Three-Loop Anomalous Dimension of the Heavy Quark Pair Production Current in Non-Relativistic QCD

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Abstract

The three-loop non-mixing contributions to the anomalous dimension of the leading order quark pair production current in non-relativistic QCD are computed. It is demonstrated that the renormalization procedure can only be carried out consistently if the dynamics of both soft and the ultrasoft degrees of freedom is present for all scales below the heavy quark mass, and if the soft and ultrasoft renormalization scales are always correlated.
I. INTRODUCTION

The lineshape scan of the threshold top pair production cross section constitutes an integral part of the top quark physics program at a future $e^+e^-$ or $\gamma\gamma$ collider.\cite{1, 2, 3} Because in the Standard Model the top quark width $\Gamma_t \approx 1.5$ GeV is larger than the typical hadronization energy $\Lambda_{QCD}$, it is expected that the lineshape is a smooth function of the c.m. energy, and that non-perturbative effects are strongly suppressed. From the rise of the cross section a precise measurement of the top quark mass will be possible, while from the shape and the normalization of the cross section one can extract the top quark Yukawa coupling $y_t$, the top width and the strong coupling.\cite{4} Past fixed order next-to-next-to-leading order (NNLO) computations of the cross section have shown that a measurement of the top quark mass in a threshold mass scheme with theoretical uncertainties of 100-200 MeV or better are feasible.\cite{5} However, in the fixed order approach the theoretical uncertainty of the normalization of the NNLO cross section were estimated at the 20% level,\cite{5} which would jeopardize competitive measurements of $y_t$, $\Gamma_t$ or $\alpha_s$.

Recently, the renormalization-group-improved $e^+e^-$ top threshold cross section was computed\cite{6, 7} in the framework of an effective theory for non-relativistic heavy quark pairs, called vNRQCD.\cite{8, 9, 10, 11, 12, 13} This effective theory describes the dynamics of heavy quarkonium systems, when the hierarchy of scales $m \gg mv \gg mv^2 \gg \Lambda_{QCD}$ is satisfied, $m$ being the mass and $v$ the average c.m. velocity of the quarks. The matching is carried out at the hard scale $\mu = m$ onto a potential-like theory with both soft and ultrasoft degrees of freedom. For scales $\mu < m$ the correlation of energy and momenta is accounted for since the ultrasoft and soft renormalization scales $\mu_U$ and $\mu_S$ are related, $\mu_U = \mu_S^2/m \equiv mv^2$. The running in vNRQCD is expressed in the dimensionless scaling parameter $\nu$. All operators (and their coefficients) are evolved from $\nu = 1$ to $\nu \sim v$ of order of the average c.m. velocity of the quarks, where matrix elements are free from large logarithmic terms. In dimensional regularization the factors of $\mu_U$ and $\mu_S$ multiplying operators in the renormalized effective Lagrangian are determined uniquely from the $\nu$ power counting in $d$ dimensions.\cite{10, 13}

In renormalization-group-improved perturbation theory the expansion of the normalized cross section $R$ takes the parametric form

$$R = \frac{\sigma_{tt}}{\sigma_{\mu^+\mu^-}} = v \sum_k \left(\frac{\alpha_s}{v}\right)^k \sum_i (\alpha_s \ln v)^i \times \left\{1 (\text{LL}); \alpha_s, v (\text{NLL}); \alpha_s^2, \alpha_s v, v^2 (\text{NNLL})\right\},$$

(1)

where $v \ll 1$ is the top quark velocity and where the indicated terms are of leading logarithmic (LL), next-to-leading logarithmic (NLL), and next-to-next-to-leading logarithmic (NNLL) order. In Refs.\cite{6, 7, 13} all logarithms were summed in the Wilson coefficients of the operators that contribute to the cross section at NNLL order except for the Wilson coefficient $c_1$ of the leading order spin-triplet current

$$J_{1,p} = \psi_p^\dagger \sigma(i\sigma_2)\chi_p^\ast,$$

(2)

for which only the NLL anomalous dimension was known.\cite{8}. In Refs.\cite{6, 7, 13} it was shown that the summation of logarithms leads to a significant reduction of the normalization uncertainties, and a theoretical uncertainty of 3% was estimated for the NNLL cross section in $e^+e^-$ annihilation. The computation of the NNLL anomalous dimension of the current $J_{1,p}$, is an important task for the determination of the full NNLL order cross section and for a cross check of the error estimate made in Refs.\cite{6, 7, 13}. 

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The evolution of $c_1(\nu)$ is obtained by integrating the anomalous dimension

$$\nu \frac{\partial}{\partial \nu} \ln [c_1(\nu)] = \gamma_{c_1}^{\mathrm{NLL}}(\nu) + \gamma_{c_1}^{\mathrm{NNLL}}(\nu) + \ldots.$$  (3)

The LL order anomalous dimension is zero. The NLL order term reads

$$\gamma_{c_1}^{\mathrm{NLL}}(\nu) = - \frac{\mathcal{V}_c^{(s)}(\nu)}{16\pi^2} \left[ \frac{\mathcal{V}_c^{(s)}(\nu)}{4} + \mathcal{V}_2^{(s)}(\nu) + \mathcal{V}_r^{(s)}(\nu) + \mathbf{S}^2 \mathcal{V}_s^{(s)}(\nu) \right]$$

$$+ \alpha_s^2(m\nu) \left[ \frac{C_F}{2} (C_F - 2 C_A) \right] + \alpha_s^2(m\nu) \left[ 3\mathcal{V}_s^{(s)}(\nu) + 2\mathcal{V}_k^{(s)}(\nu) \right],$$  (4)

where $\mathbf{S}^2 = 2$ is the squared quark total spin operator for the spin-triplet configuration.\(^1\) The terms $\mathcal{V}_c^{(s)}$ and $\mathcal{V}_2^{(s)}$, $\mathcal{V}_r^{(s)}$, $\mathcal{V}_s^{(s)}$ are the color singlet Wilson coefficients of the potentials of order $\alpha_s v^{-1} (1/k^2)$ and $\alpha_s v (1/m^2, (p^2 + p'^2)/(2m^2 k^2), \mathbf{S}^2/m^2)$, respectively\(^2\),\(^3\),\(^4\),\(^5\), while $\mathcal{V}_{k1,k2}^{(s)}$ are the Color singlet Wilson coefficients of the sum operators $\mathcal{O}_{k1}^{(1)}$, $\mathcal{O}_{k2}^{(1)}$ introduced in Ref.\(^6\). In the convention for the vNRQCD operator basis used in Refs.\(^7\),\(^8\),\(^9\),\(^10\),\(^11\),\(^12\) the potentials at order $\alpha_s^2 v^0$ contained operators having the momentum dependence $1/|k|$, $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ being the momentum transfer, which mixed into $\gamma_{c_1}$ at NLL order. For technical reasons explained below in Sec. III B we adopt in this work the convention where all $\alpha_s^2 v^0$ potentials are represented by sum operators in analogy to $\mathcal{O}_{k1}^{(1)}$, $\mathcal{O}_{k2}^{(1)}$. Instead of the $1/|k|$ potentials used in Refs.\(^7\),\(^8\),\(^9\),\(^10\),\(^11\),\(^12\), we thus have the sum operators $\mathcal{O}_{k}^{(1)}$ and $\mathcal{O}_{k}^{(T)}$ giving the contribution

$$\Delta \mathcal{C}_p = \mathcal{V}_k^{(1)} \mathcal{O}_{k}^{(1)} + \mathcal{V}_k^{(T)} \mathcal{O}_{k}^{(T)}$$

to the vNRQCD Lagrangian with Wilson coefficients $\mathcal{V}_k^{(1)}$ and $\mathcal{V}_k^{(T)}$, respectively. At LL order the coefficients are in agreement with the convention in Ref.\(^6\) and contribute to the second term on the RHS of Eq. (1). The explicit form for the operators is given in the appendix. Note that in the following we frequently refer to the sums over field indices in the sum operators as loop integrals in order to simplify the presentation.

