The Schrödinger Wave Functional and Closed String Rolling Tachyon

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ABSTRACT: In this short note we apply Schrödinger picture description of the minisuperspace approach to the closed string tachyon condensation. We will calculate the rate of produced closed string and we will show that the density of high massive closed string modes reaches the string density in time of order one in string units.

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1. Introduction

A basic issue in string theories that have tachyons in tree-level spectra is their fate. The presence of the tachyons leads to the condensation processes in given theories. Most of the results are based on the conjecture that the world-sheet models that realize these initial and final states of these condensation process are related through renormalization group (RG) flow on the world-sheet. This relation between space-time evolution and RG flow is still not well understood, especially in case of closed tachyons (For review, see [1, 2, 3].). A lot of open issues can only be studied through the construction of exact time-dependent backgrounds. For the open string tachyon such an investigation was invented by A. Sen [4, 5, 6] and further developed in [7, 8, 9, 10] (Recent works considering rolling tachyon in open string theory are [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38].)

The condensation processes of closed tachyon are very difficult in picture, mainly because their drastic effect they have on space-time itself. Some results have been obtained recently in case of certain localized closed string tachyons [39, 40, 41]. On the other hand it was shown in [42] that there are exact, time-dependent classical solutions tachyonic closed string theories describing homogeneous tachyon condensation. This corresponding CFT on the worksheet was named as timelier Liouville theory. It is described by the action

$$\tag{1.1} S = \frac{1}{4\pi} \int d^2\sigma \left( -\left( \partial X^0 \right)^2 + 4\pi \mu e^{2\beta X^0} \right).$$

This theory has negative norm boson and central charge $c = 1 - 6q^2$, $q = \beta - \frac{1}{\beta}$. This corresponds to a real dilaton with timelier slope $q$ however as in [42] we will be mainly interested in $q = 0$, $\beta = 1$. The potential term in (1.1) can be interpreted as closed string tachyon field that grows exponentially in time. At early times $X^0 \to -\infty$ this term is small and we have ordinary string action in flat space.

The main problem with (1.1) is that this action is not positive definite and hence does not fully define the associated functional integral in CFT. It is natural to define timelier theory by analytic continuation $\phi = -iX^0$ and $b = i\beta$ from the ordinary

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1We have suppressed spatial direction and set $\alpha' = 1$. 
Spacetime Liouville theory with positive definite worksheet action

\[ S = \frac{1}{4\pi} \int d^2\sigma \left( (\partial\phi)^2 + 4\pi \mu e^{2b\phi} \right). \]  

(1.2)

and background charge \( Q = b + 1/b \). The central charge is \( c = 1 + 6Q^2 \). This strategy has been used recently in [42, 43] where many interesting results can be found.

Since the worksheet action (1.1) describes the propagation of the string in the time-dependent background we can expect the closed string pair production. The rate of pair production has been calculated in [42] in minisuperspace approximation where it was shown that it diverges exponentially. According to [42] this result implies that the gas of produced closed strings will reach the string density and the string perturbation theory will break down in time of order string units. The calculation of the particle production was performed using the canonical formalism that is well known from the study of the quantum field theory in curved peacetime [45, 46, 47] where quantum fields are considered as Heisenberg operators that explicitly depend on time while the states of the system do not evolve. As companion to this approach we would like to calculate the rate of the closed string pair production in the Schrödinger representation of the quantum field theory. The reason why we perform this analysis is that the Schrödinger picture provides a simple and intuitive description of vacuum states in quantum field theory in situations where the background metric is time-dependent or in case where some parameters of quantum field theory depend on time. The vacuum states are explicitly specified single, possibly time-dependent, kernel function satisfying a differential equation with prescribed boundary conditions. This makes no reference to the assumed spectrum of excited states and so circumvents the difficulties of the conventional canonical description of a vacuum as a ”no-particle” state with respect to the creation and annihilation operators defined by a particular mode decomposition of the field, an approach which is not well suited to the time-dependent problems.

The main goal of this paper is to formulate the Schrödinger picture description of the minisuperspace approximation to the closed string rolling tachyon. Using this powerful approach to the quantum field theory in time-dependent background we will obtain an expression for the density of particles produced during the rolling tachyon evolution. Then we will estimate the time when the particle density reaches the string scale and we will show that this situation occurs very quickly after beginning of the rolling tachyon process and leads to the breakdown of string perturbation theory.

