The value of the $V_{ud}$ matrix element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix can be derived from nuclear superallowed beta decays, neutron decay, and pion beta decay. We survey current world data for all three. Today, the most precise value of $V_{ud}$ comes from the nuclear decays; however, the precision is limited not by experimental error but by the estimated uncertainty in theoretical corrections. Experimental uncertainty does limit the neutron-decay result, which, though statistically consistent with the nuclear result, is approximately a factor of three poorer in precision. The value obtained for $V_{ud}$ leads to a result that differs at the 98% confidence level from the unitarity condition for the CKM matrix. We examine the reliability of the small calculated corrections that have been applied to the data, and assess the likelihood of even higher quality nuclear data becoming available to confirm or deny the discrepancy. Some of the required experiments depend upon the availability of intense radioactive beams. Others are possible today.

1 Introduction

The Cabibbo-Kobayashi-Maskawa matrix$^{1,2}$ relates the quark eigenstates of the weak interaction with the quark mass eigenstates (unprimed)

\[
\begin{pmatrix}
    d' \\
    s' \\
    b'
\end{pmatrix} =
\begin{pmatrix}
    V_{ud} & V_{us} & V_{ub} \\
    V_{cd} & V_{cs} & V_{cb} \\
    V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\begin{pmatrix}
    d \\
    s \\
    b
\end{pmatrix}
\]

and, as such, the matrix is unitary. Thus there are many relationships among the nine elements of the matrix that can be tested by experiment. The leading element, $V_{ud}$, only depends on quarks in the first generation and so is the element that can be determined most precisely. Here we will discuss the current
status of \( V_{ud} \) and the unitarity test as it relates to the elements in the first row:

\[
V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 1. \tag{2}
\]

In examining the unitarity test, we will adopt the Particle Data Group (PDG02) recommendations for \( V_{us} \) and \( V_{ub} \). We note that for \( V_{us} \), PDG02 recommends only the value determined from \( K_{e3} \) decay, \( |V_{us}| = 0.2196\pm0.0023 \), arguing that the value obtained from hyperon decays suffers from theoretical uncertainties due to first-order SU(3) symmetry-breaking effects in the axial-vector couplings. It now appears, though, from preliminary results that the experimental result for the \( e3 \) branch of the \( K^+ \) decay may also be in doubt. For the time being, we continue to use the value of \( V_{us} \) from PDG02 but acknowledge that this result may have to change in future. As to \( V_{ub} \), its value is small, \( |V_{ub}| = 0.0036 \pm 0.0010 \), and consequently it has a negligible impact on the unitarity test, Eq. (2).

2 The value of \( V_{ud} \)

The value of \( V_{ud} \) can be determined from three distinct sources: nuclear superallowed Fermi beta decays, the decay of the free neutron, and pion beta decay. We discuss each in turn.

2.1 Nuclear superallowed Fermi beta decays

Nuclei have the singular advantage that transitions with specific characteristics can be selected and then isolated for study. One example is the superallowed \( 0^+ \rightarrow 0^+ \) beta transitions, which depend uniquely on the vector part of the weak interaction. Furthermore, in the allowed approximation, the nuclear matrix element for these transitions is given by the expectation value of the isospin ladder operator, which is independent of any details of nuclear structure and is given simply as an SU(2) Clebsch-Gordan coefficient. Thus, the experimentally determined \( f_t \)-values are expected to be very nearly the same for all \( 0^+ \rightarrow 0^+ \) transitions between states of a particular isospin, regardless of the nuclei involved. Naturally, there are corrections to this simple picture coming from electromagnetic effects, but these corrections are small – of order 1% – and calculable. Thus, if we write \( \delta_R \) as the nucleus-dependent part of the radiative correction, \( \Delta_R^\Lambda \) as the nucleus-independent part of the radiative correction, and \( \delta_C \) as the isospin symmetry-breaking correction, then the experimental \( f_t \)-value can be expressed as follows:
Table 1: Experimental results ($Q_{EC}$, $t_{1/2}$ and branching ratio, $R$) and calculated correction, $P_{EC}$, for $0^+ \rightarrow 0^+$ transitions. The other calculated corrections, $\delta_R$ and $\delta_C$, are given in Tables 3 and 4 respectively.

<table>
<thead>
<tr>
<th></th>
<th>$Q_{EC}$ (keV)</th>
<th>$t_{1/2}$ (ms)</th>
<th>$R$ (%)</th>
<th>$P_{EC}$ (%)</th>
<th>$ft$ (s)</th>
<th>$\mathcal{F}t$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{10}$C</td>
<td>1907.77(9)</td>
<td>19290(12)</td>
<td>1.4645(19)</td>
<td>0.296</td>
<td>3038.7(45)</td>
<td>3072.9(48)</td>
</tr>
<tr>
<td>$^{14}$O</td>
<td>2830.51(22)</td>
<td>70603(18)</td>
<td>99.336(10)</td>
<td>0.087</td>
<td>3038.1(18)</td>
<td>3069.4(26)</td>
</tr>
<tr>
<td>$^{26}$Al</td>
<td>4232.42(35)</td>
<td>6344.9(19)</td>
<td>$\geq 99.97$</td>
<td>0.083</td>
<td>3035.8(17)</td>
<td>3071.4(22)</td>
</tr>
<tr>
<td>$^{34}$Cl</td>
<td>5491.71(22)</td>
<td>1525.76(88)</td>
<td>$\geq 99.988$</td>
<td>0.078</td>
<td>3048.4(19)</td>
<td>3070.6(25)</td>
</tr>
<tr>
<td>$^{38}$K</td>
<td>6044.54(12)</td>
<td>931.95(64)</td>
<td>$\geq 99.998$</td>
<td>0.082</td>
<td>3049.5(21)</td>
<td>3070.9(27)</td>
</tr>
<tr>
<td>$^{42}$Sc</td>
<td>6425.58(28)</td>
<td>680.72(26)</td>
<td>99.9941(14)</td>
<td>0.095</td>
<td>3045.1(14)</td>
<td>3075.7(24)</td>
</tr>
<tr>
<td>$^{46}$V</td>
<td>7050.63(69)</td>
<td>422.51(11)</td>
<td>$\geq 99.97$</td>
<td>0.096</td>
<td>3044.6(18)</td>
<td>3074.4(27)</td>
</tr>
<tr>
<td>$^{50}$Mn</td>
<td>7632.39(28)</td>
<td>283.25(14)</td>
<td>99.942(3)</td>
<td>0.100</td>
<td>3043.7(16)</td>
<td>3072.9(28)</td>
</tr>
<tr>
<td>$^{54}$Co</td>
<td>8242.56(28)</td>
<td>193.270(63)</td>
<td>99.9955(6)</td>
<td>0.104</td>
<td>3045.8(11)</td>
<td>3072.1(27)</td>
</tr>
</tbody>
</table>

