Perspective on Afterglows: Numerically Computed Views, Lightcurves and the Analysis of Homogeneous and Structured Jets with Lateral Expansion

Jay D. Salmonson
Lawrence Livermore National Laboratory, Livermore, CA 94551

ABSTRACT

Herein I present numerical calculations of lightcurves of homogeneous and structured afterglows with various lateral expansion rates as seen from any vantage point. Such calculations allow for direct simulation of observable quantities for complex afterglows with arbitrary energy distributions and lateral expansion paradigms. A simple, causal model is suggested for lateral expansion of the jet as it evolves; namely, that the lateral expansion kinetic energy derives from the forward kinetic energy. As such the homogeneous jet model shows that lateral expansion is important at all times in the afterglow evolution and that analytical scaling laws do a poor job at describing the afterglow decay before and after the break. In particular, I find that lateral expansion does not cause a break in the lightcurve as had been predicted. A primary purpose of this paper is to study structured afterglows, which do a good job of reproducing global relationships and correlations in the data and thus suggest the possibility of a universal afterglow model. Simulations of structured jets show a general trend in which jet breaks become more pronounced with increasing viewing angle with respect to the jet axis. In fact, under certain conditions a bump can occur in the lightcurve at the jet break time. I derive scaling relations for this bump and suggest that it may be a source of some bumps in observed lightcurves such as that of GRB 000301C. A couple of lateral expansion models are tested over a range of efficiencies and viewing angles and it is found that lateral expansion can, in some cases, substantially sharpen the jet break. I show flux surface contour maps and simulated images of the afterglows which give insight into how they evolve and determine their lightcurves.

Subject headings: gamma rays: bursts — gamma rays: theory

1. Introduction

It is currently widely believed that gamma-ray bursts (GRBs) derive from narrow (half-opening angle $\theta_{0,GRB} \sim$ few degrees) jets of relativistic ejecta pointing toward the observer. One of the basic motivations for this has been to relieve the energy crisis in GRBs by reducing the necessary total energy from the inferred isotropic equivalent energy (up to several $10^{54}$ ergs) by a factor $\theta_{0,GRB}^2/2 \sim 10^{-4} - 10^{-6}$ (and thus boosting the total event rate by the reciprocal, $2/\theta_{0,GRB}^2$ to include unseen jets directed away from the observer). If the GRB emission is collimated, it is plausible that the afterglow shock is also collimated into a cone with opening angle $\theta_0$, generally thought to be larger than $\theta_{0,GRB}$.

A rich area of inquiry is then to predict and look for observable consequences of a narrow jet both in the GRB phase (e.g. ??) and in the afterglow phase (e.g.) [rohads99, sph99]. The most fundamental consequences derive from the deceleration of the afterglow as it propagates into the interstellar medium (ISM). The relativistic motion of the emitting shock causes the radiation to be beamed into an angle $1/\Gamma$ in the observer frame. At early times, this relativistic beaming angle is smaller than the physical jet opening angle, $1/\Gamma < \theta_0$, thus the emission appears to be isotropic. Eventually, as the jet decelerates, it transitions to $1/\Gamma > \theta_0$.
and the finite, non-isotropic extent of the jet becomes apparent to both the observer and to the jet itself. The observer sees a deficit of flux compared to that expected from an isotropic emitter. The jet, now being entirely causally connected, begins to expand sideways. These two effects combine to cause a break in the lightcurve (??). Roughly twenty such jet-breaks have been observed (?, see) and references therein]fksd01 and these constitute the most direct evidence for beaming to date.

Also, since the later, slower emission is beamed into a wider angle than the earlier emission, we expect to see off-axis optical and radio afterglows where no gamma-ray or X-ray burst was seen (??). To date these so-called “orphan afterglows” have yet to be positively observed.

Thus far the study of this afterglow jet-break has assumed an afterglow pointed directly at the observer with constant, homogeneous, unstructured energy and mass density \( \sim H(\theta_0 - \theta) \) across the jet surface, with a hard edge at \( \theta = \theta_0 \) where \( H() \) is the Heaviside step function. As such, from scaling laws, \( t_j \sim \theta_0^{8/3} \), the observation of a jet-break time, \( t_j \), gives direct information about the opening angle of the jet. While this model is relatively simple to calculate and is amenable to analytical calculation, it is not necessarily the easiest jet morphology for nature to produce and thus may not accurately represent a physical jet. Firstly, one certainly expects the observers viewing angle, \( \theta_v \), of the jet to vary. Furthermore, a homogeneous jet with a hard edge will be unstable to expansion and rarefaction and is thus unlikely to propagate intact and is unlikely to have been formed in the first place. Recent numerical simulations by ?) of relativistic jets emerging from stars, within the context of the ‘collapsar’ model, show substantial structure, with the most energetic material along the jet axis and decreasing with larger angles from the axis.