The mixing displayed in Eq. (4) arises from two-loop vertex diagrams containing only potential loops and insertions of the Coulomb potential, the subleading heavy quark kinetic energy operator and the $1/m$-suppressed potentials.\(^6\) At NLL order, in Eq. (4), $\alpha_s$, $\mathcal{V}_c^{(s)}$, $\mathcal{V}_2^{(s)}$, $\mathcal{V}_r^{(s)}$, $\mathcal{V}_s^{(s)}$, $\mathcal{V}_{k1,k2}^{(s)}$ and $\mathcal{V}_{k}^{(1,T)}$ need to be known at LL order for all values $\nu < 1$.\(^6\),\(^7\),\(^8\),\(^9\),\(^10\),\(^11\),\(^12\),\(^13\),\(^14\)

At NNLL order there are two classes of contributions. The first is due to the two-loop mixing shown in Eq. (4) and requires the NLL results for $\alpha_s$, $\mathcal{V}_c^{(s)}$, $\mathcal{V}_2^{(s)}$, $\mathcal{V}_r^{(s)}$, $\mathcal{V}_s^{(s)}$, $\mathcal{V}_{k1,k2}^{(s)}$ and $\mathcal{V}_{k}^{(1,T)}$, but it does not modify the form of the NLL anomalous dimension. The second class requires the computation of three-loop vertex diagrams with potential loops and either soft or ultrasoft loops that require new $c_1$ counterterms. We call this class “non-mixing contributions” since it leads to genuinely new contributions in the anomalous dimension of $c_1$. By power counting there are no contributions from diagrams with three potential loops or which have both soft and ultrasoft loops.

In this paper we present the non-mixing contributions to the NNLL anomalous dimension of $c_1$. As mentioned above, all order $\alpha_s^2 v^0$ potentials are presented by sum operators.

\(^1\) In this work we keep the explicit dependence on the total quark spin operator. Thus the results can be generalized to the spin-singlet configuration where $\mathbf{S}^2 = 0$. 

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Otherwise, we use the vNRQCD velocity renormalization group with the notations and conventions of Refs. [9, 11, 13]. We employ the \( \overline{\text{MS}} \) scheme in \( d = 4 - 2\epsilon \) dimensions.

The outline of the paper is as follows: In Sec. II, we describe the method we have used to carry out the computations and illustrate it by rederiving the NLL anomalous dimension of \( c_1 \). Based on renormalization group invariance we also predict the NNLL order \( 1/\epsilon^2 \) term of the renormalization constant of \( c_1 \). In Sec. III, we present and discuss our result for the NNLL non-mixing contributions of the anomalous dimension of \( c_1 \). Section III A and III B are devoted to the contributions involving the dynamics of ultrasoft and soft degrees of freedom, respectively. The contributions to the NNLL renormalization group equation of \( c_1 \) are determined and discussed in Sec. III C and the modifications of the formulas for the NNLL heavy quark pair production cross section at threshold in \( e^+e^- \) annihilation are discussed in Sec. III D. Our conclusions are given in Sec. IV. The paper has three appendices where we have collected formulas for the reader interested in the details of the computations.

II. METHOD OF THE COMPUTATION

The standard method to determine the renormalization constant of the current \( J_{1,p} \) consists of computing the overall UV-divergences of quark-antiquark-to-vacuum on-shell matrix elements of spin-triplet currents at a certain loop order including lower order counterterm diagrams needed to subtract the subdivergences. There are, however, technical complications in this approach due to the existence of IR-divergent Coulomb phases for on-shell quarks and from the fact that the vertex diagrams in general depend on three physical scales, the mass \( m \), the c.m. energy \( E \) and the external quark momentum \( p \). Note that imposing the on-shell condition \( p^2 = mE \) on the quark-antiquark-to-vacuum matrix elements in dimensional regularization, yields IR-divergent \( 1/\epsilon^n \) poles that are very difficult to separate from the UV-divergences. In practice it is therefore necessary to compute the quark-antiquark-to-vacuum matrix elements in an asymptotic expansion for \( p^2 - m^2 \ll mE \), where the IR-divergent Coulomb phases manifest themselves as powers of \( \ln((p^2 - mE)/mE) \). An efficient way to avoid these complications and, in addition, to reduce the number of diagrams that have to be computed is to consider current correlator graphs rather than the vertex diagrams. The correlator graphs are obtained from closing the external quark lines of the vertex diagrams with an additional insertion of the current \( J_{1,p} \). This means that one has to determine diagrams with one more loop, but it reduces the number of diagrams to be considered, eliminates the quark momentum \( p \) as an external scale and avoids the IR-divergent Coulomb phases. Moreover, since all IR-divergences cancel, using correlator graphs it is not necessary to distinguish on- and off-shell contributions since any off-shell term that could be relevant for the renormalization constant leads to scaleless integrals in dimensional regularization which automatically vanish. In this approach the three-loop (NNLL) renormalization constant of the current \( J_{1,p} \) is obtained from the subdivergences in four-loop correlator diagrams that remain after the one- and two-loop subdivergences have been subtracted. The surviving four-loop overall divergences are canceled by external vacuum-type diagrams and not related to the renormalization of the current \( J_{1,p} \). The method applies analogously at any order. Interestingly, the overall divergences of the four-loop correlator graph vanish in dimensional regularization because the graphs are non-analytic in the c.m. energy, and there are no operators in the effective theory that could absorb the overall divergences. The result for the renormalization constant obtained by this method will agree with the one obtained from quark-antiquark-to-vacuum on-shell matrix elements.
For illustration let us reconsider the computation of the NLL renormalization constant that leads to Eq. (11). The relevant correlator diagrams are shown in Fig. 1. Four-quark interactions without label refer to the Coulomb potential. Crosses on quark lines refer to insertions of the quark kinetic energy at subleading order. Here and throughout the paper, combinatorial factors and diagrams obtained by flipping the graphs left-to-right and up-to-down are to be understood. There are no one-loop subdivergences that have to be subtracted in this case. Adopting the form

\[ c_1^0 = c_1 + \delta c_1 = Z_{c_1} c_1 \]  

for the unrenormalized Wilson coefficient of the current \( J_{1,p} \) with the renormalization constant written as