Average, $\mathcal{F}t = 3072.2(8)$

$\chi^2/\nu = 0.6$

\[ ft(1 + \delta_R)(1 - \delta_C) \equiv \mathcal{F}t = \frac{K}{2G_V^2(1 + \Delta^h)} = \text{constant}, \]  

where $K = 2\pi^3 \ln 2(hc)/(m_e c^2)^6$, which has the value $K/(hc)^6 = (8120.271 \pm 0.012) \times 10^{-10}$ GeV$^{-4}$; and $G_V$ is the weak vector coupling constant: $G_V = G_F V_{ud}$, with $G_F$ being the weak coupling constant from purely leptonic muon decay. Thus, to extract $V_{ud}$ from experimental data, the procedure is to determine the $\mathcal{F}t$-values for a variety of different nuclei having the same isospin, and then to test if they are self-consistent. If they are, their average is used to determine a value for $G_V$ and, from it, $V_{ud}$.

It is now convenient to separate the radiative correction into two terms:

\[ \delta_R = \delta'_R + \delta_{NS} \]  

where the first term, $\delta'_R$, is a function of the electron’s maximum energy and the charge of the daughter nucleus, $Z$; it therefore depends on the particular nuclear decay, but is \textit{independent} of nuclear structure. To emphasize the different sensitivities of the correction terms, we re-write the expression for $\mathcal{F}t$ as

\[ \mathcal{F}t \equiv ft(1 + \delta'_R)(1 + \delta_{NS} - \delta_C) \]  

where the first correction term in brackets is independent of nuclear structure, while the second incorporates the structure-dependent terms.
To date, superallowed $0^+ \rightarrow 0^+$ transitions have been measured to ±0.1% precision or better in the decays of nine nuclei ranging from $^{10}$C to $^{54}$Co. World data on $Q$-values, lifetimes and branching ratios – the results of over 100 independent measurements – were thoroughly surveyed in 1989 and then updated several times since then, most recently at the time of the last WEIN conference. These results appear in the first three columns of Table 1. Using the calculated electron-capture probabilities given in the next column, we obtain the “uncorrected” $\mathcal{F}t$-values listed in column 5 with partial half-lives determined from the formula $t = t_{1/2}(1 + P_{EC})/R$.

We save a detailed discussion of the radiative and isospin symmetry-breaking corrections for Sec. 3. In the present context, though, it should be noted that the values for $\delta_R$ and $\delta_C$ used to derive $\mathcal{F}t$ were taken from the last column of Tables 3 and 4, respectively; each value results from more than one independent calculation. In the case of $\delta_R$, the calculations are in complete accord with one another; for $\delta_C$, we have used an average of two independent calculations with assigned uncertainties that reflect the (small) scatter between them. Thus, in a real sense, both experimentally and theoretically, the $\mathcal{F}t$-values given in Table 1 and plotted in Fig. 1 represent the totality of
current world knowledge. The uncertainties reflect the experimental uncertainties and an estimate of the relative theoretical uncertainties in $\delta_C$. There is no statistically significant evidence of inconsistencies in the data ($\chi^2/\nu = 1.1$), thus verifying the expectation of CVC at the level of $3 \times 10^{-4}$, the fractional uncertainty quoted on the average $F_t$-value.

In using this average $F_t$-value to determine $G_V$, we must account for additional uncertainty: viz

$$F_t = 3072.3 \pm 0.9 \pm 1.1,$$

(6)

where the first error is the statistical error of the fit (as illustrated in Fig. 1), and the second is an error related to the systematic difference between the two calculations of $\delta_C$ by Towner and Hardy $^8$ and by Ormand and Brown $^9$ that we have combined in reaching this result. (For a more complete discussion of how we treat these theoretical uncertainties, see reference $^5$.) We now add the two errors linearly to obtain the value we use in subsequent analysis:

$$F_t = 3072.3 \pm 2.0.$$  

(7)

The value of $V_{ud}$ is obtained by relating the vector constant, $G_V$, determined from this $F_t$ value, to the weak coupling constant from muon decay, $G_F/(\hbar c)^3 = (1.16639 \pm 0.00001) \times 10^{-5}$ GeV$^{-2}$, according to:

$$V_{ud}^2 = \frac{K}{2G_F^2(1 + \Delta_N^V)F_t}.$$  

(8)

The result obtained is

$$|V_{ud}| = 0.9740 \pm 0.0005,$$  

[Nuclear]  

(9)

where the nucleus-independent radiative correction has been set at

$$\Delta_N^V = (2.40 \pm 0.08)\%.$$  

(10)

Note this value differs slightly (but within errors) from an earlier value $^{10}$ because of the decision by Sirlin $^{11}$ to centre the cut-off parameter $m_A$, where $(m_{a_1}/2) \leq m_A \leq 2m_{a_1}$, exactly at the $a_1$-meson mass when evaluating the axial contribution to the radiative-correction loop graph.