The first semi-analytical calculations of an afterglow jet with structure, i.e. with decreasing energy density and/or Lorentz factor as a function of angle for the jet axis, was done by ?) while and analytical treatment was done by ?). Not long before, a qualitative discussion of a structured jet model was put forward by ?). It was found by ?) and ?) that a universal structured jet, viewed at different angles, will yield a range of afterglows with jet-break times relating to viewing angle \( t_j \sim \theta_0^{8/3} \). By letting the energy per solid angle of the jet decrease with angle from the jet axis, \( \theta \), as \( \epsilon \sim \theta^{-2} \), they were able to effectively reproduce the observed relation \( E \sim t_j^{-1} \) (?).

In this paper I present calculations that further refine the work by ?) and ?. By discretizing the surface of the afterglow, arbitrary functions of energy density, Lorentz factor and lateral Lorentz factor (defined in the fluid frame) can be simulated and lightcurves produced for arbitrary viewing angles \( \theta \). In so doing, I corroborate some of the key results of (??), i.e. \( t_j \sim \theta_0^{8/3} \) and \( E_{\text{tot}} \sim t_j^{-1} \) if \( \epsilon \sim \theta^{-2} \). Also, these more detailed calculations allow for a quantitative discussion of how the lightcurve breaks; in particular I find that a flattening, and even a bump in the lighcurve is possible just prior to the jet-break time. In addition I study lateral expansion and find that some of the general scaling laws that are widely used in afterglow work are incorrect.

2. Numerical Jet Calculations

The calculation presented here begins by discretizing the surface of the afterglow mapped with polar coordinates, \((\theta, \phi)\), into small elements of solid angle \( d\Omega = \cos \psi \sin \theta d\psi d\phi \) where herein we assume the afterglow is spherical, i.e. zero inclination \( \psi = 0 \), and where \( \theta = 0 \) corresponds to the jet axis. Each surface element plows into the ISM, sweeping up mass according to its cross-section \( dA = R^2 d\Omega \), decelerating, shocking and radiating. The physics of this calculation can be broken up into two parts i) the dynamical evolution of each surface element of the afterglow, as dictated by conservation of energy and momentum and ii) the radiative mechanism, which I take to be the standard synchrotron shock model (?).

To calculate the evolution of the afterglow shock one needs only to invoke conservation of energy and momentum along with an assumption of radiation losses. Herein I assume radiative losses are dynamically insignificant i.e. the evolution is adiabatic. Define the initial bulk Lorentz factor \( \Gamma_0 = \epsilon_0/M_0 c^2 \) of a surface element of the afterglow, where \( \epsilon_0 \) and \( M_0 \) are the initial energy and rest mass per solid angle. Thus following ?) and ?), the energy and radial momentum of a surface...
element of the afterglow is
\[ \Gamma_0 + f = (1 + f)\xi \Gamma \]
\[ \Gamma_0\beta = (1 + f)\xi \Gamma \beta \]
(1)
where \( \xi = (E + M)/M \) is the internal energy of the expanding shell.\(^1\)

The mass fraction, \( f \), of accumulated interstellar mass density, \( \rho_{ISM} \equiv nm_p \), where \( m_p \) is the mass of the proton, is
\[ f \equiv \frac{M(R)}{M_0} = \frac{\Gamma_0\rho_0}{\epsilon_0} \int_0^R \rho_{ISM} \frac{\Delta \Omega}{\Delta \Omega_0} \rho_0^2 \frac{dr}{r} \]
\[ = \frac{\Gamma_0\rho_0 c^2}{\epsilon_0} \Delta \Omega_0 \Delta N_e = 1.9 \times 10^{-54} \frac{\Gamma_0}{\epsilon_{52}} \Delta \Omega_0 \Delta N_e \]
(2)
where the number of electrons swept up into the shock element is
\[ \Delta N_e = \int_0^R n_0 \Delta \Omega r^2 dr \]
(3)
and the the energy per solid angle is \( \epsilon_{52} = \mathcal{E}/(10^{52}/4\pi \text{ ergs}) \), and the element solid opening angle is
\[ \Delta \Omega = \sin \theta \Delta \phi \Delta \theta \]
(4)
where the position of a surface element, \( \theta \), will evolve as the shock laterally expands due to internal pressure (Section 7). The velocity magnitude is \( \beta = |\vec{v}|/c = \sqrt{1 - 1/\Gamma^2} \) and proper time in the fluid frame is
\[ t' = \int_0^R \frac{dr}{\Gamma \beta c} \].
(5)

Eqns. (1) can be solved for the Lorentz factor
\[ \Gamma = \frac{\Gamma_0 + f}{\sqrt{1 + 2\Gamma_0 f + f^2}} \].
(6)

So by specifying \( \epsilon_0 \) and \( \Gamma_0 \) and a prescription for lateral expansion \( v_L \) (Section 7), the entire evolution of the afterglow as a function of \( R \) is determined.