This paper is organized as follows. In the next section (2) we formulate the Schrödinger picture description of the minisuperspace approach to the closed string rolling tachyon background. Then we will calculate the rate of particle production and in the asymptotic future we find agreement with [42]. As a application of this formula we will estimate the time when the density of produced closed string modes reaches string density. In conclusion (3) we outline our results.
2. Minisuperspace approximation and Schrödinger picture description

In this section we will describe the closed string tachyon condensation in the minisuperspace approximation [42]. In the minisuperspace approximation we retain only the zero mode $x^0$ of $X^0$. The on-shell condition for the closed string excitation is Klein-Gordon equation with time-dependent mass [42]

$$\left[ \frac{\partial^2}{(\partial x^0)^2} + k^2 + 2\pi\mu e^{2x^0} + 2(N_L + N_R - 2) \right] \phi(x_0) = 0 . \quad (2.1)$$

Using $\phi(x, t) = \int \frac{dk}{(2\pi)^D} \phi(t)e^{ikx}$ we can write (2.1) as \footnote{Our convention is $\eta_{\mu\nu} = \text{diag}(-1, \ldots, 1)$, $\nu = 0, \ldots, D$, $a, b, c, \ldots = 1, \ldots, D$. We also denote $x^0 = t$ and $x = (x^1, \ldots, x^D)$ and $m^2 = 2(N_L + N_R - 2)$, where $N_L, N_R$ are the left and the right-moving oscillators contributions.}

$$\left[ \partial_t^2 - \delta^{ab} \partial_a \partial_b + 2\pi\mu e^{2t} + m^2 \right] \phi(x, t) = 0 . \quad (2.2)$$

It is clear that this equation of motion can be obtained by variation of the following action

$$S = -\int dt L = -\frac{1}{2} \int dt dx \left[ \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2(t) \phi^2 \right] , \quad (2.3)$$

where

$$m^2(t) = 2\pi\mu e^{2t} + m^2 . \quad (2.4)$$

The action (2.3) describes the scalar field with time-dependent mass. The quantisation of such field can be performed in the same way as in the case of the quantum field theory in curved space-time [45, 46, 47], for recent analysis, see [8, 9, 10, 42]. As companion to this approach we apply Schrödinger picture description of the quantum field theory defined by (2.3). The similar approach was recently used for minisuperspace description of S-branes in [44] \footnote{For nice review of Schrödinger picture description of quantum field theory in curved space-time, see [48, 49, 50].}. We begin with the action (2.3) from which we obtain the canonical momentum conjugate to $\phi(t, x)$

$$\pi(t, x) = \frac{\delta L}{\delta \dot{\phi}(x, t)} = \dot{\phi}(t, x) \quad (2.5)$$

and the Hamiltonian

$$H = \int dx \left( \pi \dot{\phi} - L \right) = \frac{1}{2} \int dx \left( \pi^2 + \eta^{ab} \partial_a \phi \partial_b \phi + m^2(t) \phi^2 \right) . \quad (2.6)$$

The system can be quantised by treating the fields as operators and imposing appropriate commutation relations. This involves choice of a foliation of a space-time in a
succession of space-like hypersurfaces. We choose these to be the hypersurfaces of fixed \( t \) and impose equal-time commutation relations
\[
[\hat{\phi}(x, t), \hat{\pi}(y, t)] = i\delta(x - y), [\hat{\phi}(x, t), \hat{\phi}(y, t)] = [\hat{\pi}(x, t), \hat{\pi}(y, t)] = 0. \tag{2.7}
\]
In the Schrödinger picture we take the basis vector of the state vector space to be the eigenstate of the field operator \( \hat{\phi}(t, x) \) on a fixed \( t \) hypersurface, with eigenvalues \( \phi(x) \)
\[
\hat{\phi}(t, x) \mid \phi(x), t \rangle = \phi(x) \mid \phi(x), t \rangle. \tag{2.8}
\]
Notice that the set of field eigenvalues \( \phi(x) \) is independent of the value of \( t \) labeling the hypersurface. In this picture, the quantum states are explicit functions of time and are represented by wave functionals \( \Psi[\phi(x), t] \). Operators \( \hat{O}(\hat{\pi}, \hat{\phi}) \) acting on these states may be represented by
\[
\hat{O}(\hat{\pi}(x), \hat{\phi}(x)) = \mathcal{O} \left( -i \frac{\delta}{\delta \phi(x)}, \phi(x) \right). \tag{2.9}
\]
The Schrödinger equation which governs the evolution of the wave functional is
\[
i \frac{\partial \Psi[\phi, t]}{\partial t} = H \left( -i \frac{\delta}{\delta \phi(x)}, \phi(x) \right) \Psi[\phi, t] = \frac{1}{2} \int dx \left[ -\frac{\delta^2}{\delta \phi^2} + \eta^{ab} \partial_a \phi \partial_b \phi + m^2(t) \phi^2 \right] \Psi[\phi, t]. \tag{2.10}
\]
To solve this equation, we make the ansatz, that the vacuum functional is simple Gaussian. We therefore write
\[
\Psi_0[\phi, t] = N_0(t) \exp \left\{ -\frac{1}{2} \int dxdy \phi(x)G(x, y, t)\phi(y) \right\} = N_0(t)\psi_0(\phi, t), \tag{2.11}
\]
where \( N_0(t), G(x, y, t) \) obey following equations
\[
i \frac{\partial N_0(t)}{\partial t} = \frac{N_0(t)}{2} \int dz G(z, z, t),
\]
\[
i \frac{\partial G(x, y, t)}{\partial t} = \int dz G(z, x, t)G(y, z, t) - \left( \eta^{ab} \partial_a \partial_b + m^2(t) \right) \delta(x, y). \tag{2.12}
\]
Because the spatial sections are flat it is natural to perform a Fourier transformation on the space dependence of the field configuration
\[
\phi(x) = \int \frac{dk}{(2\pi)^D} e^{ikx} \alpha(k), \tag{2.13}
\]
where reality of \( \phi \) implies \( \alpha^*(k) = \alpha(-k) \). Similarly we can define \( \delta/\delta \alpha(k) \) as
\[
\frac{\delta}{\delta \phi(x)} = \int \frac{dk}{(2\pi)^D} e^{ikx} \frac{\delta}{\delta \alpha(k)} \tag{2.14}
\]
where
\[ \frac{\delta \alpha(k)}{\delta \alpha(k')} = (2\pi)^D \delta(k + k'). \] (2.15)