\[ Z_{c_1} = 1 + \frac{\delta z_{c_1}^{\text{NLL}}}{\epsilon} + \left( \frac{\delta z_{c_1}^{\text{NNLL,2}}}{\epsilon^2} + \frac{\delta z_{c_1}^{\text{NNLL,1}}}{\epsilon} \right) + \ldots, \]  

one obtains

\[ \delta z_{c_1}^{\text{NLL}} = \frac{1}{4} \gamma_{c_1}^{\text{NLL}}(\nu), \]

where \( \gamma_{c_1}^{\text{NLL}}(\nu) \) is given in Eq. (11). Taking into account the velocity renormalization group equations for the vNRQCD coefficients in \( d = 4 - 2\epsilon \) dimensions (\( \nu \frac{d}{dt} g_s(mv) = -\epsilon g_s(mv) + \ldots, \nu \frac{d}{dt} g_s(mv^2) = -2\epsilon g_s(mv^2) + \ldots \), etc.) one can derive Eq. (11) using that \( c_0 \) is renormalization group invariant. From the same relation and the fact that all \( 1/\epsilon^n \) \((n = 1, 2, \ldots)\) terms cancel in the renormalization group equations one also finds the coefficient of the \( 1/\epsilon^2 \) term at NNLL order \((a_s \equiv \alpha_s(mv), a_u \equiv \alpha_s(mv^2))\),

\[
\delta z_{c_1}^{\text{NNLL,2}} = -\frac{a_s^2 a_u}{24\pi} C_F \left( 2C_F^2 + 3C_A C_F + C_A^2 \right) - \frac{a_s^2}{192\pi^2} C_F \beta_0 \left( V_2^{(s)} + V_r^{(s)} + S V_s^{(s)} \right) \\
+ \frac{a_s^3}{48\pi} \beta_0 \left\{ C_A C_F + \frac{C_F^2}{4} \left[ 3 - c_D + 2C_F^2 \left( 1 - \frac{2}{3} S^2 \right) \right] - 2(3V_{k1}^{(s)} + 2V_{k2}^{(s)}) \\
- C_F \left( C^{(2)}_{2a} - C_F (C^{(2)}_{2b} + 2C^{(2)}_{2c}) \right) \right\} \\
+ \frac{a_s^3}{288\pi} C_F \left\{ 8C_A^2 + 14C_F^2 + C_A C_F \left[ \frac{29}{2} + \frac{11}{2} c_D + c_F^2 (-13 + 7S^2) \right] \right\},
\]
where $\beta_0 = 11/3C_A - 4/3Tn_l$ is the one-loop QCD beta-function, and $c_D$ and $c_F$ are the coefficients of the Darwin and the magnetic operators of the HQET action [14]. The terms $C_{2a,2b,2c}^{(2)}$ are coefficients associated to the soft 6-field operators $O_{2a,2b,2c}^{(2)}$, which were introduced in Ref. [13], and which mix into the order $\alpha_s^3$ potential operators. The formulas for the coefficients, which all depend on the scaling parameter $\nu$, are collected in App. C. Note that at NNLL order only the diagrams that lead to non-mixing contributions of the anomalous dimension of $c_1$ can also contribute to $\delta Z_{c_1}^{NNLL,2}$. Thus, Eq. (8) provides a non-trivial cross check for our result and the consistency of the effective theory under renormalization.

III. NON-MIXING CONTRIBUTIONS AT NNLL ORDER

For the non-mixing contributions to the NNLL anomalous dimension of $c_1$ one can distinguish between four-loop correlator diagrams containing ultrasoft loops and those with soft loops. Diagrams with both ultrasoft and soft loops do not exist at this order. Note that the ultrasoft and soft non-mixing contributions are separately gauge-invariant since they are induced by ultrasoft and soft gluons, respectively, which represent different degrees of freedom in the effective theory.

A. Ultrasoft non-mixing contributions at NNLL order

For the determination of the ultrasoft non-mixing contributions we adopted the Coulomb gauge, where the propagation of the longitudinal ultrasoft gluon field component $A^0$ does
not contribute. For the transverse ultrasoft gluon field $A$ the leading order couplings to quarks and the Coulomb potential have to be taken into account. The complete set of four-loop correlator diagrams is displayed in Figs. 3. The diagrams needed to cancel the one-loop subdivergences are displayed in Figs. 3a,c,h using the graphical notation from Refs. [8, 9, 13].

Here, $\delta c_2$ denotes the counterterm of the Wilson coefficient of the $v^2$-suppressed spin-triplet current $\frac{1}{m^2} \psi^\dagger \psi \sigma (i \sigma_2) \chi^* - p$. There are no two-loop subdivergences. After the removal of the one-loop subdivergences the contribution to the renormalization factor $Z_{c_1}$ from each of the diagrams in Figs. 2 is of the form

$$Z_{c_1} = \frac{\alpha_s(\mu_U)}{4\pi} \frac{\alpha_s^2(\mu_S)}{4\pi} \left[ \frac{A}{\epsilon^2} + \frac{B}{\epsilon} \right].$$  

The contributions to $A$ and $B$ as well as the totals are given in Tab. I. The results coming from graphs 2a and 2e are zero, but non-trivial and have been obtained from explicit computation. The results do not involve factors of $\rho \equiv \gamma_E - \ln(4\pi)$ because we have implemented the $\overline{MS}$ scheme by scaling each term $\mu_S$ or $\mu_U$ with a factor $e^{\rho/2}$. Note that each counterterm diagram can in general contribute to several four-loop diagrams. For example, the counterterm diagrams involving $\delta N_{2,r,s}$ are required for the subtraction of one-loop subdivergences in the four-loop graphs 2a, 2b, and 2h, whereas the diagrams with $\delta N_{k,k_2}$ have to be considered for the graphs 2a, 2b, and 2k. On the other hand, the diagram with $\delta c_2$ is needed for the graphs 2a and 2e. The graphs 2d, 2g, and 2n do not have any one- or two-loop subdivergences. From the totals given in Tab. I one finds agreement with the prediction of the ultrasoft $1/\epsilon^2$ term made in Eq. (8).

It is an interesting conceptual aspect that the divergences in the graphs in Figs. 2 can only be renormalized consistently if the correlation of soft and ultrasoft scales, $\mu_U = \mu^2_s/m = m\nu^2$ is taken into account (see also Ref. [7]). As an example, let us consider the contribution to $\delta Z_{c_1}^{NNLL,2}$ induced by the diagram Fig. 2g in some detail. The result for the diagram reads

$$\alpha_s(\mu_U) \frac{\alpha_s^2(\mu_S)}{4\pi} \frac{m^2}{E^2} \times \left[ \frac{1}{6\epsilon^2} + \frac{1}{\epsilon} \left( \frac{1}{6} \ln \left( \frac{\mu_U^2}{E^2} \right) + \frac{1}{2} \ln \left( \frac{\mu_S^2}{p^2} \right) - 2 \ln 2 + \frac{29}{18} \right) + \ldots \right].$$

![Counterterm graphs for the removal of subdivergences in graphs of Figs. 2 and 4.](image)

FIG. 3: Counterterm graphs for the removal of subdivergences in graphs of Figs. 2 and 4. Graphs with wave function renormalization constants are to be understood.
The corresponding counterterm contribution from the graph in Fig. 3c reads

\[
\text{Fig. 3c} = \alpha_s(\mu_U) \beta_0(\mu_S) \frac{m_p}{4\pi} \left[ -\frac{1}{3\epsilon^2} + \frac{1}{\epsilon} \left( -\ln \left( \frac{\mu^2}{p^2} \right) + 2 \ln 2 - \frac{7}{3} \right) + \ldots \right] \tag{11}
\]

and the sum including the \( c_1 \) counterterm graph from Fig. 3i gives

\[
\text{Fig. 2g + Fig. 3c + Fig. 3i} = -i C_A^2 C_F \frac{\alpha_s(\mu_U) \beta_0(\mu_S) m_p}{4\pi} \left[ -\frac{1 + 12\delta}{6\epsilon^2} \right.