From the value of $V_{ud}$ given in Eq. (9), the unitarity sum, Eq. (2), becomes

$$\sum_i V_{ui}^2 = 0.9968 \pm 0.0014,$$  

[Nuclear]  

(11)
which fails to meet unity by 2.2 standard deviations. In connection with this result, we note the following two points:

(a) The error bar associated with $|V_{ud}|$ in Eq. (9) is not predominantly experimental in origin. In fact, if experiment were the sole contributor, the uncertainty would be only $\pm 0.0001$. The largest contributions to the $|V_{ud}|$ error bar come from $\Delta^A_R$ ($\pm 0.0004$) and $\delta_C$ ($\pm 0.0003$).

(b) The unitarity result in Eq. (11) depends on the values of nuclear structure-dependent corrections. In Sec. 3 we will examine whether the failure to meet unitarity can be repaired by reasonable adjustments to these corrections. Our conclusion is largely negative. Other speculative possibilities are presented in Sec. 4.

2.2 Neutron decay

On the one hand, free neutron decay has an advantage over nuclear decays since there are no nuclear-structure dependent corrections to be calculated. On the other hand, it has the disadvantage that it is not purely vector-like but has a mix of vector and axial-vector contributions. Thus, in addition to a lifetime measurement, a correlation experiment is also required to separate the vector and axial-vector pieces. Both types of experiment present serious experimental challenges. The value of $V_{ud}$ is determined from the expression

$$V_{ud}^2 = \frac{K/\ln 2}{G_A^2 (1 + \Delta^A_R)(1 + 3\lambda^2)f(1 + \delta_R)\tau_n},$$

where $\lambda$ is the ratio of axial-vector and vector effective coupling constants, $\lambda = G_A^I/G_V^I$, with $G_A^I = G_A^2 (1 + \Delta^A_R)$ and $G_V^I = G_V^2 (1 + \Delta^V_R)$. Here $\Delta^A_R$ and $\Delta^V_R$ are the nucleus-independent radiative corrections. With the experimental measurement of $\lambda$ being a determination of $G_A^I/G_V^I$, the actual value of $\Delta^A_R$ is not required for the evaluation of $V_{ud}^2$. In Eq. (12), $f$ is the statistical rate function and $\delta_R$, the nucleus-dependent radiative correction evaluated for the case of a neutron. Wilkinson $^{12}$ has evaluated the product $f(1 + \delta_R)$. His value was revised by Towner and Hardy $^{13}$ who incorporated the current best $Q$-value to obtain $f(1 + \delta_R) = 1.71489 \pm 0.00002$. Lastly, $\tau_n$ is the mean lifetime for neutron decay.

A survey of world data on neutron decay appears in Table 2. The lifetime measurements, when considered as a single body of data, are statistically consistent. However the measurements of $\lambda$ are not ($\chi^2/\nu = 2.6$) and, as a consequence, the uncertainty quoted in the table for the overall average value of $\lambda$ has been scaled by a factor of 1.6. Inserting the average values for $\tau_n$ and $\lambda$ into Eq. (12), we determine the value of $V_{ud}$ to be
Table 2: Experimental results for neutron decay

<table>
<thead>
<tr>
<th>Method</th>
<th>Measured values</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_n$(s)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n$ beam</td>
<td>891 ± 9 $^{14}$</td>
<td></td>
</tr>
<tr>
<td>$n$ trap</td>
<td>877 ± 10 $^{15}$</td>
<td>887.6 ± 3.0 $^{16}$</td>
</tr>
<tr>
<td></td>
<td>888.4 ± 3.3 $^{17}$</td>
<td>882.6 ± 2.7 $^{18}$</td>
</tr>
<tr>
<td></td>
<td>889.2 ± 4.8 $^{19}$</td>
<td>885.4 ± 1.0 $^{20}$</td>
</tr>
<tr>
<td>Overall</td>
<td></td>
<td>885.6 ± 0.8</td>
</tr>
</tbody>
</table>

| $\lambda$  | $\beta$-asym.           |              |
| $\lambda$  |                          |              |
| $\beta$-asym. | −1.254 ± 0.015$^b$ $^{21}$ | −1.257 ± 0.012$^b$ $^{22}$ |
| $\beta$ and $\nu$ asym | −1.262 ± 0.005 $^{23}$ | −1.2594 ± 0.0038 $^{24}$ |
| $\lambda$  |                          |              |
| $\beta$-corr. | −1.266 ± 0.004 $^{25}$ | −1.2739 ± 0.0019 $^{26}$ |
| Overall     |                         | −1.2690 ± 0.0022 |

$^a$ Following the practice used for superallowed decay, we retain only those measurements with uncertainties that are within a factor of ten of the most precise measurement for each quantity. All such measurements, of which we are aware, that have not been withdrawn by their authors are listed.

$^b$ Corrected for weak magnetism and recoil following ref. $^{12}$.

$$|V_{ud}| = 0.9745 ± 0.0016, \quad \text{[Neutron]} \quad (13)$$

and the unitarity sum to be

$$\sum_i V_{ui}^2 = 0.9978 ± 0.0033, \quad \text{[Neutron]} \quad (14)$$

a value that agrees with unitarity and with the nuclear result, Eq. (11) which is over a factor of two more precise. In connection with this result, we note the following two points:

(a) For neutron decay, the error bar associated with $|V_{ud}|$ in Eq. (13) is some three times larger than the error bar obtained from nuclear decays, Eq. (9); however, in contrast with the latter case, it is predominantly experimental in origin. The largest theoretical contribution to the $|V_{ud}|$ error bar comes (at the level of $±0.0004$) from $\Delta^\nu_R$, a correction that is common to both nuclear and neutron decays.

