillator. $V_{j,in,R}$, which described a particle coming from $I^-_R$, now diverges at $\infty$ in the complex plane; $V_{j,in,R}$, which described a particle coming from the horizon, now diverges at the origin of the complex plane. The same issue will arise in the Milne orbifold case.

5.3 The Milne orbifold case

We now make some comments on the continuation of the Lorentzian orbifold $\mathbb{R}^{1,1}/\text{boost}$ to Euclidean space. We will propose an analytic continuation but clearly more work is needed to establish its terms of usage. Under Wick rotation $t = -i\tau$, the lightcone coordinates $X^\pm$ rotate into a pair of complex coordinates and its complex conjugate $Z, \bar{Z}$. Wick rotating the boost parameter $\beta = i\theta$, as in the electric case the orbifold action now identifies $(Z, \bar{Z}) \rightarrow (e^{i\theta}Z, e^{-i\theta}\bar{Z})$, i.e. points on the Euclidean plane related by a rotation of angle $\theta$. This cannot be the usual rotation orbifold however, for the following reasons:

i) The quantum symmetry of the Minkowskian orbifold is $U(1)$ (acting by multiplying the $w$-th twisted sector by a character $e^{i\alpha w}$). On the other hand, if $\beta = 2\pi k/N$, then the standard orbifold by a rotation $e^{2\pi ik/N}$ has $N$ twisted sectors and a quantum symmetry $\mathbb{Z}_N$.

ii) If $\beta = 2\pi r$ for some irrational $r$ (a case that we will refer to as an “irrational rotation angle”), we do have an infinite number of twisted sectors, but the action on the angular coordinate is ergodic, and smooth untwisted states have to be rotationally invariant. This disagrees with the integer quantization of the angular momentum in the Milne orbifold.
iii) Twisted sectors in the usual rotation orbifold describe normalizable states localized at the fixed points. The Lorentzian orbifold on the other hand requires non-normalizable vertex operators with negative Euclidean energy.  

In order to address i) and ii), it is useful to recall that twisted sectors with irrational rotation angle have been recently encountered in a covariant treatment of closed strings in a gravitational wave supported by Neveu-Schwarz flux. This wave background admits a simple description in terms of a Wess-Zumino-Witten model based on a non-semi-simple algebra, the extended Heisenberg algebra $H_4$. Using a free field representation, one may represent the vertex operators of closed string states with $p^+ \neq 0$ by the product of an exponential of the light-cone coordinates $e^{i(p^+ u + p^- v)}$, times a twist field $H_\theta$ that creates twisted boundary conditions on the transverse coordinates $(Z, \bar{Z})$, with $\theta = 2\pi p^+$. Correlation functions for up to four twist fields with arbitrary level of excitation have been computed using current algebra techniques. A crucial feature in the discussion is the existence of “spectrally flowed states”, which appear when the rotation angle $\theta$ exceeds $2\pi$, thereby lifting the field identification $\theta \sim \theta + 2\pi$ and allowing a smooth description independent of the rationality of $\theta$. These states have a simple physical interpretation as long strings winding around the center of the transverse plane, which are stabilized by a balance between tensive energy and flux.

While this construction does provide a computational framework to treat twist fields with a continuous rotation angle, it still is not directly useful for our purposes, as the natural observables involve normalizable states in transverse space. In order to deal with non-normalizable wave functions relevant for the Milne Universe, one may instead consider the double analytic continuation of the Nappi-Witten background,

$$ ds^2 = -2dUd\bar{U} + dX^+dX^- - \frac{1}{4}\mu^2 X^+X^-dU^2, \quad H = \mu dUdX^+dX^- $$  \hspace{1cm} (5.9)

where $U$ is a complex coordinate which arises by Wick rotation of the light-cone coordinates $(u,v)$. Unfortunately, the non-reality of the metric and flux casts doubts on the consistency of this model.

Given these difficulties, we now suggest an alternative construction of the irrational rotation orbifold, which is smooth in the parameter $\beta$, has a $U(1)$ quantum symmetry for any value of $\beta$ and contains observables for any integer angular momentum $j$. Our procedure is as follows: we first augment the Minkowski orbifold by a topological sector on the world-sheet which is trivial in Minkowski space. After rotating the new world-sheet theory to Euclidean space, we find that the topological sector now contains information – it is the custodian of the $U(1)$ quantum symmetry even when the rotation angle is rational. The price to pay is that vertex operators are no longer local invariant operators, but involve Wilson lines on the world-sheet.