In k space, the Hamiltonian is
\[
H = \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \left[ -\frac{\delta^2}{\delta \alpha(k) \alpha(-k)} + \Omega_k^2(t) \alpha(k) \alpha(-k) \right],
\] (2.16)

where
\[ \Omega_k^2(t) = m^2(t) + \omega_k^2, \quad \omega_k^2 = k^2 + m^2. \] (2.17)

For each k the integrand in (2.16) represents a harmonic oscillator with the time-dependent frequency \( \Omega_k^2(t) \). After performing the Fourier transformation for the kernel \( G(x, y, t) \)
\[
G(x, y, t) = \int \frac{d^Dk}{(2\pi)^D} e^{ik(x-y)} \tilde{G}(k, t)
\] (2.18)
the kernel equation (2.12) reduces to
\[
i \frac{\partial \tilde{G}(k, t)}{\partial t} = \tilde{G}^2(k, t) - \Omega_k^2(t).
\] (2.19)

This equation can be solved with the ansatz
\[ \tilde{G}(k, t) = -\frac{i}{\tilde{\psi}_k(t) / \tilde{\psi}_k(t)}, \] (2.20)

where \( \tilde{\psi}_k(t) \) obeys
\[ \ddot{\psi}_k + \Omega_k^2(t) \psi_k = 0. \] (2.21)

Now the vacuum state functional has the form
\[
\Psi[\phi, t] = N_0(t) \exp \left( \frac{i}{2} \int \frac{d^Dk}{(2\pi)^D} \alpha(-k) \frac{\dot{\psi}_k}{\psi_k} \alpha(k) \right),
\] (2.22)

where
\[ N_0(t) = e^{-i \int_0^t dt' E_0(t')} , \quad E_0(t) = \frac{1}{2} V \int \frac{d^Dk}{(2\pi)^D} \tilde{G}(k, t). \] (2.23)

In the previous expression \( V \) is spatial volume of D+1 dimensional space-time. For closed string rolling tachyon the equation (2.21) has the form
\[
\left[ \dot{\psi}_k^2 + 2\pi \mu e^{2t} + \omega_k^2 \right] \psi_k(t) = 0.
\] (2.24)

The solution of the previous equation is
\[ \psi_k^{in} = \left( \frac{\pi \mu}{2} \right)^{i \omega_k/2} \frac{\Gamma(1 - i \omega_k)}{\sqrt{2\omega_k}} J_{i \omega_k} \left( \sqrt{2\pi \mu e^t} \right). \] (2.25)