(b) Currently, the theoretical uncertainty on $\Delta^\nu_R$ dominates the nuclear result for $|V_{ud}|$. As experimental results for the neutron improve, $\Delta^\nu_R$ will eventually dominate the neutron result too. Therefore, so long as $\Delta^\nu_R$ remains at its current level of uncertainty, the neutron results will never be able to test
unitarity with significantly better precision than the nuclear decays do now, in spite of their independence of $\delta_C$. They will, of course, be able to provide an important test of whether there are some systematic problems with the nuclear-dependent corrections, which are not now anticipated in the theoretical uncertainty quoted in Eq. (9).

2.3 Pion beta decay

Like neutron decay, pion beta decay has an advantage over nuclear decays in that there are no nuclear structure-dependent corrections to be made. It also has the same advantage as the nuclear decays in being a purely vector transition, in its case $0^+ \rightarrow 0^+$, so no separation of vector and axial-vector components is required. Its major disadvantage, however, is that pion beta decay, $\pi^+ \rightarrow \pi^0 e^+ \nu_e$, is a very weak branch, of the order of $10^{-8}$. This results in severe experimental limitations. For the pion beta decay, the value of $V_{ud}$ is determined from the expression

$$V_{ud}^2 = \frac{K/\ln 2}{G_F^2 (1 + \Delta V_R) f_1 f_2 f (1 + \delta_R) \tau_m / BR},$$

(15)

where $f$ is the approximate statistical rate function

$$f = \frac{1}{30} \left( \frac{\Delta}{m_e} \right)^5$$

(16)

with $\Delta$ being the pion mass difference ($\Delta = m_{\pi^+} - m_{\pi^0}$) and $m_e$, the electron mass. The mass difference is known with high precision from the work of Crawford et al.\cite{29} to be $\Delta = (4.5936 \pm 0.0005)$ MeV. The factors $f_1$ and $f_2$ are corrections to $f$, and are easily calculable functions \cite{30} of $\Delta/m_{\pi^+}$ with values of $f_1 = 0.941039$ and $f_2 = 0.951439$. The nucleus-dependent radiative correction evaluated for the case of the pion is $\delta_R = (1.05 \pm 0.15)\%$. Finally, $\tau_m$ and $BR$ are the pion mean lifetime and branching ratio, respectively.

Two precise lifetime measurements\cite{31,32} were published in 1995 and the PDG02 average is

$$\tau_m = (2.6033 \pm 0.0005) \times 10^{-8} \text{ s}. \quad (17)$$

The branching ratio is principally from McFarlane et al.\cite{30}:

$$BR = (1.025 \pm 0.034) \times 10^{-8}. \quad (18)$$

Inserting Eqs. (17) and (18) into Eq. (15), we determine the value of $V_{ud}$ to be
\[ |V_{ud}| = 0.9670 \pm 0.0161, \]  
\[ \text{[Pion]} \]  
(19)
and the unitarity sum
\[ \sum_i V_{ai}^2 = 0.9833 \pm 0.0311, \]  
\[ \text{[Pion]} \]  
(20)
satisfying the unitarity condition but with comparatively large uncertainty. The error on \(|V_{ud}|\) is entirely due to the uncertainty in the pion branching ratio. There is an experiment currently underway at PSI designed to improve this branching ratio by a factor of eight. If successful, the error on \(|V_{ud}|\) would be reduced to \(\pm 0.0022\), still less precise than the current results from both neutron and nuclear decays. Of course, ultimately it too will be limited by the theoretical uncertainty on \(\Delta V\).

3 Theoretical corrections in nuclear decays

In Sec. 2.1 we noted that the value of \(V_{ud}\) determined from nuclear decays – the most precise result available – resulted in the unitarity test among the elements of the first row of the CKM matrix not being satisfied by two standard deviations. Here, we discuss the theoretical corrections involved in the determination, and assess whether the failure to meet unitarity can be removed by reasonable adjustments in these calculations. To restore unitarity, the calculated radiative corrections for all nine nuclear transitions would have to be shifted downwards by 0.3\% (i.e. as much as one-quarter of their current value), or the calculated isospin symmetry-breaking correction shifted upwards by 0.3\% (over one-half their value), or some combination of the two.

3.1 Radiative corrections

As mentioned in Sec. 2.1, the radiative correction is conveniently divided into terms that are nucleus-dependent, \(\delta_R\), and terms that are not, \(\Delta V_R\). These are written

\[ \delta_R = \delta'_R + \delta_{NS} \]
\[ \delta'_R = \frac{\alpha}{2\pi} \left[ g(E_m) + \delta_2 + \delta_3 \right] \]
\[ \Delta V_R = \frac{\alpha}{2\pi} \left[ 4 \ln(m_Z/m_p) + \ln(m_p/m_A) + 2C_{\text{Born}} \right] + \cdots, \]  
(21)
Table 3: Calculated nucleus-dependent radiative correction, $\delta_R$, in percent units, and the component contributions as identified in Eq. (21).

<table>
<thead>
<tr>
<th>Element</th>
<th>$\frac{\alpha}{2\pi}\overline{g}(E_m)$</th>
<th>$\frac{\alpha}{2\pi}\delta_2$</th>
<th>$\frac{\alpha}{2\pi}\delta_3$</th>
<th>$\delta'_{NS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{10}$C</td>
<td>1.468</td>
<td>0.182</td>
<td>0.005</td>
<td>-0.360(35)</td>
</tr>
<tr>
<td>$^{14}$O</td>
<td>1.286</td>
<td>0.227</td>
<td>0.008</td>
<td>-0.250(50)</td>
</tr>
<tr>
<td>$^{26m}$Al</td>
<td>1.110</td>
<td>0.325</td>
<td>0.021</td>
<td>0.009(20)</td>
</tr>
<tr>
<td>$^{34}$Cl</td>
<td>1.002</td>
<td>0.388</td>
<td>0.034</td>
<td>-0.085(15)</td>
</tr>
<tr>
<td>$^{38m}$K</td>
<td>0.964</td>
<td>0.413</td>
<td>0.041</td>
<td>-0.100(15)</td>
</tr>
<tr>
<td>$^{42}$Sc</td>
<td>0.939</td>
<td>0.448</td>
<td>0.049</td>
<td>0.030(20)</td>
</tr>
<tr>
<td>$^{46}$V</td>
<td>0.903</td>
<td>0.468</td>
<td>0.057</td>
<td>-0.040(7)</td>
</tr>
<tr>
<td>$^{50}$Mn</td>
<td>0.873</td>
<td>0.494</td>
<td>0.065</td>
<td>-0.042(7)</td>
</tr>
<tr>
<td>$^{54}$Co</td>
<td>0.843</td>
<td>0.507</td>
<td>0.073</td>
<td>-0.029(7)</td>
</tr>
</tbody>
</table>