+ \frac{1}{\epsilon} \left( \frac{1}{6} \ln \left( \frac{\mu^2}{E^2} \right) - \left( \frac{1}{2} + 2A\delta \right) \ln \left( \frac{\mu_S^2}{p^2} \right) + 4A\delta \left( \ln 2 - 1 \right) - 2B\delta - \frac{13}{18} \right) + \ldots \left. \right] \tag{12}
\]

where \( \delta = (C_A^2 C_F)^{-1} \). This leads to the following contribution for the \( c_1 \) counterterm

\[
A = -\frac{1}{12} C_A^2 C_F,
\]

\[
B = \frac{1}{6} \left[ \ln \left( \frac{m \mu_U}{\mu_S} \right) - \ln 2 - \frac{7}{6} \right] C_A^2 C_F. \tag{13}
\]
FIG. 4: Four-loop graphs for the calculation of the soft non-mixing contributions of the NNLL anomalous dimension of $c_1$.

FIG. 5: (a) Six-field operators that contribute to the soft running of $\mathcal{O}^{1,T}_{k,k_1,k_2}$ through the graphs in (b).

The dependence on $\ln \mu_S$ and $\ln \mu_U$ in $B$ vanishes only if the correlation $\mu_U = \mu_S^2/m$ is accounted for in Fig. 2g as well as in the counterterm graphs needed to cancel the subdivergences. This is the case for all graphs in Figs. 2 that contribute to $B$. This demonstrates that the correlation of the soft and the ultrasoft renormalization scales, which is unambiguously from the mass dimension and the $v$ counting of the operators (see Ref. [10, 13]), is also needed to ensure the renormalizability and consistency of the effective theory. It also shows that the correlation of the soft and ultrasoft scales for all scales below $m$ is an integral property of the effective theory for nonrelativistic dynamic (i.e. non-static) heavy quark pairs.

B. Soft non-mixing contributions at NNLL order

The four-loop correlator diagrams relevant for the soft non-mixing contributions are displayed in Figs. 4 where we have used again the graphical notations of Refs. [8, 9, 13]. Diagrams with insertions of two soft vertices involving the functions $U^{(\sigma)}$, $W^{(\sigma)}$, $Y^{(\sigma)}$, $Z^{(\sigma)}$ [9, 10] are shown in Figs. 4a-e. Insertions of the 6-field operators $\mathcal{O}_{2,2A,2c}^{(2)}$ from Ref. [13] are shown in Fig. 4f. In Fig. 4g there are also insertions of soft 6-field sum operators (see Fig. 5a), which, upon closing the soft lines (Fig. 5b), contribute to the soft running of the operators $\mathcal{O}^{1,T}_{k,k_1,k_2}$. (The ultrasoft running of the operators $\mathcal{O}^{1,T}_{k,k_1,k_2}$ was discussed in detail in Ref. [13].) The four-quark matrix elements of these operators, with the intermediate sum and the soft loop integration being carried out, are given in App. B. For a part of these matrix elements, as explained in App. B we used results obtained earlier in Ref. [16] based on the asymptotic expansion of QCD diagrams. To ensure consistency of the effective the-
A − B

...for the quarks is defined as $S_{\alpha}$. All for that the dynamics of soft and ultrasoft degrees of freedom are both needed simultaneously under renormalization at non-trivial subleading order. In particular, the agreement shows the common convention of non-relativistic quantum mechanics and has also been adopted for the vNRQCD action [9, 10]. We also note that other schemes for the quark spin in the effective theory are possible.

Like for the ultrasoft contributions, we find agreement with the prediction of the soft $1/\epsilon^2$ terms made in Eq. (8). This demonstrates the consistency of the vNRQCD action under renormalization at non-trivial subleading order. In particular, the agreement shows that the dynamics of soft and ultra-soft degrees of freedom are both needed simultaneously for all scales below heavy quark mass $m$.

<table>
<thead>
<tr>
<th>Graphs</th>
<th>$A'$</th>
<th>$B'$</th>
</tr>
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<tbody>
<tr>
<td>a</td>
<td>$\frac{7}{36} C_F^3 + \frac{3}{21} C_A C_F^2 (1 - \frac{1}{3} C_F^2 (6 + S^2))$</td>
<td>$\frac{5}{6} C_F^3 - \frac{1}{18} C_A C_F^2 (11 - \frac{17}{36} C_F^2 (3 - 4 S^2))$</td>
</tr>
<tr>
<td>b</td>
<td>$-\frac{1}{360 \pi \epsilon} \beta_0 C_F (\gamma^{(s)}_2 + \gamma^{(s)}_3 + S^2 \gamma^{(s)}_4)$</td>
<td>$\frac{1}{18} (\beta_0 - 4 C_A) C_F (\gamma^{(s)}_2 + \gamma^{(s)}_3 + S^2 \gamma^{(s)}_4)$</td>
</tr>
<tr>
<td>c+d+e</td>
<td>$\frac{1}{24} \beta_0 C_F^2 c$</td>
<td>$-\frac{1}{18} (\beta_0 - 4 C_A) C_F^2$</td>
</tr>
<tr>
<td>f</td>
<td>$-\frac{1}{12} \beta_0 C_F (C_{2 a}^{(2)} - C_F C_{2 b}^{(2)} - 2 C_F C_{2 c}^{(2)})$</td>
<td>$-\frac{1}{18} (\beta_0 + 8 C_A) C_F (C_{2 a}^{(2)} - C_F C_{2 b}^{(2)})$</td>
</tr>
<tr>
<td>g</td>
<td>$\frac{1}{15} \beta_0 C_F (C_A - \frac{1}{2} C_F) + \frac{1}{9} C_A C_F (2 C_F + C_A)$</td>
<td>$\frac{1}{15} \beta_0 C_F (C_F - \frac{37}{9} C_A) + \frac{1}{9} C_A C_F (C_F + \frac{101}{16} C_A)$</td>
</tr>
<tr>
<td></td>
<td>$-\frac{1}{6} \beta_0 (3 \gamma^{(s)}<em>{k1} + 2 \gamma^{(s)}</em>{k2})$</td>
<td>$+ \frac{1}{6} (7 \beta_0 - 16 C_A) \gamma^{(s)}<em>{k1} + \frac{1}{36} (25 \beta_0 - 64 C_A) \gamma^{(s)}</em>{k2}$</td>
</tr>
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TABLE II: Contributions to Eq. (14) from the graphs in Fig. 4.
C. Solution of the anomalous dimension at NNLL order