In order to calculate the observed flux, \( F_\nu \), at a given frequency, \( \nu \), note that the intensity transforms as \( I = \Gamma^3 \delta^3 \) where the Doppler factor is
\[ \delta = [\Gamma(1 - \beta^2 \cdot \hat{n})]^{-1} \]
(7)
where \( \beta = v/c \) and \( \hat{n} \) is the unit vector pointing toward the observer. Following \( ? \), here I focus on the power-law branch of the spectrum between the peak, \( \nu_m \), and cooling, \( \nu_c \), frequencies. As such, the proper intensity at the proper peak frequency, \( \nu_m' \), is
\[ I_{\nu_m'} = \frac{P_{\nu_m'} \Delta N_e}{4\pi R^2 \Delta \Omega} \]
(8)
where the shock element has surface area, \( R^2 \Delta \Omega \), and the proper power per electron radiated at, \( \nu_m' \), is \( ? \)
\[ P_{\nu_m'} = 5.4 \times 10^{-24} \left( \frac{\phi}{0.59} \right) \nu_0^{1/2} \beta_{B,2}^{1/2} \]
\[ \times \Gamma \beta^2 \text{ ergs s}^{-1} \text{ Hz}^{-1} \text{ electron}^{-1} \].
(9)
where the factor \( \beta^2 \) has been included here to make this equation valid in the non-relativistic limit \( ? \). The observed intensity at a frequency, \( \nu \), is
\[ I_\nu = I_{\nu_m'} \left( \frac{\nu}{\nu_m} \right)^{-\alpha} \delta^3 \]
(11)
and integrating over the population of radiating electrons, the flux will go like

$$F_\nu = \int \frac{P'_\nu \Delta A}{4\pi R^2 \Delta \Omega} \left( \frac{\nu}{\nu_m} \right)^{-\alpha} \frac{\nu^3 (1+z)}{D_L^2} \Delta N_e$$  \hfill (12)

where $D_L$ is the luminosity distance. The surface area of an afterglow element as seen by the observer, $\Delta A$, is calculated self consistently in the code by projecting the elements onto a surface perpendicular to the observer line of sight. The minimum electron frequency is

$$\nu'_m = 3.5 \times 10^9 \left( \frac{x_p}{0.64} \right) \epsilon_{e,-1} n_0 \epsilon_{B,-2}^{1/2} \Gamma^3 \text{ Hz}$$  \hfill (13)

and $\nu' = \nu/\delta$ where we take $\nu = 4.4 \times 10^{14}$ Hz for the R-band. So

$$\left( \frac{\nu}{\nu_m} \right)^{-\alpha} = \left[ 7.96 \times 10^{-6} \left( \frac{x_p}{0.64} \right)^{1/2} \epsilon_{e,-1} n_0 \epsilon_{B,-2}^{1/2} \Gamma^3 \delta \right]^\alpha$$  \hfill (14)

and thus eqn. (11) scales like $L_\nu \propto \Gamma^{1+\delta} \propto R \nu^{-\alpha} \propto \Gamma^{3+\delta} + 4 t \nu^\alpha$, where $t \propto R/\Gamma/\delta$, thus demonstrating the explicit scalings and consistency with 7? For $\alpha = 1/2$ the observed flux is

$$F_\nu = 6.4 \times 10^{-57} \left( \frac{\phi_p}{0.59} \right) \left( \frac{x_p}{0.64} \right)^{1/2} \epsilon_{e,-1} n_0 \epsilon_{B,-2}^{3/4}$$

$$\times \frac{2}{1+z} \frac{D_L^2(1)}{D_L^2(z)} \int \Gamma^{5/2} \delta^{7/2} \beta^2 \frac{\Delta A}{R^2 \Delta \Omega} \Delta N_e \text{ mJy}$$  \hfill (15)

where a cosmology, $(\Omega_0, \Lambda) = (0.3, 0.7)$, was used with $H_0 = 65$ km s$^{-1}$ Mpc$^{-1}$ to give $D_L(1) = 2.2 \times 10^{28}$ cm. It is important to note that by explicitly evolving $\Gamma$ (eqn. 6) and $F_\nu$ (eqn. 15) in terms of the number of swept up electrons, $N_e \propto$ volume (eqn. 3), this formulation consistently accomodates sideways expansion of the jet (Section ??). Finally, the flux lightcurve as a function of observer time, $t_{\text{obs}}$, is calculated by

$$t_{\text{obs}} = (1+z) \int_0^R \left( 1 - \beta \cdot \hat{n} \right) \frac{dr}{\beta c}$$  \hfill (16)