where the ellipses represent further small terms of order 0.1%. In these equations, $E_m$ is the maximum electron energy in beta decay, $m_z$ the $Z$-boson mass, $m_{a_1}$ the $a_1$-meson mass, and $\delta_2$ and $\delta_3$ the order-$Z\alpha^2$ and $-Z^2\alpha^3$ contributions respectively. The function $g(E_e, E_m)$, which is a function of electron energy, was first defined by Sirlin \cite{Sirlin33} as part of the order-$\alpha$ universal photonic contribution arising from the weak vector current; it is here averaged over the electron spectrum to give $g(E_m)$. Finally, the terms $C_{\text{Born}}$ and $\delta_{NS}$ come from the order-$\alpha$ axial-vector photonic contributions: the former accounts for single-nucleon contributions, while the latter covers two-nucleon contributions and is consequently dependent on nuclear structure.

Calculated values for all four components of $\delta_R$ are given in Table 3. There have been two independent calculations\cite{Sirlin34,Sirlin35} of both $\delta_2$ and $\delta_3$; they are completely consistent with one another if proper account is taken of finite-size effects in the nuclear charge distribution. The values listed in Table 3 are our recalculations\cite{Note5} using the formulas of Sirlin\cite{Sirlin34} but incorporating a Fermi charge-density distribution for the nucleus. Note that we have followed Sirlin in assigning an uncertainty equal to $(\alpha/2\pi)\delta_3$ as an estimate of the error made in stopping the calculation at that order. Also appearing in the table are our adopted values for $\delta_{NS}$, which we have taken from our recent re-evaluation\cite{Note8}.

In assessing the changes in $\delta_R$ that would be required in order to restore unitarity, it is helpful to rewrite Eq. (21) in terms of the typical values taken by its components: viz

$$\delta_R \simeq 1.00 + 0.40 + 0.05.$$ (22)
Thus, \( \delta'_{R} \simeq 1.4\% \). If the failure to obtain unitarity in the CKM matrix with \( V_{ud} \) from nuclear beta decay is due to the value of \( \delta'_{R} \), then all \( \delta'_{R} \) values must be reduced to \( \sim 1.1\% \). This is not likely. The leading term, 1.00%, involves standard QED and is well verified. The order-\( Z\alpha^2 \) term, 0.40%, while less secure has been calculated twice\(^{34,35} \) independently, with results in accord.

Taking a similar approach for the nucleus-independent radiative correction, we write

\[
\Delta V_R = 2.12 - 0.03 + 0.20 + 0.1\% \simeq 2.4\%,
\]

of which the first term, the leading logarithm, is unambiguous. Again, to achieve unitarity of the CKM matrix, \( \Delta V_R \) would have to be reduced to 2.1%: \( i.e. \) all terms other than the leading logarithm must sum to zero. This also seems unlikely.

### 3.2 Isospin symmetry-breaking corrections

Because the leading terms in the radiative corrections are so well founded, attention has focussed more on the isospin symmetry-breaking correction. Although smaller than the radiative correction, the isospin symmetry-breaking correction is clearly sensitive to nuclear-structure issues. It comes about because Coulomb and charge-dependent nuclear forces destroy isospin symmetry between the initial and final states in superallowed beta-decay. The consequences are twofold: there are different degrees of configuration mixing in the two states, and, because their binding energies are not identical, their radial wave functions differ. Thus, we accommodate both effects by writing \( \delta_C = \delta_{C1} + \delta_{C2} \). Constraints can be placed on the calculation of \( \delta_{C1} \) by insisting that it reproduce the measured coefficients of the isobaric mass multiplet equation. Constraints are placed on \( \delta_{C2} \) by insisting that the asymptotic forms of the proton and neutron radial functions match the known separation energies.

The results of several calculations for \( \delta_C \) are shown in Table 4. The values in the first column are those calculated by the methods developed by Towner, Hardy and Harvey\(^7 \) and refined in more recent publications\(^{39,40,8} \). They result from shell-model calculations to determine \( \delta_{C1} \), and full-parentage expansions in terms of Woods-Saxon radial wave functions to obtain \( \delta_{C2} \). Ormand and Brown, whose values\(^9 \) for \( \delta_C \) appear in column 2, also employed the shell model for calculating \( \delta_{C1} \) but derived \( \delta_{C2} \) from a self-consistent Hartree-Fock calculation. Both of these independent calculations for \( \delta_C \) reproduce the measured coefficients of the relevant isobaric multiplet mass equation, the known proton and neutron separation energies, and the measured \( ft \)-values of the weak non-analogue \( 0^+ \rightarrow 0^+ \) transitions\(^{40} \) where they are known. In our
Table 4: Calculated isospin symmetry-breaking correction, $\delta_C$, in percent units.

