Enhancement of Non-Gaussianity after Inflation

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Abstract

We study the evolution of cosmological perturbations on large scales, up to second order, for a perfect fluid with generic equation of state. Taking advantage of super-horizon conservation laws, it is possible to follow the evolution of the non-Gaussianity of perturbations through the different stages after inflation. We find that a large non-linearity is generated by the gravitational dynamics from the original inflationary quantum fluctuations. This leads to a significant enhancement of the tiny intrinsic non-Gaussianity produced during inflation in single-field slow-roll models.

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1 INTRODUCTION

Inflation is the simplest and most successful mechanism proposed to date for the causal generation of primordial cosmological perturbations on cosmologically relevant scales [1]. The gravitational amplification of the primordial perturbations is supposed to seed structure formation in the Universe and produce Cosmic Microwave Background (CMB) anisotropies in agreement with observational data [2]. Due to the smallness of the primordial cosmological perturbations, their generation and evolution have usually been studied within linear theory [3]. After the seminal work by Tomita [4], only recently second-order perturbation theory [5–7] has been employed to evaluate specific physical observables generated during inflation [8, 9]. The importance of an accurate determination of higher-order statistics as the bispectrum comes from the fact that they allow to search for the signature of non-Gaussianity in the primordial perturbations which is usually parametrized by a dimensionless non-linear parameter $f_{NL}$. Indeed, a number of present and future CMB experiments, such as WMAP [10] and Planck, have enough resolution to either constrain or detect non-Gaussianity of CMB anisotropy data with high precision [11].

The main result of the second-order analysis performed in [8, 9] is that single-field slow-roll models of inflation give rise to a level of intrinsic non-Gaussianity which – at the end of the inflationary stage – is tiny, being first-order in the slow-roll parameters.

The goal of this paper is to study the post-inflationary evolution on super-horizon scales of the primordial non-linearity in the cosmological perturbations. We perform a fully relativistic analysis of the dynamics of second-order perturbations for a perfect fluid with generic equation of state taking advantage of the super-horizon conservation of the second-order gauge-invariant curvature perturbation recently discussed in Refs. [12, 13] (see also [8, 14, 15]). Our main result is that the post-inflationary evolution gives rise to an enhancement of the level of non-Gaussianity on super-horizon scales. Once again, inflation provides the key generating mechanism to produce super-horizon seeds, which are later amplified by gravity.

The plan of the paper is as follows. In Section 2 we provide the second-order expansion of the metric and of the energy-momentum tensor, assuming that the source term is represented by a perfect fluid with constant equation of state. In Section 3 we solve the perturbed Einstein equations up to first order around a Friedmann-Robertson-Walker background. The body of the paper is contained in Section 4, where we derive the super-horizon evolution equations of the second-order gravitational potential and density perturbations. Section 5 contains a brief discussion of our findings.

2 PERTURBATIONS OF A FLAT ROBERTSON-WALKER UNIVERSE UP TO SECOND ORDER

In order to study the perturbed Einstein equations, we first write down the perturbations on a spatially flat Robertson-Walker background following the formalism of Refs. [5, 6].
We shall first consider the fluctuations of the metric, and then the fluctuations of the energy-momentum tensor. Hereafter greek indices run from 0 to 3, while latin indices label the spatial coordinates from 1 to 3. If not otherwise specified we will work with conformal time $\tau$, and a prime will stand for a derivative with respect to $\tau$.

2.1 The metric tensor

The components of a perturbed spatially flat Robertson-Walker metric can be written as

\[
g_{00} = -a^2(\tau) \left(1 + 2\phi^{(1)} + \phi^{(2)}\right),
\]

\[
g_{0i} = a^2(\tau) \left(\dot{\omega}_i^{(1)} + \frac{1}{2}\omega_i^{(2)}\right),
\]

\[
g_{ij} = a^2(\tau) \left[(1 - 2\psi^{(1)} - \psi^{(2)})\delta_{ij} + \left(\dot{\chi}_{ij}^{(1)} + \frac{1}{2}\chi_{ij}^{(2)}\right)\right],
\]