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9In particular, this seems to rule out the localized tachyon considered in [50].

10Conical singularities with an irrational angle also arose in stringy investigations of black hole and Rindler space thermodynamics [33, 34], where they were assumed to reduce to the usual rotation orbifold at rational values of the deficit angle.
In more detail, we replace the usual Lorentzian path integral over $X^\pm$ by

$$\int [dX^+] [dX^-] [dA] [d\theta] \ e^{i \int \left[ \partial_\mu X^+ \phi + \beta A_\mu X^+ \right] \left( \partial_\nu X^- - \beta A_\nu X^- \right) \eta^{\mu \nu} + \theta F_{01} \right] \prod_{i=1}^k \left( \Sigma_n e^{2\pi i n_i \int_{\gamma_i} A} \right) $$

(5.10)

where $\gamma_i$, $i = 1 \ldots k$, denote the non-trivial 1-cycles of the Riemann surface, and $\theta$ is a dynamical field taking values in $\mathbb{R}$. This action is gauge invariant under a non-compact Abelian gauge group,

$$X^\pm \rightarrow e^{\pm \beta \alpha} X^\pm, \quad A \rightarrow A - d\alpha$$

(5.11)

However, we do not impose any identification on the coordinates $X^\pm$, but instead use the gauge field to implemented the twist. The last factor in the path integral is in fact not a two-dimensional integral of Lagrangian density, however it can be absorbed into the latter if we write $\int \theta F$ as $\int d\theta A$ and assume $\theta$ to be periodic: upon integrating by parts, one pick up lines in which $\theta$ jumps by its allowed periodicity, giving rise to the integrals of $A$ over 1-cycles. Notice that this prescription is modular invariant. The sum $\Sigma_n e^{2\pi i n_i \int_{\gamma_i} A}$ serves to restrict the holonomies around any non-trivial cycle to be integers. Let us for example consider the torus, represented as usual by a parallelogram in the complex plane with modular parameter $\tau = \tau_1 + i\tau_2$. The path integral splits into a sum over two integers $m_a$ and $m_b$ such that $\int_{\gamma_a} A = m_a$ and $\int_{\gamma_b} A = m_b$, where $\gamma_a$ and $\gamma_b$ are the a and b cycles of the torus. More specifically we can choose a gauge

$$A = d\phi(w, \bar{w}), \quad \phi(w, \bar{w}) = m_a \Re(w) + \frac{m_b - m_a \tau_1}{\tau_2} \Im(w)$$

(5.12)

The new variables $U^\pm = e^{\pm \beta \phi} X^\pm$ now have a free field Lagrangian density and are twisted by $m_a$ and $m_b$ actions of the boost operator along the two cycles. We then get the standard orbifold partition function. Finally we rotate to Euclidean target space, $X^+ \rightarrow Z$, $X^- \rightarrow \bar{Z}$, $\beta \rightarrow i\theta$. We obtain an Abelian gauge theory coupled to a unit charge scalar field, with the only subtlety that the gauge group is non-compact.

We can now examine some of the issues raised above. If $\beta = 2\pi/N$ for some $N$, the model we are discussing is still distinct from the $\mathbb{R}^2/\mathbb{Z}_N$ orbifold. The latter would correspond to a compact $U(1)$ gauge field. In our case the non-compactness of the Abelian gauge group still enforces selection rules between the different sector. For example in the pants diagrams it would still be true that $F = 0$ implies $\int A_{in-cycle} = \int A_{out-cycle-1} + A_{out-cycle-2}$, rather than this same equality modulo $N$.