<table>
<thead>
<tr>
<th></th>
<th>TH$^a$</th>
<th>OB$^b$</th>
<th>SVS$^c$</th>
<th>NBO$^d$</th>
<th>Adopted$^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{10}$C</td>
<td>0.18</td>
<td>0.15</td>
<td>0.00</td>
<td>0.12</td>
<td>0.17(3)</td>
</tr>
<tr>
<td>$^{14}$O</td>
<td>0.32</td>
<td>0.15</td>
<td>0.29</td>
<td>0.24(3)</td>
<td></td>
</tr>
<tr>
<td>$^{26}$mAl</td>
<td>0.27</td>
<td>0.30</td>
<td>0.27</td>
<td>0.29(3)</td>
<td></td>
</tr>
<tr>
<td>$^{34}$Cl</td>
<td>0.64</td>
<td>0.57</td>
<td>0.33</td>
<td>0.61(3)</td>
<td></td>
</tr>
<tr>
<td>$^{38}$mK</td>
<td>0.62</td>
<td>0.59</td>
<td>0.33</td>
<td>0.61(3)</td>
<td></td>
</tr>
<tr>
<td>$^{42}$Sc</td>
<td>0.49</td>
<td>0.42</td>
<td>0.44</td>
<td>0.46(3)</td>
<td></td>
</tr>
<tr>
<td>$^{46}$V</td>
<td>0.43</td>
<td>0.38</td>
<td></td>
<td>0.41(3)</td>
<td></td>
</tr>
<tr>
<td>$^{50}$Mn</td>
<td>0.51</td>
<td>0.35</td>
<td></td>
<td>0.43(3)</td>
<td></td>
</tr>
<tr>
<td>$^{54}$Co</td>
<td>0.61</td>
<td>0.44</td>
<td>0.49</td>
<td>0.53(3)</td>
<td></td>
</tr>
</tbody>
</table>

$^a$ Ref. 8;  $^b$ Ref. 9;  $^c$ Ref. 41;  $^d$ Ref. 42.

$^e$ Average of OB and TH; assigned uncertainties reflect the relative scatter between these calculations.

In Sec. 2.1, we have used the average of these two sets of $\delta_C$ values: our adopted values appear in the last column of the table.

Two more recent calculations provide a valuable check that these $\delta_C$ values are not suffering from severe systematic effects. Sagawa, van Giai and Suzuki 41 have added RPA correlations to a Hartree-Fock calculation that incorporates charge-symmetry and charge-independence breaking forces in the mean-field potential to take account of isospin impurity in the core; the correlations, in essence, introduce a coupling to the isovector monopole giant resonance. The calculation is not constrained, however, to reproduce known separation energies. Finally, a large shell-model calculation has been mounted for the $A = 10$ case by Navrátíl, Barrett and Ormand 42. Both of these two new works have produced values of $\delta_C$ very similar to, but actually smaller than those used in our analysis, i.e. worsening rather than helping the unitarity problem.

The typical value of $\delta_C$ is of order 0.4%. If the unitarity problem is to be solved by improvements in $\delta_C$, then all $\delta_C$ values must be increased so that the typical value is raised to around 0.7%. There is no evidence whatsoever for such a shift from recent works.
4 Speculative suggestions to resolve unitarity problem

Since it is concluded in Sec. 3 that reasonable adjustments to the theoretical corrections, $\delta_R$, $\Delta^*_V$ and $\delta_C$, are unlikely to resolve the unitarity problem posed by the nuclear result for $V_{ud}$, we turn in this section to some more speculative alternatives. We discuss four suggestions, two that do not require extensions to the Standard Model and two that do. All but one require the introduction of important new physics to explain the apparent discrepancy.

4.1 Saito-Thomas correction

One suggestion to resolve the unitarity problem was made some years ago by Thomas. It is based on a quark-meson coupling model, in which nuclear matter consists of non-overlapping nucleon (MIT) bags bound by the self-consistent exchange of $\sigma$ and $\omega$ mesons in the mean-field approximation. The model is extended to include an isovector-vector meson ($\rho$) and an isovector-scalar meson ($\delta$). The coupled equations to be solved are very similar to those of Quantum Hadrodynamics, but involve boundary conditions at the bag radius. As a consequence of a coupling to meson fields the quark mass in a medium becomes an effective one,

$$m_i^* = m_i - (V_\sigma \pm \frac{1}{2}V_\delta)$$

with the upper sign for the up quark. Here $V_\sigma$ and $V_\delta$ are strengths of $\sigma$-meson and $\delta$-meson mean fields. At the quark level, the conserved vector current (CVC) hypothesis is broken if the up and down quarks have different masses, but for a free nucleon this breaking is second order in $(m_d - m_u)$. In a nuclear medium, however, this breaking may become first order in $(m_d^* - m_u^*)$, but a critical constraint of the calculation is to show it reverts to $(m_d - m_u)^2$ at zero density.

Saito-Thomas (ST) solve the coupled equations in an infinite nuclear-matter approximation, the solutions being functions of the matter density, $\rho_B$. Let $\psi_{i/j}$ be the wavefunction of quark $i$ bound in nucleon $j$ in nuclear matter; then, the following overlap integral can be defined between quark $i$ bound in a proton and quark $i'$ bound in a neutron:

$$I_{ii'}(\rho_B) = \int_{\text{Bag}} dV \psi^\dagger_{i/p} \psi_{i'/n}.$$  (25)

Calculations indicate that $I_{ii'}(\rho_B)$ varies linearly in $\rho_B$, being unity at zero density. What is required for beta decay is the product of three overlap integrals.
\[ |I_{ud}|^2 \times |I_{uu}|^2 \times |I_{dd}|^2 \equiv 1 - \delta_{C}^{\text{quark}} \]  

and this product also is approximately linear in the density. The results of the calculations \(43\) are

\[ \delta_{C}^{\text{quark}} = b \times \left( \frac{\rho_B}{\rho_0} \right), \]  

where \(\rho_0\) is the saturation density for equilibrium nuclear matter, and \(b\) is in the range \((0.15 - 0.20)\%\) for bag radii in the range \((0.6 - 1.0)\) fm. Thus \(\delta_{C}^{\text{quark}}\), at densities of \(\rho_0/2\), lies in the range 0.075\% to 0.10\% while, at \(\rho_0\), it lies between 0.15\% and 0.20\%. If such a correction were to be applied to the results from nuclear superallowed transitions, the corresponding value for \(V_{ud}^2\) would then be increased by these amounts.