(2.1)

where the scale factor $a$ is a function of the conformal time $\tau$. The standard splitting of the perturbations into scalar, transverse (i.e. divergence-free) vector parts, and transverse trace-free tensor parts with respect to the 3-dimensional space with metric $\delta_{ij}$ can be performed in the following way:

\[
\dot{\omega}_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)},
\]

(2.2)

\[
\dot{\chi}_{ij}^{(r)} = D_{ij}\chi^{(r)} + \partial_i \chi_{ij}^{(r)} + \partial_j \chi_i^{(r)} + \chi_{ij}^{(r)},
\]

(2.3)

where $(r) = (1),(2)$ stand for the order of the perturbations, $\omega_i$ and $\chi_i$ are transverse vectors ($\partial^i \omega_i^{(r)} = \partial^i \chi_i^{(r)} = 0$), $\chi_{ij}^{(r)}$ is a symmetric transverse and trace-free tensor ($\partial^i \chi_{ij}^{(r)} = 0$, $\chi_i^{(r)} = 0$) and $D_{ij} = \partial_i \partial_j - (1/3) \delta_{ij} \partial_k \partial_k$ is a trace-free operator. Here and in the following latin indices are raised and lowered using $\delta_{ij}$ and $\delta^{ij}$, respectively.

For our purposes the metric in Eq. (2.1) can be simplified. In fact, first-order vector perturbations are zero; moreover, the tensor part gives a negligible contribution to second-order perturbations. Thus, in the following we can neglect $\omega_i^{(1)}$, $\chi_{(1)i}$ and $\chi_{(1)ij}$. However the same is not true for the second order perturbations. In the second-order theory the second-order vector and tensor contributions can be generated by the first-order scalar perturbations even if they are initially zero [6]. Thus we have to take them into account and we shall use the metric

\[
g_{00} = -a^2(\tau) \left(1 + 2\phi^{(1)} + \phi^{(2)}\right),
\]

\[
g_{0i} = a^2(\tau) \left(\partial_i \omega^{(1)} + \frac{1}{2}\partial_i \omega^{(2)} + \frac{1}{2}\omega_i^{(2)}\right),
\]

\[
g_{ij} = a^2(\tau) \left[(1 - 2\psi^{(1)} - \psi^{(2)})\delta_{ij} + D_{ij} \left(\chi^{(1)} + \frac{1}{2}\chi^{(2)}\right)\right.
\]

\[
+ \left.\frac{1}{2} \left(\partial_i \chi_i^{(2)} + \partial_j \chi_j^{(2)} + \chi_{ij}^{(2)}\right)\right].
\]

(2.4)
The contravariant metric tensor is obtained by requiring (up to second order) that \( g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda \) and it is given by

\[
\begin{align*}
g^{00} &= -a^{-2}(\tau) \left( 1 - 2\phi^{(1)} - \phi^{(2)} + 4 \left( \phi^{(1)} \right)^2 - \partial^i \omega^{(1)} \partial_i \omega^{(1)} \right), \\
g^{0i} &= a^{-2}(\tau) \left[ \partial^i \omega^{(1)} + \frac{1}{2} \left( \partial^i \omega^{(2)} + \omega^{i(2)} \right) + 2 \left( \psi^{(1)} - \phi^{(1)} \right) \partial^i \omega^{(1)} - \partial^i \omega^{(1)} D^j_k \chi^{(1)} \right], \\
g^{ij} &= a^{-2}(\tau) \left[ \left( 1 + 2\psi^{(1)} + \psi^{(2)} + 4 \left( \psi^{(1)} \right)^2 \right) \delta^{ij} - D^{ij} \left( \chi^{(1)} + \frac{1}{2} \chi^{(2)} \right) \right. \\
&\quad - \frac{1}{2} \left( \partial^i \chi^{j(2)} + \partial^j \chi^{i(2)} + \chi^{ij(2)} \right) - \partial^i \omega^{(1)} \partial^j \omega^{(1)} \\
&\quad \left. - 4\psi^{(1)} D^{ij} \chi^{(1)} + D^{ik} \lambda^{(1)} D^j_k \chi^{(1)} \right].
\end{align*}
\] (2.5)

Using \( g_{\mu\nu} \) and \( g^{\mu\nu} \) one can calculate the connection coefficients and the Einstein tensor components up to second order in the metric fluctuations. They are given in the Appendix A of Ref. [8]. From now on, we will adopt the Poisson gauge \([16]\) which is defined by the condition \( \omega = \chi = \chi_i = 0 \). Then, one scalar degree of freedom is eliminated from \( g_{0i} \) and one scalar and two vector degrees of freedom from \( g_{ij} \). This gauge generalizes the so-called longitudinal gauge to include vector and tensor modes and contains a solenoidal vector \( \omega^{(2)}_i \).