From the results shown in Tabs. I and II we find the following result for the non-mixing contributions of the NNLL anomalous dimension of \( c_1 (a_s \equiv \alpha_s(m_\nu), a_u \equiv \alpha_s(m_\nu^2)) \),

\[
\gamma_{c_1, nm}^{\text{NNLL}}(\nu) = \frac{a_s^2 a_u}{\pi} \left[ 2 C_F (2C_F^2 - C_A C_F - C_A^2) \ln 2 - \frac{1}{9} C_F (38C_F^2 + 27C_A C_F + C_A^2) \right] \\
+ \frac{a_s^2}{24\pi^2} (\beta_0 - 4C_A) C_F (V_2^{(s)} + V_r^{(s)} + S^2 V_s^{(s)}) \\
- \frac{a_s^3}{12\pi} (\beta_0 + 8C_A) C_F (C_{2a}^{(2)} - C_F(C_{2b}^{(2)} - 4C_{2c}^{(2)})) + 4 \frac{a_s^3}{\pi} C_A C_F^2 C_{2c}^{(2)} \\
+ \frac{a_s^3}{4\pi} (7\beta_0 - 16C_A) V_{k1}^{(s)} + \frac{a_s^3}{24\pi} (25\beta_0 - 64C_A) V_{k2}^{(s)} \\
- \frac{a_s^3}{48\pi} \beta_0 \left\{ \frac{37}{2} C_A C_F + C_F^2 \left[ 9 + c_D - \frac{2}{3} c_F \left( 3 - 2 S^2 \right) \right] \right\} \\
+ \frac{a_s^3}{48\pi} \left\{ \frac{101}{2} C_A C_F + C_A C_F^2 \left[ 13 + \frac{11}{3} c_D - \frac{1}{3} c_F \left( 5 + 8S^2 \right) \right] + 60C_F^3 \right\}.
\] (15)

Let us write the solution of Eq. (8) for \( \nu < 1 \) as

\[
\ln \left[ \frac{c_1(\nu)}{c_1(1)} \right] = \xi_{\text{NLL}}(\nu) + \left( \xi_{\text{nm}}^{\text{NNLL}}(\nu) + \xi_{\text{nm}}^{\text{NNLL}}(\nu) \right) + \ldots. \quad (16)
\]

The NLL order term \( \xi_{\text{NLL}}(\nu) \) was determined in Refs. [13, 17]. From Eq. (15) we find the following form for the NNLL non-mixing term,

\[
\xi_{\text{nm}}^{\text{NNLL}}(\nu) = b_2 \alpha_s(m)^2 (1 - z) + b_3 \alpha_s(m)^2 (1 - z^2) + b_4 \alpha_s(m)^2 \left[ 1 - z + 2 \ln(w) \right] \\
+ b_5 \alpha_s(m)^2 \left[ \frac{5}{2} - 2z - \frac{1}{2} z^2 + (4 - z^2) \ln(w) \right] \\
+ b_6 \alpha_s(m)^2 \left[ 1 - z^2 - 2C_A/\beta_0 \right] + b_7 \alpha_s(m)^2 \left[ 1 - z^2 - 13C A/(6\beta_0) \right],
\] (17)
where the coefficients $b_i$ read

\[
\begin{align*}
 b_2 &= \frac{C_F^2 (\beta_0 - 4C_A)}{6\beta_0^2 (6\beta_0 - 13C_A)(\beta_0 - 2C_A)} \left\{ \frac{6\beta_0^2}{2} \left[ 2C_F + C_A(S^2 - 3) \right] \right. \\
& \quad + \beta_0 C_A \left. \left[ -74C_F + C_A(42 - 13S^2) \right] + C_A^2(100C_F - 9C_A) \right\}, \\
 b_3 &= \frac{C_F}{3744C_A \beta_0^2} \left\{ -3\beta_0^2 \left[ 481C_A^2 + 350C_A C_F - 64C_F^2 \right] \\
& \quad + \beta_0 C_A \left[ 3939C_A^2 + 2474C_A C_F + 2648C_F^2 \right] + 48C_A^2 C_F \left[ 100C_F - 9C_A \right] \right\}, \\
 b_4 &= \frac{2C_F}{9\beta_0} \left\{ C_A^2 \left[ 1 + 18 \ln 2 \right] + 9C_A C_F \left[ 3 + 2 \ln 2 \right] - 2C_F^2 \left[ -19 + 18 \ln 2 \right] \right\}, \\
 b_5 &= \frac{C_F}{18\beta_0^2} \left\{ \beta_0 \left[ 17C_A^2 + 30C_A C_F + 4C_F^2 \right] - 32C_A \left[ C_A^2 + 3C_A C_F + 2C_F^2 \right] \right\}, \\
 b_6 &= \frac{C_F^2 (5C_A + 8C_F)}{144(\beta_0 - 2C_A)(\beta_0 - C_A)} \left\{ \beta_0^2 \left[ -3 + 4S^2 \right] + 6\beta_0 C_A \left[ 9 - 10S^2 \right] \\
& \quad + 4C_A^2 \left[ -33 + 32S^2 \right] \right\}, \\
 b_7 &= -\frac{C_F^2 (5C_A + 8C_F)}{39C_A (6\beta_0 - 13C_A)(12\beta_0 - 13C_A)} \left\{ 18\beta_0^2 - 171\beta_0 C_A + 385C_A^2 \right\}. \quad (18)
\end{align*}
\]

Note that $b_4$ originates exclusively from the ultrasoft corrections determined in Sec. 111 A.

One can also determine the NNLL mixing contributions of the anomalous dimension at the hard scale $\nu = 1$ since the corresponding matching conditions of the couplings appearing in Eq. (11) are available. [10, 13] The result reads

\[
\gamma_{c_{1,m}}^{\text{NNLL}}(\nu = 1) = \frac{\alpha_s(m)^3}{48\pi} C_F^2 \left[ C_A \left( 16S^2 - 3 \right) + 4C_F \left( 5 - 2S^2 \right) - \frac{16}{5} T \right]. \quad (19)
\]

From Eq. (19) we can determine the $\alpha_s(m)^3 \ln \nu$ term of the NNLL mixing contribution, which is the first term in the expansion in terms of $\alpha_s(m)$,

\[
\xi_{m}^{\text{NNLL}}(\nu) = \gamma_{c_{1,m}}^{\text{NNLL}}(1) \ln \nu + \mathcal{O}(\alpha_s^4 \ln^2 \nu). \quad (20)
\]

Numerically the NNLL non-mixing contributions to $c_1$ are rather large and entirely dominated by the first term in Eq. (19), which originates from the ultrasoft corrections. In Tab. 111 the values for $\xi_{\nu}^{\text{NNLL}}(\nu)$ and $\xi_{m}^{\text{NNLL}}(\nu)$ are displayed for different $\nu$ for the top and the bottom quarks. For top quarks we find that the NNLL non-mixing contributions are of the same size as the NLL terms for the relevant region $\nu \sim v \approx 0.2$. Here, the new NNLL order corrections shift $c_1$ by about $+5\%$. However, it is not yet possible to draw phenomenological conclusions for the normalization of the top threshold cross section in $e^+e^-$ collisions from this result, since the NNLL mixing contributions for $\nu < 1$ are still unknown. Because these corrections involve two ultrasoft loop integrations arising in the NLL running of the coefficients in Eq. (1) they could potentially be large as well. It is therefore an important task to determine the mixing corrections for all scales below $m$.