However, in their original work, the authors themselves admit \(43\) that their results are merely qualitative, having been derived from a model that deals only with nuclear matter. More recently, they have also recognized \(45\) that a different model actually produces a considerably smaller effect. Furthermore, there remains the question of whether a simple addition of the Saito-Thomas correction, \(\delta_{C}^{\text{quark}}\), to the \(\delta_{C}\) already computed in a nucleons-only calculation is the correct approach at all, since it may lead to double counting. In the nucleons-only case, certain parameters are adjusted to reproduce Coulomb observables such as the \(b\)- and \(c\)-coefficients of the isobaric mass multiplet equation, and proton and neutron separation energies. Thus, the parameters become effective ones, which in some unquantifiable way, actually contain quark effects.

In summary, the Saito-Thomas correction continues to be unspecified for finite nuclei and now appears for several reasons to be a much smaller effect than was originally proposed. It seems clear that this effect, if it is applicable at all, would be insufficient in itself to restore unitarity.

### 4.2 Ad hoc parameterized fit

Wilkinson \(46\) has suggested that data like those displayed in Fig. 1 might be better fitted by a quadratic function in \(Z\), representing a further correction of speculative origin. He considers the value of this fitted function at \(Z = 0\) to be the appropriate \(Ft\)-value to take forward in the determination of \(V_{ud}\). Alternatively, he has also considered the option of disregarding the structure-dependent corrections entirely and then fitting the uncorrected \(ft\)-values with a similar function quadratic in \(Z\). There is no physical justification whatsoever for incorporating any \(Z\)-dependent corrections beyond those already accounted for in \(\delta_{R}\) and \(\delta_{C}\). Furthermore, there is no statistically significant indication
from the $F_t$-value data in Fig. 1 that one is required. Consequently, until such time as new $F_t$-value data can demonstrate a clear residual $Z$-dependence, this suggested explanation for the nuclear unitarity result must be regarded as unsatisfactory.

4.3 Right-hand currents

Because experimental evidence at low energies favors maximal parity violation in weak interactions, that condition has been built into the Standard Model. However, one class of possible extensions to the Standard Model would restore parity symmetry at higher energies through the introduction of additional heavy charged gauge bosons that are predominantly right-handed in character. Under certain specific conditions, such models could explain the nuclear results described in Sec. 2.1.

For example, an extension known as the manifest left-right symmetric model \(^{47}\) leads to a revised form \(^{13}\) for Eq. (8) which includes a mixing angle $\zeta$:

\[
V_{ud}^2(1 - 2\zeta) = \frac{K}{2G_F^2(1 + \Delta_V^Z)F_t}.
\]

If, under these conditions, we require that $V_{ud}^2$ must satisfy unitarity, \textit{viz}

\[
V_{ud}^2 = 1 - V_{us}^2 - V_{ub}^2,
\]

then, with the $F_t$ value taken from Eq. (7), we can derive a value for the mixing angle of $\zeta = 0.0015 \pm 0.0007$. Within the context of the manifest left-right symmetric model, this is a fully satisfactory result; not surprisingly, it is two standard deviations away from the Standard Model’s pure $V - A$ value.

4.4 Scalar interaction

If the Standard Model of pure $V - A$ weak interactions is extended instead to include scalar and tensor interactions, then the expression for the beta spectrum shape would include an additional term coming from the scalar interaction, which is inversely proportional to the electron energy. Since the $F_t$ values include an integral over the spectrum shape, the presence of a non-zero scalar interaction would be reflected by a correction to the $F_t$ values that is inversely proportional to the decay energy of the parent nucleus. To test this possibility, we fit the data in Fig. 1 by the function

\[
F_t = k [1 + b_F \gamma(W^{-1})],
\]
where $k$ and $b_F$ are parameters determined from the fit. The latter, $b_F$, is known as the Fierz interference term; a non-zero value for this term would signal the presence of a scalar interaction. In Eq. (30), $\gamma^2 = (1 - (\alpha Z)^2)$, $\alpha$ is the fine-structure constant, $Z$ the atomic number of the daughter nucleus, and $\langle W^{-1} \rangle$ is the value of $1/W$ averaged over the electron spectrum, where $W$ is the electron energy in electron rest-mass units. The value obtained for the Fierz interference term from the data in Fig. 1 is

$$b_F = -0.0027 \pm 0.0029. \quad (31)$$

A negative sign is expected for a positron emitter. The 90% confidence level upper limit is

$$|b_F| < 0.0075. \quad (32)$$

To proceed to $V_{ud}$, we need a value of $\mathcal{F}t$ extrapolated to the $Z = 0$ limit. To this end, we fit the nine values of $\gamma\langle W^{-1} \rangle$ by a second-order polynomial in $Z$ and use this polynomial to provide the value of $\gamma\langle W^{-1} \rangle$ at $Z = 0$. With $k$ and $b_F$ taken from the fit, Eq. (31), we obtain

$$\mathcal{F}t(0) = 3067.5 \pm 2.9$$

$$|V_{ud}| = 0.9748 \pm 0.0006 \quad \text{[Nuclear + bF]}$$

$$\sum_i V_{ui}^2 = 0.9983 \pm 0.0016. \quad \text{[Nuclear + bF]} \quad (33)$$

This result is now in accord with unitarity but, of course, requires the presence of a non-zero scalar coupling to achieve that goal.

We close this section by noting that all four suggested explanations we have presented for the non-unitarity result in Eq. (11) are entirely speculative and should be considered with caution.