### 2.2 Energy-momentum tensor of the fluid

Since after inflation and reheating the Universe enters a radiation-dominated phase and, subsequently, a matter- and dark energy-dominated phases, we shall consider a generic fluid characterized by an energy density \( \rho \) and pressure \( P \) with energy-momentum tensor

\[
T_{\mu\nu} = (\rho + P) u^\mu u_\nu + P \delta^\mu_\nu,
\] (2.6)

where \( u^\mu \) is the four-velocity vector subject to the constraint \( g^{\mu\nu} u_\mu u_\nu = -1 \). At second order of perturbation theory it can be decomposed as

\[
u^\mu = \frac{1}{a} \left( \delta^\mu_0 + v^{(1)}_\mu + \frac{1}{2} v^{(2)}_\mu \right).
\] (2.7)

For the first- and second-order perturbations, we get

\[
\begin{align*}
v^0_{(1)} &= -\psi^{(1)}, \\
v^0_{(2)} &= -\phi^{(2)} + 3 \left( \psi^{(1)} \right)^2 + v^{(1)}_i v^{(1)}_i.
\end{align*}
\] (2.8)

Similarly, we obtain
\[ u_0 = a \left( -1 - \phi^{(1)} - \frac{1}{2} \phi^{(2)} + \frac{1}{2} \left( \psi^{(1)} \right)^2 - \frac{1}{2} v_i^{(1)} v_i^{(1)} \right), \]
\[ u_i = a \left( v_i^{(1)} + \frac{1}{2} v_i^{(2)} - 2 \psi^{(1)} v_i^{(1)} + \frac{1}{2} \omega_i^{(2)} \right). \] (2.9)

The energy density \( \rho \) can be split into a homogeneous background \( \rho_0(\tau) \) and a perturbation \( \delta \rho(\tau, x^i) \) as follows

\[ \rho(\tau, x^i) = \rho_0(\tau) + \delta \rho(\tau, x^i) = \rho_0(\tau) + \delta^{(1)} \rho(\tau, x^i) + \frac{1}{2} \delta^{(2)} \rho(\tau, x^i), \] (2.10)

where the perturbation has been expanded into a first and a second-order part, respectively. The same decomposition can be adopted for the pressure \( P \).

Using the expression (2.10) into Eq. (2.6) and calculating \( T^\mu_\nu \) up to second order we find

\[ T^\mu_\nu = T^{\mu(0)}_\nu + \delta^{(1)} T^{\mu}_\nu + \delta^{(2)} T^{\mu}_\nu, \] (2.11)

where \( T^{\mu(0)}_\nu \) corresponds to the background value, and

\[ T^{0(0)}_0 + \delta^{(1)} T^{0}_0 = - \rho_0 - \delta^{(1)} \rho, \] (2.12)
\[ \delta^{(2)} T^{0}_0 = - \frac{1}{2} \delta^{(2)} \rho - (1 + w) \rho_0 v_i^{(1)} v_i^{(1)}, \] (2.13)

\[ T^{i(0)}_0 + \delta^{(1)} T^{i}_0 = - (1 + w) \rho_0 v_i^{(1)}; \] (2.14)
\[ \delta^{(2)} T^{i}_0 = - (1 + w) \rho_0 \left[ \left( \psi^{(1)} + \frac{\delta^{(1)} \rho}{\rho_0} \right) v_i^{(1)} + \frac{1}{2} v_i^{(2)} \right], \] (2.15)

\[ T^{i(0)}_j + \delta^{(1)} T^{i}_j = w \rho_0 \left( 1 + \frac{\delta^{(1)} \rho}{\rho_0} \right) \delta^i_j, \] (2.16)
\[ \delta^{(2)} T^{i}_j = (1 + w) \rho_0 v_i^{(1)} v_j^{(1)} + \frac{1}{2} w \delta^{(2)} \rho \delta^i_j. \] (2.17)

In the previous expressions we have made the assumption that the pressure \( P \) can be expressed in terms of the energy density as \( P = w \rho \) with constant \( w \).