To better compare physical timescales, the redshift dependence is removed from the times plotted in this paper; $t \equiv t_{\text{obs}}/(1+z)$. Thus we have caste the calculation of the afterglow lightcurve into the state variables of the problem: $R$, $\Omega$, $\Gamma$, $\delta$.

A calculation proceeds as follows. An initial afterglow is specified by $E(\theta, \phi)$ and $\Gamma_0(\theta, \phi)$ and a lateral expansion prescription and is allowed to plow into the ISM by incrementing the radius by $\Delta R/R \sim$ a few percent. The intensity, eqn. (11) and observer time, eqn. (16) are saved at each surface element. Thus a lattice of $(I_\nu, t)$ pairs are evaluated in $(\theta, \phi, R)$ space. Intensity is then interpolated on dataslices of constant observer time $t$. Finally, the total flux, $F_\nu$, at each observation is derived from eqn. (15) where surface areas, $dA$, are calculated from a projection of the positions of the observed intensities onto the observer plane of view.

3. Review of Homogeneous Jet Model

It is worthwhile to begin this discussion with a brief review of the homogeneous jet model, which is amenable to analytical calculations and thus allows for comparison and validation of numerical results with known results. The apparent surface area of the afterglow goes like $dA \propto (R \theta_A)^2$ where $\theta_A$ is the angular size of the effective viewable aperture onto the afterglow surface

$$\theta_A \approx \begin{cases} 1/\Gamma \quad &\theta_v + 1/\Gamma \ll \theta_0 \\ \theta_0 \quad &\theta_v + 1/\Gamma \gg \theta_0 \\ \text{(relativistic beaming dominated)} &\text{ (physical jet extent dominated)} \end{cases}$$  \hfill (17)

Also, one can divide the Doppler factor (eqn. 7) into asymptotic limits

$$\delta \approx 2 \left[ \frac{\Gamma}{1/(\Gamma \theta_c^2)} \right] \quad \theta_c \gg 1/\Gamma + \theta_0 \quad \text{ .}$$  \hfill (18)

Using eqn. (15) the flux at a given frequency is

$$F_\nu \sim \Gamma^{1+3\alpha} \delta^{3+3} R^3 \theta_A^2$$  \hfill (19)

and from eqn. (16) the observer time goes like

$$t \approx \frac{R}{\Gamma \delta c}$$  \hfill (20)

Before the afterglow shock has reached its deceleration radius, $R_d$, it coasts freely, $\Gamma \approx \Gamma_0$, and
so $\delta \approx \text{const.}$, thus $F_{\nu} \sim R^{3} \sim t^{3}$, where $t \sim R$. After the shock passes the deceleration radius, i.e. $R > R_d$, the radius and Lorentz factor are related by $R \propto \Gamma^{-2/3}$ (eqns. 2, 6). In this regime there exist asymptotic power-law slopes for $F(t)$ only in the simple cases where one of the three scales, $\theta_0, \theta_e, 1/\Gamma$, dominates over the other two. These three cases are summarized in Table 1 and can be seen in Fig. ??.

One characteristic timescale of the afterglow model is the deceleration time

$$t_d = \frac{R_d}{\Gamma \delta c} \approx 0.2 \left( \frac{E_{52}/n}{100 \gamma_0} \right)^{1/3} \Gamma^{-8/3} \left( 1 + \Gamma \delta^2 \theta_e^2 \right) s$$

(21)

where I define $R_d = (3E_0/4\pi \Gamma_0^2 \delta_{\text{ISM}}^2)^{1/3}$ as the radius at which $f = 1/\Gamma_0$ (eqn. 2). For jets viewed well off axis, $\theta_e \gg 1/\Gamma, \theta_0$, the flux (eqn. 19) varies like $F_d \propto \delta^{-7/2}$. Since $t \propto 1/\delta$, then $F_d \propto t^{-7/2}$.