5 Future experimental prospects

It is evident from the foregoing discussion that the current world data on $V_{ud}$ are tantalizingly close to producing a definitive result on unitarity. The nuclear measurements have achieved the highest experimental precision but they are now constrained by theoretical uncertainties. The neutron and pion measurements are, as yet, experimentally less precise than the nuclear ones, but they are free from one of the more important sources of theoretical uncertainty, $\delta_C$. All three classes of measurements are now being extended and improved at
a number of laboratories, and there are good prospects for considerably reduced error bars within a few years. In combination with the re-visitation of the $K_{e3}$ decay $^4$, these results should settle the uncertainty over $\delta_C$ and determine whether the deviation from unitarity, apparent from the nuclear result, is real or not. This is ample argument to justify considerable experimental activity. At the same time, it should not be forgotten that the full impact of these experimental advances will be diluted until there are theoretical improvements in the correction $\Delta_V^R$ – the dominant source of theoretical uncertainty in all cases. Ultimately, to make significant improvements in the unitarity test, there will have to be advances in both theory and experiment.

Of the three classes of experiment, that focusing on pion decay is currently farthest from the precision required for a meaningful unitarity test. We noted in Sec. 2.3 that a considerable improvement is anticipated in the foreseeable future, but the result will not reach a precision higher than that already achieved for neutron decay. There is considerable optimism, however, that the neutron measurements themselves can be improved as new experiments with ultra-cold neutrons come to fruition. The outcome can be expected seriously to challenge the nuclear experiments for experimental precision, but will take at least a few more years to do so.

As to the nuclear experiments, the nine superallowed transitions whose $ft$ values are known to within a fraction of a percent have been the subject of intense scrutiny for at least the past three decades. All except $^{10}$C have the special advantage that the superallowed branch from each is by far the dominant transition in its decay ($> 99\%$). This means that the branching ratio for the superallowed transition can be determined to high precision from relatively imprecise measurements of the other weak transitions, which can simply be subtracted from 100%. Given the quantity of careful measurements already published, are there reasonable prospects for significant improvements in these decay measurements in the near future? Given the uncertainty in the theoretical corrections, which experiments can shed the most light on the efficacy of these corrections?

If we begin by accepting that it is valuable for experiment to be at least a factor of two more precise than theory, then an examination of the world data shows that the Q-values for $^{10}$C, $^{14}$O, $^{26}_{\text{m}}$Al and $^{46}$V, the half-lives of $^{10}$C, $^{34}$Cl and $^{38}_{\text{m}}$K, and the branching ratio for $^{10}$C can all bear improvement. Such improvements will soon be feasible. The Q-values will reach the required level (and more) as mass measurements with new on-line Penning traps become possible; half-lives will likely yield to measurements with higher statistics as high-intensity beams of separated isotopes are developed for the new radioactive-beam facilities; and, finally, an improved branching-ratio mea-
surement on $^{10}$C has already been made with Gammasphere and simply awaits analysis $^{49}$. Qualitative improvements will also come as we increase the number of superallowed emitters accessible to precision studies. The greatest attention recently has been paid to the $T_z = 0$ emitters with $A \geq 62$, since these nuclei are expected to be produced at new radioactive-beam facilities, and their calculated structure-dependent corrections, $\delta C - \delta_{NS}$, (see Eq.(5)) are predicted to be large $^{9,41,8}$. They could then provide a valuable test of the accuracy of $\delta C$ calculations. It is likely, though, that the required precision will not be attainable for some time to come. First, with current Penning-trap technology, these nuclei are too short lived ($t_{1/2} \leq 100$ms) for their masses to be measured with the required precision. Second, the decays of these nuclei are of higher energy and each must involve numerous weak Gamow-Teller transitions that together will account for significant intensity in addition to the superallowed transition $^{50}$. Branching-ratio measurements will thus be very demanding, particularly with the limited intensities likely to be available initially for these rather exotic nuclei. Finally, lifetime measurements will often be similarly constrained by statistics, although not in every case. The half-life of $^{74}$Rb was recently obtained $^{51}$ with fully adequate precision.

More accessible in the short term will be the $T_z = -1$ superallowed emitters with $18 \leq A \leq 38$. There is good reason to explore them. For example, the calculated value $^{8}$ of the total structure-dependent correction, $\delta C - \delta_{NS}$, for $^{30}$S decay, though smaller than the values expected for the heavier nuclei, is actually 1.13% – about a factor of two larger than for any other case currently known – while $^{22}$Mg has a rather low value of 0.51%. If such large differences are confirmed by the measured $ft$-values, then it will do much to increase our confidence in the calculated isospin symmetry-breaking corrections. To be sure, these decays will provide a challenge, particularly in the measurement of their branching ratios, but the required precision should be achievable with isotope-separated beams that are currently available. In fact, such experiments are already underway at the Texas A&M cyclotron. Results for the decay of $^{22}$Mg are currently being prepared for publication.

6 Conclusions

The current world data on superallowed $0^+ \rightarrow 0^+$ beta decays lead to a self-consistent set of $Ft$-values that agree with CVC but differ provocatively, though not yet definitively, from the expectation of CKM unitarity. There are no evident defects in the calculated radiative and isospin symmetry-breaking corrections that could remove the problem, so, if any progress is to be made in
firmly establishing (or eliminating) the discrepancy with unitarity, additional experiments are required. We have indicated what some relevant nuclear experiments might be.

In the past decade there have been significant improvements in the measurements of the neutron lifetime and beta asymmetry, and further improvements are promised in the near future. It is likely that these studies in neutron decay will soon approach the results from nuclear superallowed decays in precision; and when there, they will have the advantage in that their results are not dependent on nuclear-structure corrections. So far, though, the neutron result is quite consistent with the nuclear one.

Clearly, there is strong motivation to pursue experiments, not only on the neutron and nuclear front but also in re-visiting the $K_{e3}$ decay, since, if firmly established, a discrepancy with unitarity would indicate the need for important new physics.

Acknowledgments

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