3 Basic First-Order Einstein Equations on Large-Scales

Our starting point are the perturbed Einstein equations \( \delta G^\mu_\nu = \kappa^2 \delta T^\mu_\nu \) in the Poisson gauge. Here \( \kappa^2 \equiv 8 \pi G_N \). At first-order, the \((0-0)\)- and \((i-0)\)-components of Einstein equations are
\[ \frac{1}{a^2} \left[ 6 \mathcal{H}^2 \phi^{(1)} + 6 \mathcal{H} \psi^{(1)'} - 2 \nabla^2 \psi^{(1)} \right] = -\kappa^2 \delta^{(1)} \rho, \]  
\[ \frac{2}{a^2} \left( \mathcal{H} \partial^2 \phi^{(1)} + \partial^2 \psi^{(1)'} \right) = -\kappa^2 (1 + w) \rho_0 v^{(1)}, \]

where we have indicated by $\mathcal{H} = \frac{\dot{a}}{a}$ the Hubble rate in conformal time. These equations, together with the non-diagonal part of the $(i - j)$-component of Einstein equations, give $\psi^{(1)} = \phi^{(1)}$ and, on super-horizon scales,

\[ \psi^{(1)} = -\frac{1}{2} \frac{\delta^{(1)} \rho}{\rho_0} = \frac{3(1 + w)}{2} \mathcal{H} \frac{\delta^{(1)} \rho}{\rho_0}. \]  

The continuity equation yields an evolution equation for the large-scale energy density perturbation

\[ \delta^{(1)} \rho' + 3 \mathcal{H} (1 + w) \delta^{(1)} \rho - 3 \psi^{(1)'} (1 + w) \rho_0 = \frac{2}{3} \frac{\rho_0}{\mathcal{H}^2} \nabla^2 \left( \psi^{(1)'} + \mathcal{H} \psi^{(1)} \right). \]  

This equation, together with the the background continuity equation $\rho_0' + 3 \mathcal{H} (1 + w) \rho_0 = 0$, leads to the conservation on large-scales of the first-order gauge-invariant curvature perturbation [3]

\[ \zeta^{(1)} = -\psi^{(1)} - \mathcal{H} \frac{\delta^{(1)} \rho}{\rho_0}. \]

Indeed, both the density perturbation, $\delta \rho$ and the curvature perturbation, $\psi$, are in general gauge-dependent. Specifically, they depend upon the chosen time-slicing in an inhomogeneous universe. The curvature perturbation on fixed time hypersurfaces is a gauge-dependent quantity: after an arbitrary linear coordinate transformation at first-order, $t \rightarrow t + \delta t$, it transforms as $\psi^{(1)} \rightarrow \psi^{(1)} + \mathcal{H} \delta t$. For a scalar quantity, such as the energy density, the corresponding transformation is $\delta \rho^{(1)} \rightarrow \delta \rho^{(1)} - \rho_0' \delta t$. However the gauge-invariant combination $\zeta^{(1)}$ can be constructed which describes the density perturbation on uniform curvature slices or, equivalently the curvature of uniform density hypersurfaces. On large scales $\zeta^{(1)'} \simeq 0$. Using Eq. (3.3) and the background continuity equation, we can determine

\[ \psi^{(1)} = -\frac{3(1 + w)}{5 + 3w} \zeta^{(1)}, \]  

which is useful to relate the curvature $\psi^{(1)}$ during either the matter or the radiation epoch to the gauge-invariant curvature perturbation $\zeta^{(1)}$ at the end of the inflationary stage. Indeed, since $\zeta^{(1)}$ is constant, we can write
\[
\psi^{(1)} = \frac{3(1 + w)}{5 + 3w} \zeta^{(1)}, \tag{3.7}
\]

where the subscript "I" means that \( \zeta^{(1)} \) is computed during the inflationary stage.