Let us now consider the observables in the untwisted sector, for an irrational value of the rotation angle. The standard local untwisted vertex operators

$$\Psi_{p^+, p^-}(X^+, X^-) = \int dw \ e^{i(p^- X^+ e^{-\beta w} + p^+ X^- e^{\beta w}) + i\epsilon w}, \quad p^+, p^- > 0$$

(5.13)

\[11\] A similar construction was used in [52, 53, 54, 55].

\[12\] Here we are using a Euclidean world-sheet and a Minkowski target space.

\[13\] In this specific gauge there is still a single gauge degree of freedom for the entire surface. This fact is irrelevant here but it will complicate the notion of observables. We will turn to this point shortly.

\[14\] In the compact $U(1)$ case, the identification of the holonomy modulo $N$ is implemented by a large gauge transformation.
satisfy $\Psi_l(e^{\alpha\beta}X^+, e^{-\alpha\beta}X^-) = e^{2\pi i l \alpha} \Psi_l(X^+, X^-)$, and hence are not gauge invariant. This remains true when we go to the Euclidean continuation and replace $X^+$ and $X^-$ by $Z, \bar{Z}$. However we can still compute S-matrix elements in the following way. Define an operator

$$
\Psi_l^\gamma(Z, \bar{Z}) = P(e^{2\pi i l \int_{\gamma} A}) \Psi_l(Z, \bar{Z})
$$

(5.14)

where $\gamma$ is any path from a fixed reference point, say 0, on the Riemann surface. Note that this definition is actually independent of the choice of the trajectory $\gamma$, thanks to the flatness of the gauge connection $A$ and the quantization law of the holonomies $m_a$ and $m_b$. The operator $\Psi_l^\gamma$ is still not gauge invariant – it rotates under gauge transformations at point 0. However, if we have several such operators then the product $\prod_i \Psi_l^\gamma_i$ is gauge invariant if $\Sigma_l i = 0$. This is not a new restriction on the model and we expect it to occur in the orbifold. Hence we can use these operators to define a gauge invariant $n$-point function.

We shall leave twisted vertex operators for future work.

6. Discussion

In this work, we have studied the kinematics of charged open strings in a constant electric field, and of twisted closed strings on the Lorentzian orbifold $\mathbb{R}^{1,1}/\text{boost}$. Despite being, to our knowledge, unrelated under any duality, these two situations share many formal similarities at the level of first quantized string theory. In particular, drawing from the well understood dynamics of charged particles in an electric field, we proposed an alternative quantization prescription for the open/closed string zero-modes, which does lead to physical states in the charged/twisted sectors. Despite the lack of a globally time-like Killing vector, we were also able to give an analytic continuation to an Euclidean background, in analogy with the continuation from an electric to a magnetic field. We described the zero-mode wave functions in a variety of useful representations, and outlined the corresponding vertex operators, by analytic continuation to the Euclidean space. In contrast to the usual observables in a magnetic field or rotation orbifold, the Euclidean vertex operators describing Lorentzian scattering states are non-normalizable operators diverging in the spatial direction where the particle is coming from. They may be expressed in terms of a continuous spectrum of twist fields in the CFT of a free complex boson, which remain to be constructed.

While clarifying much of the kinematics, these results are only a first step toward understanding the dynamics, which we plan to pursue in a future publication. In particular, the analogy with the electric field falls short of telling the true fate of the Milne Universe. While twisted state pair production takes place in both cases, it happens homogeneously throughout space-time in the electric field case, leading to complete screening in finite time. In the Milne case, the pair production diverges on the light-cone as well as in the whiskers, with no effect in principle at past or future infinity. Yet, the relation between the electric field $\nu$ and the contraction rate of the orbifold $\beta$ strongly suggests that the Milne singularity may be smoothed under condensation of twisted states, and possibly

\footnote{Note that the vertex operators are singular at $Z = \bar{Z} = 0$.}
lead to a smooth transition from Big Crunch to Big Bang [2]. Indeed, once produced, the twisted closed strings contribute an energy that grows linearly with the radius of the Milne universe, thus mimicking the effect of a two-dimensional positive cosmological constant: the resulting inflation may thus be sufficient to prevent the circle to reach zero-size. If so, the non-perturbative instability toward large black hole formation raised in [14] may never be reached. A crucial element for this scenario to hold is of course that the recombination rate of the twisted strings be lower than their production rate. Note also that untwisted states may also be produced, much like uncharged particles in Rindler space. One would however expect their effect at the singularity to be negligible, as they become infinitely massive there.