For next purposes we also write the second solution of the previous equation
\[ \psi_k^{out}(t) = \sqrt{\frac{\pi}{2}} (ie^{i \omega_k})^{-1/2} H_{-i \omega_k}^{(2)} \left( \sqrt{2\pi \mu e^t} \right). \] (2.26)
The relation between these two modes is
\[ \psi_{\text{out}}^k = A \psi_{\text{in}}^k + B \psi_{\text{in}}^{\ast k}, \quad (2.27) \]
where
\[ A = e^{\pi \omega_k + \pi i/2} B^{\ast} = \sqrt{\omega_k \pi} e^{\pi \omega_k / 2 - i \pi / 4} \left( \frac{(\pi \mu / 2 - i \omega_k / 2)}{\sinh \pi \omega_k \Gamma(1 - i \omega_k)} \right). \quad (2.28) \]

Since for \( t \to -\infty \) the mass term approaches standard flat space expression it is natural to demand that the vacuum state functional \( \Psi[\phi, t] \) approaches the usual positive frequency Minkowski vacuum state
\[ \psi_0(\phi, -\infty) = \exp \left( -\frac{1}{2} \int \frac{dk}{(2\pi)^D} |\phi(k)|^2 \omega_k \right). \quad (2.29) \]

This boundary condition for \( \Psi[\phi, t] \) implies following form of the kernel
\[ \tilde{G}(k, t) = -i \frac{\dot{\psi}_{\text{in}}^{\ast k}(t)}{\psi_{\text{in}}^{\ast k}(t)}. \quad (2.30) \]

To see this note that for \( t \to -\infty \) we have
\[ \lim_{t \to -\infty} \psi_{\text{in}}^{\ast k}(t) = \frac{1}{\sqrt{2\omega_k}} e^{i \omega_k t}, \quad \lim_{t \to -\infty} \tilde{G}(k, t) = \omega_k. \quad (2.31) \]

As in [42] we would like to determine the rate of the particle production in the rolling tachyon background. In the same way as in [44] we introduce an operator of number of particles with momentum \( k \)
\[ N(k, t) = \frac{1}{2\Omega_k(t)} \left[ -\frac{\delta^2}{\delta \alpha(-k) \alpha(k)} + \Omega_k^2(t) \alpha(k) \alpha(-k) - \Omega_k(t)(2\pi)^D \delta_k(0) \right]. \quad (2.32) \]

To support the claim that \( N(k, t) \) is natural operator of the number of particles with momentum \( k \) at time \( t \) note that the Hamiltonian can be written as
\[ H(t) = \int \frac{dk}{(2\pi)^D} \Omega_k(t) \left[ N(k, t) + \frac{V_D}{2} \right] + V_D = (2\pi)^D \delta_k(0) \quad (2.33) \]
which has the form of the collection of Hamiltonians of harmonic oscillators with the time-dependent frequency \( \Omega_k(t) \) where the operator \( N(k, t) \) counts the number of excited modes.