4 Basic second-order Einstein equations on large-scales and non-Gaussianity

In order to determine the non-Gaussianity of the cosmological perturbations after inflation, we have to derive the behaviour on large-scales of the metric and the energy density perturbations at second order. Again, our starting point are the Einstein equations perturbed at second order \( \delta^{(2)} G_{\mu \nu} = \kappa^2 \delta^{(2)} T_{\mu \nu} \) in the Poisson gauge. The second-order expression for the Einstein tensor \( \delta^{(2)} G_{\mu \nu} \) can be found in any gauge in the Appendix A of Ref. [8] and we do not report it here.

- The \((0-0)\)-component of Einstein equations (see Eq. (A.39) in Ref. [8]) leads to

\[
3 \mathcal{H}^2 \phi^{(2)} + 3 \mathcal{H} \psi^{(2)'} - \nabla^2 \psi^{(2)} - 12 \mathcal{H}^2 (\psi^{(1)})^2 - 3 (\nabla \psi^{(1)})^2 \\
-8 \psi^{(1)} \nabla^2 \psi^{(1)} - 3 (\psi^{(1)})^2 = \kappa^2 a^2 \delta^{(2)} T^{0}_{0}, \tag{4.1}
\]

which, on large super-horizon scales reduces to

\[
\phi^{(2)} = -\frac{1}{2} \frac{\delta^{(2)} \rho}{\rho_0} + 4 (\psi^{(1)})^2. \tag{4.2}
\]

- A relation between the gravitational potentials at second-order \( \psi^{(2)} \) and \( \phi^{(2)} \) can be obtained from the traceless part of the \((i-j)\) components of Einstein’s equations (see Eqs. (A.42) and (A.43) in Ref. [8]). We find

\[
\psi^{(2)} - \phi^{(2)} = -4 (\psi^{(1)})^2 - \nabla^{-2} \left( 2 \partial^i \psi^{(1)} \partial_i \psi^{(1)} + 3 \mathcal{H}^2 \psi^{(1)} v^{(1)} \right) \\
+ 3 \nabla^{-4} \partial_i \partial^i \left( 2 \partial^j \psi^{(1)} \partial_j \psi^{(1)} + 3 \mathcal{H}^2 v^{(1)} \right). \tag{4.3}
\]

This constraint is the second-order equivalent of the linear constraint \( \psi^{(1)} = \phi^{(1)} \) in the Poisson gauge.

- In order to close the system and fully determine the variables \( \psi^{(2)} \), \( \phi^{(2)} \) and \( \delta^{(2)} \rho \), we use the energy conservation at second-order and the divergence of the \((i-0)\)-component of Einstein equations (see Eq. (A.40) in Ref. [8])\footnote{Notice that Eq. (4.4) generalizes Eq. (5.33) of Ref. [13] and corrects a sign misprint in front of the fourth term of that equation.}

\[
\text{Notice that Eq. (4.4) generalizes Eq. (5.33) of Ref. [13] and corrects a sign misprint in front of the fourth term of that equation.}
\]
\[\delta^{(2)}\rho' + 3\mathcal{H}(1 + w) \delta^{(2)}\rho - 3(1 + w) \rho_0 \psi^{(2)}' - 6(1 + w) \psi^{(1)}' \left[\delta^{(1)}\rho + 2\rho_0 \psi^{(1)}\right] = -2(1 + w)\rho_0 \left(v^i_i v^i_{(1)}\right)' - 2(1 + w)(1 - 3w)\mathcal{H}\rho_0 v^i_i v^i_{(1)} + 4(1 + w)\rho_0 \partial_i \psi^{(1)} v^i_{(1)} + 2\rho_0 \mathcal{H}^2 \left(\psi^{(1)} \nabla^2 \psi^{(1)}' - \psi^{(1)} \nabla^2 \psi^{(1)}\right). \] (4.4)