For bottom quarks the NNLL non-mixing contributions are more than five times larger than the NLL terms for the relevant region $\nu \sim v \approx 0.3-0.4$. This is not unexpected
because for bottomonium systems the binding energy $\sim mv^2$ is already of order $\Lambda_{\text{QCD}}$. Thus our result seems to affirm that for $b\bar{b}$ states non-perturbative effects have a rather strong influence, and that the vNRQCD description ceases to work even for the ground state.\cite{13} However, also for the bottom quark case the knowledge of the NNLL mixing contributions might be useful to gain further insight into this issue.

$$m = 175 \text{ GeV}$$  

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TABLE III: Numerical values for $\xi_{\text{NLL}}^{\nu}$ and $\xi_{\text{NNLL}}^{\nu}$. The values for $m$ are pole masses. The numbers are obtained by evaluation of the analytic results using four-loop running for $\alpha_s$ and taking $\alpha_s^{(5)}(175 \text{ GeV}) = 0.107$ and $\alpha_s^{(4)}(4.8 \text{ GeV}) = 0.216$ as input.

### D. Production Cross Section at Threshold and Comparison

Due to the convention we use for the order $\alpha_s v^0$ potentials, our formulas for the NNLL order vector current correlator $A_1$, which is used to express the quark pair production cross section in $e^+e^-$ annihilation, slightly differs from Ref.\cite{13}. This also affects the two-loop matching condition for $c_1$. The difference arises because the corresponding matrix elements are UV-divergent and the $d$-dependent contributions that arise from summing the intermediate indices in the operators $O_k^{1,T}$ in dimensional regularization (see App. A) lead to modifications of the UV-finite terms. Altogether, we now need four different types of corrections to the current correlator, $\delta G_{\text{CACF}}^k$, $\delta G_{\text{CF2}}^k$, $\delta G^{k1}$ and $\delta G^{k2}$ to account for the corrections originating from the order $\alpha_s^2 v^0$ potentials. The results for $\delta G^{k1}$ and $\delta G^{k2}$ were given in Ref.\cite{13}. The corrections $\delta G_{\text{CACF}}^k$ and $\delta G_{\text{CF2}}^k$ arise from the terms proportional to $C_A C_F$ and $C_F^2$, respectively, in the color singlet combination of the operators $O_k^1$ and $O_k^T$. 

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and have the form

\[ \delta G_{\text{CAF}}^k(a, v, m, \nu) = -\frac{m^2}{8\pi a} \left\{ i\nu - a \left[ \ln \left( \frac{-i\nu}{\nu} \right) - \frac{5}{4} + \ln 2 + \gamma_E + \Psi \left( 1 - \frac{i\alpha}{2\nu} \right) \right] \right\}^2 + \frac{m^2}{8\pi a} \left[ -v^2 + \frac{a^2}{16} \left( \frac{1}{\epsilon^2} - \frac{3}{\epsilon} - 11 \right) \right], \]

\[ \delta G_{\text{CF2}}^k(a, v, m, \nu) = -\frac{m^2}{8\pi a} \left\{ i\nu - a \left[ \ln \left( \frac{-i\nu}{\nu} \right) - 1 + \ln 2 + \gamma_E + \Psi \left( 1 - \frac{i\alpha}{2\nu} \right) \right] \right\}^2 + \frac{m^2}{8\pi a} \left[ -v^2 + \frac{a^2}{16} \left( \frac{1}{\epsilon^2} - \frac{2}{\epsilon} - 12 \right) \right]. \]

(21)

The vector current correlator \( A_1 \) at NNLL order then reads

\[ A_1(v, m, \nu) = 6 N_c \left[ G^c(a', v, m, \nu) + \left( V_2(s)(\nu) + 2V_s^{(s)}(\nu) \right) \delta G^a(a, v, m, \nu) + V_c^{(s)}(\nu) \right] \delta G^{b}\delta G^{c} + \delta G^{\text{kin}}(a, v, m, \nu) \]

\[ - C_A C_F \alpha_s^2(m\nu) \delta G_{\text{CAF}}^k(a, v, m, \nu) + \frac{C_F^2}{2} \alpha_s^2(m\nu) \delta G_{\text{CF2}}^k(a, v, m, \nu) \]

\[ + \alpha_s^2(m\nu) V_k^{(s)}(a, v, m, \nu) + \alpha_s^2(m\nu) V_k^{(s)}(a, v, m, \nu) \]

\[ a = -\frac{1}{4\pi} V_c^{(s)}(\nu), \quad a' = -\frac{1}{4\pi} V_{ceff}^{(s)}(\nu), \]

(22)

where \( G^c, \delta G^a, \delta G^b, \delta G^{\text{kin}} \) and \( \delta G^{k1}, \delta G^{k2}, V_c^{(s)}, V_{ceff}^{(s)} \) were given in Refs. [7] and [13], respectively. The two-loop matching condition for \( c_1 \) now reads

\[ c_1(1) = 1 - \frac{2C_F}{\pi} \alpha_s(m) + \alpha_s^2(m) \left[ C_F^2 \left( \frac{\ln 2}{3} - \frac{31}{24} - \frac{2}{\pi^2} \right) + C_A C_F \left( \frac{\ln 2}{2} - \frac{5}{8} + \frac{\kappa}{2} \right) \right], \]

(23)

where the constant \( \kappa \) was determined in Ref. [18].

Recently, the order \( \alpha_s^3 \ln \alpha_s \) corrections to the heavy quarkonium partial width into a lepton pair were computed using an asymptotic expansion of QCD diagrams close to threshold [19]. In this work the summation of higher order logarithms was not attempted. The results were partly based on three-loop quark-antiquark-to-vacuum matrix elements in the off-shell limit \( mE \neq 0, p^2 = 0 \) [20]. With the result for the total cross section of quark pair production at threshold in \( e^+e^- \) annihilation in Refs. [6, 7, 13], and including the new formulas given above, the order \( \alpha_s^3 \ln \alpha_s \) corrections can also be derived by expanding out the summations contained in the Wilson coefficients for energies on the bound state poles. The contributions proportional to \( S^2 = S(S+1) \) in our result do not agree with Ref. [19] because there the terms \( S(S+1) \) do not refer to the three-dimensional quark spin operators [20] and are defined in a different scheme for the quark spin. For the \( \alpha_s^3 \ln \alpha_s \) corrections to the heavy quarkonium partial width into a lepton pair we find a discrepancy, which would correspond to the term \( \frac{1}{3} C_A^3 C_F - \frac{4}{3} C_A C_F^2 (2 - 5 \ln 2) \) being added to the constant \( C_1 \) of Ref. [19]. This term has the same sign and approximately the same size as the constant \( C_1 \) itself and affects the error estimates made in Ref. [19].