This dynamical slow-down of the contraction rate may perhaps be understood at the level of low energy field theory, by going to the T-dual picture. The collapsing geometry of Milne turns into a trumpet-like geometry whose circle opens up to infinite radius in finite time. The string coupling also blows up there, but the initial coupling (or more precisely the coupling at some fixed point in the trumpet) may be made as small as desired such that dilaton gradient is localized arbitrarily close to $T = 0$. Winding modes are now ordinary momentum modes, and it is not surprising that they should be pair-produced. Furthermore, the situation resembles the standard FRW cosmology of an expanding universe with a gas of particles. The effect of the latter, for any sensible matter, would be to decrease the expansion rate of the universe. It may also be interesting to probe the geometry by localized S-brane probes, although an early investigation suggested that such probes might see an even more singular geometry after twisted string condensation [28].

A related question is the fate of the “whiskers”, which is tied to the issue of closed time-like curves. While usually banned on general relativity grounds, they seem to occur naturally in many string backgrounds, for which they provide a natural asymptotic region possibly suitable for holography [4]. Several proposals for shielding CTC’s in string theory have been made in [56, 57, 39, 58], and it would be interested to study their fate under winding mode pair creation. Let us also note that the occurrence of non-normalizable modes in the Euclidean continuation of the electric field and Lorentzian orbifold suggest a possible holographic interpretation, analogous to one of the proposals for holography in plane waves [54].

Assuming CTC’s are still admissible in string theory, the dynamics in one whisker may after all not be that complicated. If the analogy to electric field holds, then positively charged winding states are produced in the left whisker while negatively charged winding states go into the right one. Each whisker therefore sees only one of the two correlated particles, so that the state in one whisker may be described by a density matrix of uncorrelated single winding string states. In string theory this would correspond to considering the addition of single-particle twisted states, and at the end sum over all such backgrounds. The pair creation problem is now transformed into single particle creation, which could be treated using CFT techniques, maybe along the lines of [60].

Irrespective of the details of the dynamics, it is clear that correlated two-particle states will play a important role. This highlights a serious deficiency in the standard perturbative string theory approach: we do not know how to take such effects into account systemati-
cally. While condensation of coherent states may be incorporated by the Fischler-Susskind mechanism, squeezed states and other multi-particle states require the development of new tools, be it closed string field theory or non-local string theories, \[ 51, 52, 53, 54, 55 \]. These are some of the obstacles that any string theory description of time-dependent backgrounds will have to address.

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A. Light-cone quantization in an electric field

In view of the fact that charged particles in an electric field are accelerated to the velocity of the light, it is natural to try and quantize them on the light front. The world-sheet Hamiltonian

\[ M^2 = a^+_0 a^-_0 + a^-_0 a^+_0 \]  

(A.1)
can then be viewed as the generator of dilations in the \((a^+_0, a^-_0)\) phase space, which allows for very simple eigenfunctions\(^{16}\). Diagonalizing the generator \( P^- = -k_+ \) and working in the \(a^-_0 = -k_+ - \nu x^-\) representation, we obtain

\[ f_{k_+}(x^+, x^-) = (2\nu)^{-\frac{1}{4} + \frac{M^2}{4\nu}} \Gamma \left( \frac{1}{4} - \frac{iM^2}{4\nu} \right) (k_+ + \nu x^-)^{-\frac{1}{2} - \frac{iM^2}{4\nu}} e^{ix^+(k_+ + \frac{1}{2}\nu x^-)} \]  

(A.2)

This basis of functions is most appropriate to expand the modes at a fixed (early) \( x^- \) time, \( i.e. \) for incoming electrons. Equivalently, we may diagonalize the light-cone momentum \( P^+ = -k_- \) and work in a \(a^+_1 = \nu x^+ - k^-\) representation, obtaining:

\[ f_{k_-}(x^+, x^-) = (2\nu)^{\frac{1}{4} + \frac{M^2}{4\nu}} \Gamma \left( \frac{1}{4} + \frac{iM^2}{4\nu} \right) (k_- - \nu x^+)^{-\frac{1}{2} + \frac{iM^2}{4\nu}} e^{ix^-(k_- - \frac{1}{2}\nu x^+)} \]  

(A.3)

which is appropriate to expand the modes at a fixed (late) \( x^+ \) time, \( i.e. \) for outgoing positrons. The two basis are related by exchange of position \( a^+_0 \) and momentum \( a^-_0 \), hence by Fourier transform:

\[ \int \frac{dk_+}{\sqrt{2\pi\nu}} f_{k_-}(\bar{x}^+, \bar{x}^-) e^{ik_+k_-/\nu} = f_{k_+} \]  

(A.4)
as can be easily checked using the usual integral representation of the Gamma function

\[ A^{-2s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dz}{z^{s+1}} e^{-\pi A/z} \]. The S-matrix is thus equal to the Fourier transform\(^{17}\). A full

\(^{16}\)This quantization scheme was first proposed in a footnote of \[ 28 \], but perhaps too hastily dismissed.