This equation can be rewritten in a more suitable form

\[
\left[\psi^{(2)} + \mathcal{H} \frac{\delta^{(2)}\rho}{\rho_0} + (1 + 3w)\mathcal{H}^2 \left(\frac{\delta^{(1)}\rho}{\rho_0}\right)^2 - 4\mathcal{H} \left(\frac{\delta^{(1)}\rho}{\rho_0}\right) \psi^{(1)}\right]' = \frac{2}{3} \left(v^i_i v^i_{(1)}\right)' \\
+ \frac{2}{3}(1 - 3w)\mathcal{H} v^i_i v^i_{(1)} - \frac{4}{3} \partial_i \psi^{(1)} v^i_{(1)} + \frac{16}{27 (1 + w)^2 \mathcal{H}} \psi^{(1)} \nabla^2 \psi^{(1)} \\
- \frac{2}{3 (1 + w) \mathcal{H}^2} \left[\left(1 - \frac{8}{9 (1 + w)}\right) \psi^{(1)} \nabla^2 \psi^{(1)}' - \left(1 - \frac{4(1 + 3w)}{9 (1 + w)}\right) \psi^{(1)} \nabla^2 \psi^{(1)}\right] \\
+ \frac{8(1 + 3w)}{27 (1 + w)^2 \mathcal{H}^3} \left[\frac{(\nabla^2 \psi^{(1)})^2}{3} - \psi^{(1)} \nabla^2 \psi^{(1)} + \frac{\nabla^2 \psi^{(1)} \nabla^2 \psi^{(1)}}{3 \mathcal{H}}\right], \] (4.5)

where the argument on the left-hand side can be further simplified to

\[
\psi^{(2)} + \mathcal{H} \frac{\delta^{(2)}\rho}{\rho_0} - (5 + 3w)\mathcal{H}^2 \left(\frac{\delta^{(1)}\rho}{\rho_0}\right)^2 = \psi^{(2)} + \mathcal{H} \frac{\delta^{(2)}\rho}{\rho_0} - \frac{4}{5 + 3w} \left(\zeta^{(1)}\right)^2 \] (4.6)

and the final form has been obtained employing Eq. (3.7) [17]. From Eqs. (4.5) and (4.6) we find

\[
\psi^{(2)} + \mathcal{H} \frac{\delta^{(2)}\rho}{\rho_0} - (5 + 3w)\mathcal{H}^2 \left(\frac{\delta^{(1)}\rho}{\rho_0}\right)^2 = C + \int^{\tau} d\tau' S(\tau'), \] (4.7)

where \(C\) is a constant in time, \(C' = 0\), on large-scales and

\[
S = \frac{2}{3}(1 - 3w)\mathcal{H} v^i_i v^i_{(1)} - \frac{4}{3} \partial_i \psi^{(1)} v^i_{(1)} + \frac{16}{27 (1 + w)^2 \mathcal{H}} \psi^{(1)} \nabla^2 \psi^{(1)} \\
- \frac{2}{3 (1 + w) \mathcal{H}^2} \left[\left(1 - \frac{8}{9 (1 + w)}\right) \psi^{(1)} \nabla^2 \psi^{(1)}' - \left(1 - \frac{4(1 + 3w)}{9 (1 + w)}\right) \psi^{(1)} \nabla^2 \psi^{(1)}\right] \\
+ \frac{8(1 + 3w)}{27 (1 + w)^2 \mathcal{H}^3} \left[\frac{(\nabla^2 \psi^{(1)})^2}{3} - \psi^{(1)} \nabla^2 \psi^{(1)} + \frac{\nabla^2 \psi^{(1)} \nabla^2 \psi^{(1)}}{3 \mathcal{H}}\right]. \] (4.8)
4.1 Determination of the non-linearity parameter

Since we are interested in the determination of the non-linear parameter \( f_{\phi}^{NL} \) after the inflationary stage, it is convenient to fix the constant \( C \) by matching the conserved quantity at the end of inflation \((\tau = \tau_I)\)

\[
C = \psi^{(2)}_I + \mathcal{H}_I \frac{\delta^{(2)} \rho_I}{\rho_{oi}} - 2 \left( \zeta^{(1)}_I \right)^2 , \tag{4.9}
\]

where we have used the fact that during inflation \( w_I \simeq -1 \).