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IV. CONCLUSION

In this work we have determined, within the framework of vNRQCD, the non-mixing contributions to the NNLL anomalous dimension of the leading spin-triplet current \( J_{1,p} \), which describes the production of non-relativistic quark-antiquark pair in \( e^+e^- \) annihilation. Our result for the \( 1/\epsilon^2 \) terms of the three-loop renormalization constant of the current is consistent with the constraints of renormalization group invariance imposed on the already known NLL UV-divergences. This demonstrates that both soft and ultrasoft degrees of freedom need to be present in the effective theory for all scales below the heavy quark mass \( m \) as predicted by vNRQCD. Moreover, a sequence of different effective theories with an intermediate uncorrelated matching scale is not consistent under renormalization. Our results also show that the NNLL order renormalization constant of the current is independent of the renormalization scales for the soft and ultrasoft degrees of freedom, only if their correlation is accounted for, as given unambiguously from the mass dimension and the \( v \) counting of the operators in vNRQCD. We have derived an updated expression for the NNLL total cross section of heavy quark pair production at threshold in \( e^+e^- \) annihilation.

Acknowledgments

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APPENDIX A: CONVENTION FOR THE POTENTIAL OPERATORS \( \mathcal{O}_k^{(1,T)} \)

The order \( \alpha_s^2v^0 \) potential sum operators \( \mathcal{O}_k^{(1,T)} \) used in this work are obtained by matching to the one-loop \( \bar{Q}Q \) scattering amplitude in full QCD in the threshold expansion and carrying out only the \( dq^0 \) integration of the loop momentum \( q = (q^0, \mathbf{q}) \) for the contributions of order \( \alpha_s^2v^0 \). The explicit expression for the operators is \((g_s = g_s(m_S))\)

\[
\mathcal{O}_k^{(1)} = -\frac{g_s^{1/2}}{4m} \sum_{p,p',q} C_1 \left[ g_0 - g_2 \right] \left[ \psi_{p'}^\dagger \psi_p \chi_{-p'}^\dagger \chi_{-p} \right],
\]

\[
\mathcal{O}_k^{(T)} = -\frac{g_s^{1/2}}{4m} \sum_{p,p',q} \left[ -\frac{1}{4}(C_d - C_A) g_0 + C_A g_1 + \frac{1}{4}(C_d + C_A) g_2 \right] \left[ \psi_{p'}^\dagger T^A \psi_p \chi_{-p'}^\dagger \chi_{-p} \right],
\]

where the functions \( g_i \) have the form

\[
g_0 = \left[ (\mathbf{q} - \mathbf{p})^2 + (\mathbf{q} - \mathbf{p'})^2 - (\mathbf{p} - \mathbf{p'})^2 \right] \frac{(\mathbf{q} - \mathbf{p})^2 + (\mathbf{q} - \mathbf{p'})^2}{(\mathbf{q} - \mathbf{p})^4 (\mathbf{q} - \mathbf{p'})^4},
\]

\[
g_1 = \frac{(\mathbf{q} - \mathbf{p})^2 + (\mathbf{q} - \mathbf{p'})^2 + (\mathbf{p} - \mathbf{p'})^2}{(\mathbf{q} - \mathbf{p})^2 (\mathbf{q} - \mathbf{p'})^2 (\mathbf{p} - \mathbf{p'})^2},
\]

\[
g_2 = \frac{\mathbf{q}^2 - \mathbf{p}^2}{(\mathbf{q} - \mathbf{p})^4 (\mathbf{q} - \mathbf{p'})^2} + \frac{\mathbf{q}^2 - \mathbf{p'}^2}{(\mathbf{q} - \mathbf{p})^2 (\mathbf{q} - \mathbf{p'})^4}.
\]

\[(A1)\]
They give the contribution $\Delta \mathcal{L}_p = \mathcal{V}_k^{(1)} \mathcal{O}_k^{(1)} + \mathcal{V}_k^{(T)} \mathcal{O}_k^{(T)}$ to the vNRQCD Lagrangian. The Wilson coefficients read

$$\mathcal{V}_k^{(1)}(\nu) = \mathcal{V}_k^{(T)}(\nu) = 1.$$  \hspace{1cm} (A3)

Note that the tilde over the renormalization scales $\mu_U$ and $\mu_S$ refers to the $\overline{\text{MS}}$ definition $\mu_U, S = e^{\rho/2} \mu_{u,S}$, where $\rho = \gamma_E - \ln(4\pi)$. When $\mathcal{O}_k^{(1,T)}$ are inserted in four-quark matrix elements or used in the Schrödinger equation with the sum over the intermediate index $L$, described in App. A for $O$ running of the operators $O$, Wilson coefficients read

$$\Delta L = \sum \text{over intermediate index}$$

They give the contribution $\Delta L$ to the vNRQCD Lagrangian. The Wilson coefficients read

$$\mathcal{V}_k^{(1)}(\nu) = \mathcal{V}_k^{(T)}(\nu) = 1.$$  \hspace{1cm} (A3)

APPENDIX B: 4-QUARK MATRIX ELEMENTS OF 6-FIELD OPERATORS

The soft running of the operators $\mathcal{O}_k^{(1,T)}$, $\mathcal{O}_k^{(1)}$ and $\mathcal{O}_k^{(T)}$ originates from four-quark matrix elements of 6-field sum operators (see Fig. 5a) in close analogy to the operators $\mathcal{O}_k^{(2)}$, $\mathcal{O}_k^{(2,A,2\epsilon)}$, which contribute to the soft running of the spin-independent order $\alpha_s v (1/m^2, (p^2 + p'^2)/(2m^2k^2), S^2/m^2)$ potentials. \[^{13}\] The divergences in these 4-quark matrix elements have already been deduced earlier, but for the computation of the NNLL anomalous dimension of $c_1$ their full form is required. To be definite we call the 6-field operators $\mathcal{O}_k^{(1,T)}$, $\mathcal{O}_k^{(1)}$ and $\mathcal{O}_k^{(T)}$. The operators $\mathcal{O}_k^{(1,T)}$ and $\mathcal{O}_k^{(1)}$, $\mathcal{O}_k^{(T)}$ are responsible for the soft running of the operators $\mathcal{O}_k^{(1,T)}$ and $\mathcal{O}_k^{(1)}$, $\mathcal{O}_k^{(T)}$, respectively. They give the contribution