\(^{17}\)A similar statement holds true in the context of the \( c = 1 \) string \[ 56 \].
specification of the characteristic value problem would involve specifying both the incoming electrons and positrons at \(x^\pm = -\infty\), and reading off the outgoing particles at \(x^\pm = +\infty\), as in [23, 24].

The canonical commutation relations are now easy to determine:

\[
\begin{align*}
[a_{k^+}, a^*_{k^+}'] &= \text{sgn}(k_+ + \nu x^-)(2\nu)^{-1/2}|\Gamma \left( -\frac{1}{4} - \frac{iM^2}{4\nu} \right)|^2 \delta(k_+ - k_+') \\
[a_{k^-}, a^*_{k^-}'] &= \text{sgn}(k_- - \nu x^+)(2\nu)^{-1/2}|\Gamma \left( -\frac{1}{4} + \frac{iM^2}{4\nu} \right)|^2 \delta(k_- - k_-')
\end{align*}
\] (A.5) (A.6)

For positive \(k^+ + \nu x^-\), \(a^*_{k^+}\) therefore creates an electron in the incoming Hilbert space, while for negative value it annihilates one. Similarly, for negative \(k_- - \nu x^+\), \(a^*_{k_-}\) creates an outgoing electron, while for positive value it annihilates one. These rules have actually a very simple physical origin: semi-classically, the modes \(f_k^\pm\) correspond to charged particles following the hyperbola

\[2\nu(\bar{x}^\pm - a'(k_\pm))(k_\pm \pm \nu \bar{x}^\mp) = \pm L\] (A.7)

where \(a(k_\pm)\) is the phase used to construct the wave packet. For \(k^+ + \nu x^- > 0\), the electron world-line does intersect the characteristic line at \(x^-\), and reaches \(x^+ = +\infty\) at a later time. For \(k^+ + \nu x^- < 0\) however, the electron has already reached \(x^+ = +\infty\) before the initial time surface; at that time, a new positron has been emitted from \(x^+ = -\infty\), and its world-line intersects the characteristic line at \(x^-\).

**B. Correlation function of non-normalizable vertex operators**

Here we briefly discuss how correlation functions of non-normalizable operators of the form \(e^{\mathcal{O}_\tau}\) may be computed in an Euclidean theory. One may proceed in two equivalent ways,

1. One may regard the insertion of \(e^{\mathcal{O}_\tau(z, \bar{z})}\) at a point of world-sheet as imposing the boundary behavior of the fields near \((z, \bar{z})\), and compute the Euclidean path integral with these boundary conditions.

2. One may expand \(\mathcal{O}_\tau = e^{\mathcal{O}_\tau}\) as a superposition of normalizable modes \(\mathcal{O}_{ip} = e^{ip\tau}\) (and in principle, descendants as well, although that will not be the case here), after introducing an infrared cut-off.

The motivation for approach (2) is that upon inserting \(\mathcal{O}_\tau\) at a point on the world-sheet, the state on a circle of radius \(\nu\) around it is an admissible state in the CFT (this is also the way to make contact with the first approach), and hence can be written as sum over operators \(\mathcal{O}_{ip}\) using the state/operator correspondence of the CFT. One can then use the standard OPEs to compute the required Green’s function. Clearly, a crucial problem is in enforcing the infrared cut-off in a way consistent with conformal invariance.