Another key timescale is the jet-break time, $t_j$, which occurs when the shock has decelerated to a Lorentz factor $\Gamma \sim 1/\theta_0$

$$t_j = \frac{5}{4} \left( \Gamma_0 \theta_0 \right)^{8/3} t_d \approx 8.5 \left( \frac{E_{52}/n}{100 \gamma_0} \right)^{1/3} \theta_0^{8/3} \min$$

(22)

where $\theta_{0,1} = \theta_0/1^\circ$. The factor 5/4 derives from eqn. (16) and the fact that the jet-break time occurs when $1/\Gamma \sim \theta_0$ is observed at the edge of the jet rather than at the center. This is a factor of 5/2 greater than estimates derived at the jet center (?; e.g.) sph99. For $\theta_e > \theta_0$ the jet break at time $t_j$ is largely washed out, thus a more pertinent timescale in this regime is that at which the flux is a maximum. This occurs roughly when $\theta_e \sim 1/\Gamma \gg \theta_0$, where $\delta \approx 2\Gamma/(1 + \theta_e^2 \Gamma^2) \sim \Gamma$ so $F_{\text{max}} \sim \Gamma^4 \sim t^{-3/2}$, where $t \sim \Gamma^{-8/3}$.

The final phase of the afterglow is when the shock motion becomes non-relativistic, $\beta \ll 1$. This regime is beyond the purpose of this paper, however it is necessary to analytically describe the non-relativistic lightcurve behavior of the present model so to understand the asymptotic behavior of the simulations. For $\beta \ll 1$ eqns. (1) yield $\beta \propto R^{-3}$, eqn. (15) goes like $F_{\nu} \propto R^2 \beta^2$. and eqn. (16) becomes $t \propto R/\beta$. Thus the non-relativistic lightcurve of the present model is $F_{\nu,\text{non-rel.}} \propto t^{-3/4}$. Notice that this is independent of the spectral slope. This non-relativistic model is likely incomplete; for instance (?) suggest modifying the shock amplification of the shock-frame magnetic field from $B' \propto \gamma$ to $B' \propto \sqrt{\gamma (\gamma - 1)}$. Such modifications can produce substantially different behavior of the afterglow lightcurve, including a break at the relativistic/nonrelativistic transition (?). Since our understanding of such field generation is still incomplete (for example ?), here I choose not to make any such modifications. Let it suffice to understand that all post-break lightcurves modeled in this paper will asymptotically approach $F_{\nu} \propto t^{-3/4}$ as they approach the non-relativistic regime (for example, see late times of the bottom plot of figs. ??, ??, ??, ??).

In order to validate the results presented here, I compared with existing solutions. Comparing these homogeneous afterglow solutions with those of ?), I find very good agreement when one accounts for three parameter differences. As mentioned above, eqn. (11) is valid for observed frequencies above the synchrotron frequency, $\nu > \nu_m$. This simplification is necessary to remove unwanted spectral breaks when attempting to study the cause and quality of dynamical jet breaks. This limit corresponds to the case when parameter $\phi \gg 1$ in ?). Secondly, ?) take the velocity of the emitting electrons behind the shock to be $\Gamma/\sqrt{2}$, where I have assumed it to be $\Gamma$. I have tested both choices (the former can be implemented by effectively replacing $\Gamma \rightarrow \Gamma/\sqrt{2}$ in eqn. (15)) and find a quantitative difference in the flux surface, but the difference is minimal for the integrated flux; so the shape of the lightcurves is not significantly affected by this choice, as can be seen from

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Table 1: The three asymptotic limits where one of the three angular scales, $\theta_0, \theta_e, 1/\Gamma$, dominates the other two. Asymptotic expressions for $\theta_{\Lambda}$ and $\delta$ are given by eqns. 17 and 18 respectively. In the last column, the expression in parenthesis is for the spectral slope $\alpha = 1/2$ (eqn. 11).

<table>
<thead>
<tr>
<th>limit</th>
<th>$\theta_{\Lambda}$</th>
<th>$\delta$</th>
<th>Flux ($\alpha = 1/2$)</th>
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<td>$\theta_0 \gg \theta_e, 1/\Gamma$</td>
<td>$1/\Gamma$</td>
<td>$2\Gamma$</td>
<td>$t^{-3\alpha/2}$ ($t^{-3/4}$)</td>
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<td>$1/\Gamma \gg \theta_e, \theta_0$</td>
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<td>$2\Gamma$</td>
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<tr>
<td>$\theta_e \gg \theta_0, 1/\Gamma$</td>
<td>$\theta_0$</td>
<td>$2/(\Gamma \theta_e^2)$</td>
<td>$t^{3(2-\alpha)}$ ($t^{9/2}$)</td>
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