$\Delta \mathcal{L}_p = \mathcal{V}_k^{(1)} \mathcal{O}_k^{(1)} + \mathcal{V}_k^{(T)} \mathcal{O}_k^{(T)} + \mathcal{V}_k^{(1)} \mathcal{O}_k^{(1)} + \mathcal{V}_k^{(1)} \mathcal{O}_k^{(T)}$ to the vNRQCD Lagrangian. The operators $\mathcal{O}_k^{(1)}$ and $\mathcal{O}_k^{(T)}$ arise from a matching computation at two loops similar to the one described in App. A for $\mathcal{O}_k^{(1,T)}$. In analogy to Eq. (A3) one finds that $\mathcal{V}_k^{(1)}(\nu) = \mathcal{V}_k^{(T)}(\nu) = 1$. The leading order 4-quark matrix element for the unrenormalized color-singlet combination of $\mathcal{O}_k^{(1,T)}$ with the sum over intermediate indices and the soft loop being carried out (Fig. 5b)
can be derived in an expansion in $\epsilon$ from results given Ref. [16] and reads

$$
\langle i(\tilde{O}_k^{(1)} - C_F \tilde{O}_k^{(T)}) \rangle = i \frac{\pi \alpha_s(m \nu)^3 \bar{m}_S^2 \mu^4 \beta}{m(k^2)^{7/2-n}} C_F \left\{ \frac{1}{\epsilon} \left[ \frac{\beta_0}{4} (2C_A - C_F) + \frac{2}{3} C_A (C_A + 2C_F) \right] + \beta_0 \left[ C_A \left( \frac{25}{48} + \ln 2 \right) + C_F \left( \frac{1}{3} - \frac{1}{2} \ln 2 \right) \right] 
- C_A^2 \left( \frac{15}{16} - \frac{4}{3} \ln 2 \right) - C_A C_F \left( 3 - \frac{8}{3} \ln 2 \right) + \mathcal{O}(\epsilon) \right\},
$$

(B1)

which is sufficient for the determination of the renormalization constant for $c_1$. The divergent $\beta_0$ term is responsible for the running of $\alpha_s$ contained in the definition of $\mathcal{O}_k^{(s)}$ (Eq. (A6)), while the other divergence is related to the evolution of $\mathcal{V}_{k_1}^{(1)}$ and $\mathcal{V}_{k_2}^{(T)}$.

The renormalization of $\tilde{O}_{k_1}^{(1)}$ and $\tilde{O}_{k_2}^{(T)}$ is in complete analogy to the one of the operators $\mathcal{O}_{k_1}^{(1)}$ and $\mathcal{O}_{k_2}^{(T)}$. Thus one finds $\tilde{\mathcal{V}}_{k_1}^{(1)}(\nu) = \mathcal{V}_{k_1}^{(1)}(\nu)$ and $\tilde{\mathcal{V}}_{k_2}^{(T)}(\nu) = \mathcal{V}_{k_2}^{(T)}(\nu)$ at LL order, and the coefficients vanish at the hard scale. The leading order four quark matrix elements have the form

$$
\langle i\tilde{O}_{k_1}^{(1)} \rangle = -2^{6-n} i \frac{\alpha_s^2(m \nu)^4 \bar{m}_S^6}{m(k^2)^{7/2-n}} \frac{(1 - n)^2 (12 - 6n + n^2)}{n(n - 2) \cos(\frac{n \pi}{2})} \left[ C_A (4n - 1) - 4 T n \right] 
\times f\left( \frac{5 - n}{2}, 1 \right) f\left( \frac{n - 2}{2}, 1/2 \right) I \otimes \bar{I},
$$

$$
\langle i\tilde{O}_{k_2}^{(T)} \rangle = -2^{4-n} i \frac{\alpha_s^2(m \nu)^4 \bar{m}_S^6}{m(k^2)^{7/2-n}} \frac{(1 - n)^2 (17 - 9n + 2n^2)}{n(n - 2) \cos(\frac{n \pi}{2})} \left[ C_A (4n - 1) - 4 T n \right] 
\times f\left( \frac{5 - n}{2}, 1 \right) f\left( \frac{n - 2}{2}, 1/2 \right) T^A \otimes \bar{T}^A.
$$

(B2)

The divergences are responsible for the running of the coefficient $\mathcal{V}_c^{(T)}$ contained in the definition of $\mathcal{O}_{k_1}^{(1)}$ and $\mathcal{O}_{k_2}^{(T)}$. [13]

APPENDIX C: COLLECTION OF WILSON COEFFICIENTS

The gauge invariant HQET coefficients that appear in this work are [14]

$$
c_F(\nu) = z^{-C_A/\beta_0}, \quad c_D(\nu) = z^{-2C_A/\beta_0} + \left( \frac{20}{13} + \frac{32 C_F}{13 C_A} \right) \left[ 1 - z^{-13C_A/(6\beta_0)} \right].
$$

(C1)
For the color singlet channel the coefficients of the order \( \alpha_s\nu \) potentials relevant for our result are \([9, 13]\)

\[
\mathcal{V}_1^{(s)}(\nu) = -4\pi \ C_F \alpha_s(m) \ z \left[ 1 - \frac{8C_A}{3\beta_0} \ln(w) \right], \\
\mathcal{V}_2^{(s)}(\nu) = \pi C_F \alpha_s(m) \ (z-1) \left[ \frac{33}{13} + \frac{32C_F}{13C_A} + \frac{9C_A}{13\beta_0} - \frac{100C_F}{13\beta_0} \right] \\
- \frac{8\pi C_F(3\beta_0-11C_A)(5C_A+8C_F)}{13C_A(6\beta_0-13C_A)} \alpha_s(m) \left[ z^{1-(13C_A)/(6\beta_0)} - 1 \right] \\
- \frac{\pi C_F(\beta_0-5C_A)}{(\beta_0-2C_A)} \alpha_s(m) \left[ z^{1-2C_A/\beta_0} - 1 \right] - \frac{16\pi C_F(C_A-2C_F)}{3\beta_0} \alpha_s(m) \ z \ln(w), \\
\mathcal{V}_3^{(s)}(\nu) = \frac{-2\pi C_F}{(2C_A-\beta_0)} \alpha_s(m) \left[ C_A + \frac{1}{3}(2\beta_0-7C_A) \ z^{(1-2C_A/\beta_0)} \right], \tag{C2}
\]

whereas for the order \( \alpha_s\nu^0 \) potentials they read

\[
\mathcal{V}_{k1}^{(s)}(\nu) = \frac{8C_AC_1}{3\beta_0} \ln(w), \quad \mathcal{V}_{k2}^{(s)}(\nu) = \frac{2CA_CF(C_A+C_d)}{3\beta_0} \ln(w). \tag{C3}
\]

The coefficients associated to the 6-field operators \(O_{2A_2A_2e}^{(2)}\) read \([13]\)

\[
C_{2a}^{(2)}(\nu) = \frac{4C_1}{3\beta_0} \ln(w), \quad C_{2b}^{(2)}(\nu) = \frac{3C_A-C_d-4C_F}{3\beta_0} \ln(w), \\
C_{2c}^{(2)}(\nu) = \frac{-4C_A}{3\beta_0} \ln(w). \tag{C4}
\]

For the results we used the definitions

\[
z = \frac{\alpha_s(m\nu)}{\alpha_s(m)}, \quad w = \frac{\alpha_s(m\nu^2)}{\alpha_s(m\nu)}. \tag{C5}
\]

For SU\( (N_c)\)-QCD the color coefficients that appear are

\[
C_A = N_c, \quad C_F = \frac{N_c^2-1}{2N_c}, \quad T = \frac{1}{2}, \quad C_d = 8C_F - 3C_A, \quad C_1 = \frac{1}{2}C_FC_A - C_F^2. \tag{C6}
\]

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