In flat space one may carry out this procedure and verify that it does give the correct results. The computation of the correlator \(\langle e^{\mathcal{O}_1 \tau + ip_1 x} \ldots e^{\mathcal{O}_n \tau + ip_n x} \rangle_{\mathcal{O}_\tau}\) proceeds as follows.
Focusing on the $\tau$ CFT we use the regulated expression above to replace the correlator by
\[
\lim_{s \to 0} \int \ldots \int dp_1 \ldots dp_n \Pi_{i=1}^n \left( \int dz_i e^{-s z_i^2 + E_i \tau_i} \times e^{-ip_i \tau_i} \right) \left( e^{i p_1 \tau(z_1, z_1)} \ldots e^{i p_n \tau(z_n, z_n)} \right) E \quad \text{(B.1)}
\]
The last expression is the familiar $\delta(\Sigma p_i) \Pi|z_i - z_j|^{p_i p_j}$, and the delta function removes an integration over momenta. We have to convolute this expression with
\[
\int dz_i e^{-s z_i^2 + E_i \tau_i} \times e^{-ip_i \tau_i} \propto \frac{1}{\sqrt{s}} e^{\frac{1}{2}(E_i + ip_i)^2} \quad \text{(B.2)}
\]
In the limit $s \to 0$ we use steepest descent integration, when we put the $E_i$ on shell and require $\Sigma_i E_i = 0$, to localize the integral to $p_i = -i E_i$. The end result is the same as the standard analytic continuation\(^\text{\dag}\).

C. Parabolic cylinder functions

In this appendix, we assemble some useful formulas involving parabolic cylinder and Whittaker functions.

We start with parabolic cylinder functions $D_p(u)$, which form a basis of solutions of the Schrödinger equation for the harmonic oscillator with an arbitrary (non quantized) energy $E := p + 1/2$:
\[
-\partial^2_u + \left( \frac{1}{4} u^2 - E \right) = 0 \quad \text{(C.1)}
\]
Solutions can be expressed as linear combinations of two among the set
\[
D_p(u), D_p(-u), D_{-p}(u), D_{-p}(-u) \quad \text{(C.2)}
\]
Normalizable solutions occur for integer $p$, and can be expressed in terms of the usual Hermite polynomials,
\[
D_n(z) = 2^{-n/2} e^{-z^2/4} H_n(z/\sqrt{2}) \quad \text{(C.3)}
\]
For general $p$ however, the parabolic cylinder functions are non-normalizable. As $z \to \infty$ with fixed argument $\theta$, they admit the asymptotic expansion
\[
D_p \sim e^{-z^2/4} z^p (1 + O(z^{-2})) \quad |\theta| < \frac{3}{4} \pi \quad \text{(C.4)}
\]
\[
D_p \sim e^{-z^2/4} z^p (1 + O(z^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{i\pi p} e^{z^2/4} z^{-p-1} (1 + O(z^{-2})) \quad \frac{\pi}{4} < \theta < \frac{5\pi}{4} \quad \text{(C.4)}
\]
\[
D_p \sim e^{-z^2/4} z^p (1 + O(z^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-i\pi p} e^{z^2/4} z^{-p-1} (1 + O(z^{-2})) \quad -\frac{5\pi}{4} < \theta < \frac{\pi}{4} \quad \text{(C.4)}
\]
Eigenmodes of the inverted harmonic oscillator can be obtained by analytic continuation $u \to e^{i\pi/4} u$, while at the same time rotating the energy $E \to e^{-i\pi/2} E$, leading to
\[
-\partial^2_u + \left( \frac{1}{4} u^2 - E \right) = 0 \quad \text{(C.5)}
\]
\(^\text{\dag}\)It is also easy to track the factors of $s$. From $N$ integrations over $\tau_i$, and from $N - 1$ integral over $p_i$ around the saddle point we obtain a total coefficient of $s^{-1/2}$ which is nothing but a single power of the volume, as it should be.
Solutions are now linear combinations of any two eigenmodes among
\[
\psi_L^- = D_{-\frac{1}{2}+iE}(e^{-3\pi i/4}u), \quad \psi_L^+ = D_{-\frac{1}{2}-iE}(e^{3\pi i/4}u), \\
\psi_R^- = D_{-\frac{1}{2}+iE}(e^{\pi i/4}u), \quad \psi_R^+ = D_{-\frac{1}{2}-iE}(e^{-\pi i/4}u)
\]
(C.6)
(corresponding to a purely outgoing wave toward \( u \to -\infty \), incoming wave from \( u \to -\infty \),
incoming wave from \( u \to +\infty \) and outgoing wave to \( u \to \infty \), respectively.