We would like to calculate the vacuum expectation value of \( \langle N(k, t) \rangle \). The calculation of \( \langle N(k, t) \rangle \) is completely the same as in case of S-brane dynamics [44] where it was shown that
\[ \langle N(k, t) \rangle = (2\pi)^D \delta_k(0) \frac{\Omega_k(t) - \tilde{G}(k, t) \left( \Omega_k(t) - \tilde{G}^{\ast}(k, t) \right)}{2\Omega_k(t) \left( \tilde{G}(k, t) + \tilde{G}^{\ast}(k, t) \right)}. \quad (2.34) \]
We can also define the vacuum expectation value of the spatial density of the number of particles as
\[
\langle N(k, t) \rangle \equiv \frac{\langle N(k, t) \rangle}{V} = \frac{\left( \Omega_k(t) - \bar{G}(k, t) \right) \left( \Omega_k(t) - \bar{G}^*(k, t) \right)}{2\Omega_k(t)(\bar{G}(k, t) + \bar{G}^*(k, t))}.
\] (2.35)
Let us calculate \( \langle N(k, t) \rangle \) for the vacuum state functional \( \Psi^\text{in}[\phi, t] \) in the limit \( t \to \infty \). In the completely the same way as in [44] we obtain that the density of the number of particle created with momentum \( k \) is equal to
\[
\langle N(k, t) \rangle = |B_k|^2.
\] (2.36)
Now using the fact that
\[
B_k^* = \sqrt{\frac{\omega_k \pi}{2}} e^{-\pi\omega_k/2} e^{-\frac{\pi\mu}{\omega_k}} \frac{\left( \frac{\pi\mu}{\omega_k} \right)^{-i\omega_k/2}}{\sinh \pi\omega_k \Gamma(1 - i\omega_k)},
\]
\[
|\Gamma(1 + i\omega_k)|^2 = \frac{\pi\omega_k}{\sinh(\pi\omega_k)},
\]
\[
|B_k|^2 = \frac{e^{-\pi\omega_k}}{2 \sinh 2\pi\omega_k} = \frac{1}{e^{\frac{\omega_k}{4\pi}} - 1}, \quad T = \frac{1}{2\pi}
\] (2.37)
we get the result that even if \( \Psi^\text{in}[\phi, t] \) is pure state the number of particle produced at far future is the same as in the thermal state of temperature \( T = \frac{1}{2\pi} \). Then the vacuum expectation value of the Hamiltonian \(^4\) at far future is
\[
\langle H(t) \rangle = \int \frac{dk}{(2\pi)^D} \Omega_k(t) \langle N(k, t) \rangle = V_D \sqrt{2\pi\mu} \int \frac{dk}{(2\pi)^D} e^{\frac{2\mu}{\omega_k}} \frac{1}{e^{\frac{\omega_k}{4\pi}} - 1},
\] (2.38)
where \( T_H = \frac{1}{4\pi} \) is Hadegorn temperature. However we must stress that the previous expression is not quiet correct since we did not consider a degeneracy of of closed string modes with initial energy \( \omega_k \). If we denote the degeneracy of closed string modes with initial energy \( \omega_k \) as \( \rho(\omega_k) \) then the total energy density that is dumped in the string pairs created in the rolling tachyon background is equal to
\[
E_{\text{tot}} = \sqrt{2\pi\mu} \int \frac{dk}{(2\pi)^D} \rho(\omega_k) e^{\frac{2\mu}{\omega_k}} \frac{1}{e^{\frac{\omega_k}{4\pi}} - 1}.
\] (2.39)
For large \( \omega_k \) we have \( \rho(\omega_k) \sim e^{\frac{2\mu}{\omega_k}} \) and consequently
\[
E_{\text{tot}} \sim \int d\omega_k \omega_k^{D-1} e^{\frac{2\mu}{\omega_k}} e^{\frac{2\mu}{\omega_k}} - e^{\frac{2\mu}{\omega_k}}\).
\] (2.40)
We immediately see that this integral diverges exponentially. As was firstly mentioned in [42] this divergence is much more stronger then in case of S-branes where the exponentials canceled and the divergence is at most by power law. This result implies
\footnote{We omit the zero-point energy contribution.}
that the gas of pair of produced closed strings will reach the string density in a time of one in string units. To see this let us again consider the vacuum expectation value of particle density \( \langle N(k, t) \rangle \) given in (2.35). As opposite to the previous case when we calculated the rate of produced closed strings at asymptotic future we will study this expression in the far past when we can write

\[
\psi^{in*}(t) \sim \frac{e^{i\omega_k t}}{\sqrt{2\omega_k}} \left(1 - \frac{\pi \mu e^{2t}}{2(1 + i\omega)}\right) \tag{2.41}
\]

and consequently

\[
\tilde{G}(k, t) = -i \frac{\dot{\psi}^{in*}(t)}{\psi^{in*}(t)} = \omega_k + \frac{i\pi \mu e^{2t}}{(1 + i\omega_k)} , \Omega_k(t) = \omega_k + \frac{\pi \mu e^{2t}}{\omega_k} . \tag{2.42}
\]

Using these results we can estimate (2.34) at far past as

\[
\langle N(k, t) \rangle \sim C_k e^{4t} , \tag{2.43}
\]

where \( C_k \) is some numerical factor containing powers of \( \omega_k \). Since the density of states for high-energy modes is \( \rho \sim e^{\omega_k/T_H} \) we obtain an estimate for the density of high-energy particles at far past as

\[
\langle N_{tot}(k, t) \rangle \sim \rho(\omega_k) \langle N(k, t) \rangle \sim e^{\frac{\omega_k}{T_H} + 4t} . \tag{2.44}
\]

We see that this expression will reach the string density \( \langle N_{tot}(k, t) \rangle \sim 1 \) for

\[
\frac{\omega_k}{T_H} + 4t \sim 0 \Rightarrow t \sim -\frac{\omega_k}{T_H} . \tag{2.45}
\]

The previous result implies that for high-energy modes the time when the density of produced closed string pairs reaches the string density is very close to the beginning of the rolling tachyon process roughly of order one in string units.

### 3. Conclusion

In this short note we have formulated the Schrödinger picture description of the quantum field theory that arises from the minisuperspace approach to the closed string rolling tachyon which was considered previously in [42, 43]. We have also calculated the rate of produced closed strings in rolling tachyon background using Schrödinger representation of quantum field theory and we have found completely agreement with [42].

To conclude this paper we would like to stress that even if this paper brings the modest contribution to the research of the tachyon condensation and time-dependent processes in string theory we hope that Schrödinger formulation of quantum field theory could be very useful for addressing these questions.

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