More generally, in order to analyze the charged Klein Gordon equation in Rindler
coordinates we are interested in solutions to Whittaker’s equation
\[
\partial_z^2 + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{1}{z^2} - \frac{\mu^2}{z^4}\right)W(z) = 0
\]
(C.8)
Solutions with plane wave behavior at \( z \to +\infty \) (resp. \( z \to 0 \)) are the Whittaker functions
\[
W_{k,\mu}(z) \sim e^{-z/2}z^k, \quad M_{k,\mu}(z) \sim z^{\mu+\frac{1}{2}}
\]
(C.9)
These functions are not independent, but satisfy
\[
W_{k,\mu} = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - k\right)} M_{k,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - k\right)} M_{k,-\mu}(z) = W_{k,-\mu}
\]
(C.10)
For \( \mu = -1/4 \) we recover the parabolic cylinder functions,
\[
D_p(z) = 2\frac{\Gamma}{\sqrt{\pi}} W_{\frac{1}{4}+\frac{p}{2}}(\frac{z^2}{2}) z^{-1/2}
\]
(C.11)
Finally, writing \( W = e^{-z^2/2}z^{1/2+\mu}F \) one may rewrite (C.8) as the Kummer equation
\[
\left[z\partial_z^2 + (1 - 2\mu - z)\partial_z - \left(\frac{1}{2} + \mu - k\right)\right]F = 0
\]
allowing to express Whittaker functions in terms of confluent hypergeometric functions,
\[
M_{k,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z^2/2} \frac{\Gamma}{\Gamma\left(\frac{1}{2} - k\right)} \Gamma\left(\frac{3}{2} - k\right) \frac{\Gamma}{\Gamma\left(\frac{1}{2} + k\right)} \frac{\Gamma}{\Gamma\left(\frac{1}{2} - \mu - k\right)} \frac{\Gamma}{\Gamma\left(\frac{1}{2} + \mu - k\right)} F_{1}(\mu - k + \frac{1}{2}, 2\mu + 1, z)
\]
(C.13)
\[
W_{k,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z^2/2} \frac{\Gamma}{\Gamma\left(\frac{1}{2} - k\right)} \Gamma\left(\frac{3}{2} - k\right) \frac{\Gamma}{\Gamma\left(\frac{1}{2} + k\right)} \frac{\Gamma}{\Gamma\left(\frac{1}{2} - \mu - k\right)} \frac{\Gamma}{\Gamma\left(\frac{1}{2} + \mu - k\right)} F_{2}(\mu - k + \frac{1}{2}, -\frac{1}{2} - \mu - k, -1/z)
\]
(C.14)
\[
D_p(z) = 2\frac{\Gamma}{\sqrt{\pi}} e^{-z^2/4} \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - p\right)} \frac{\Gamma}{\Gamma\left(\frac{1}{2} + p\right)} F_{1}(\mu - k + \frac{1}{2}, 2\mu + 1, z) - \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - p\right)} \frac{\Gamma}{\Gamma\left(\frac{1}{2} + p\right)} F_{1}(\mu - k + \frac{1}{2}, -\frac{1}{2} - \mu - k, -1/z)
\]
(C.15)
It is often convenient to use the integral representations,
\[
\begin{align*}
\text{i} F_1(\alpha, \beta, z) &= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta - \alpha)} \int_0^1 e^{zt} t^{\alpha-1} (1-t)^{\beta-\alpha-1} dt \\
\text{i} F_1(\alpha, \beta, z) &= \frac{z^{1-\beta}}{B(\alpha, \beta - \alpha)} \int_0^z e^{v} v^{\alpha-1} (z-v)^{\beta-\alpha-1} dv \\
\text{i} F_1(\alpha, \beta, z) &= \frac{2^{1-\beta} e^{\frac{z}{2}}}{B(\alpha, \beta - \alpha)} \int_{-1}^1 e^{\frac{z}{2}v} (1+v)^{\alpha-1} (1-v)^{\beta-\alpha-1} dv
\end{align*}
\]
(C.16, 17, 18)
where the last two equations are valid for \( 0 < \Re(\alpha) < \Re(\beta) \).
References