The inflationary quantity \( \left( \psi^{(2)}_I + \mathcal{H}_I \frac{\delta^{(2)} \rho_I}{\rho_{oi}} \right) \) has been computed in Refs. [8,9]

\[
\psi^{(2)}_I + \mathcal{H}_I \frac{\delta^{(2)} \rho_I}{\rho_{oi}} \simeq (\eta - 3\epsilon) \left( \zeta^{(1)}_I \right)^2 + \mathcal{O}(\epsilon, \eta) \text{ (non-local terms)} , \tag{4.10}
\]

in terms of the slow-roll parameters \( \epsilon = 1 - \mathcal{H}_I'/\mathcal{H}_I^2 \) and \( \eta = 1 + \epsilon - (\varphi''/\mathcal{H}_I \varphi') \) where \( \mathcal{H}_I \) is the Hubble parameter during inflation and \( \varphi \) is the inflaton field driving the exponential growth of the scale factor during inflation [1]. Since during inflation the slow-roll parameters are tiny, we can safely disregard the intrinsically second-order terms originated from the inflationary epoch.

Combining Eqs. (4.2), (4.3), (4.7) and (4.9), we find

\[
\phi^{(2)} = 2 \left( \psi^{(1)} \right)^2 + \frac{3(1 + w)}{5 + 3w} \int^{\tau} d\tau' \mathcal{S}(\tau') \nonumber \\
+ \frac{3(1 + w)}{5 + 3w} \left[ \nabla^{-2} \left( 2 \partial^i \psi^{(1)} \partial_i \psi^{(1)} + 3 \left( 1 + w \right) \mathcal{H}^2 \psi^{(1)}_i v^{(1)}_i \right) \right. \nonumber \\
- \left. 3 \nabla^{-4} \partial_i \partial^j \left( 2 \partial^i \psi^{(1)} \partial_j \psi^{(1)} + 3 \left( 1 + w \right) \mathcal{H}^2 \psi^{(1)}_i v^{(1)}_j \right) \right] . \tag{4.11}
\]

As the gravitational potential \( \psi^{(1)} \) on super-horizon is generated during inflation, it is clear that the origin of the non-linearity traces back to the inflationary quantum fluctuations.

The total curvature perturbation will then have a non-Gaussian \((\chi^2)\)-component. For instance, the lapse function \( \phi = \phi^{(1)} + \frac{1}{2} \phi^{(2)} \) can be expressed in momentum space as

\[
\phi(k) = \phi^{(1)}(k) + \frac{1}{(2\pi)^3} \int d^3k_1 d^3k_2 \delta^{(3)}(k_1 + k_2 - k) f^{\phi}_{NL}(k_1, k_2) \phi^{(1)}(k_1) \phi^{(1)}(k_2) , \tag{4.12}
\]

where we have defined an effective “momentum-dependent” non-linearity parameter \( f^{\phi}_{NL} \). Here the linear lapse function \( \phi^{(1)} = \psi^{(1)} \) is a Gaussian random field. The gravitational potential bispectrum reads

\[
\langle \phi(k_1) \phi(k_2) \phi(k_3) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \left[ 2 f^{\phi}_{NL}(k_1, k_2) \mathcal{P}_\phi(k_1) \mathcal{P}_\phi(k_2) + \text{cyclic} \right] , \tag{4.13}
\]
where $\mathcal{P}_\phi(k)$ is the power-spectrum of the gravitational potential. From Eq. (4.13) and after some manipulations on the non-local terms, we read the corresponding non-linearity parameter for scales entering the horizon during the matter-dominated stage

$$f_{\phi}^{NL} \simeq -\frac{1}{2} + \frac{k_1 \cdot k_2}{k^2} \left(1 + 3 \frac{k_1 \cdot k_2}{k^2}\right) + \frac{3}{10} \int_{\tau_I}^{\tau} d\tau' S(\tau'), \quad (4.14)$$

where $k = |k_1 + k_2|$.

The non-Gaussianity provided by expression (4.14) will add to the known Newtonian and relativistic second-order contributions which are relevant on sub-horizon scales, such as the Rees-Sciama effect [18], whose detailed analysis has been given in Refs. [19].

5 Conclusions

In this paper we have provided a framework to study the evolution of non-linearities present in the primordial cosmological perturbations seeded by inflation on super-horizon scales. The tiny non-Gaussianity generated during the inflationary epoch driven by a single scalar field gets enhanced in the post-inflationary stages giving rise to a non-negligible signature of non-linearity in the gravitational potentials. On the other hand, there are many physically motivated inflationary models which can easily accomodate for a primordial value of $f_{NL}$ larger than unity. This is the case, for instance, of a large class of multi-field inflation models which leads to either non-Gaussian isocurvature perturbations [20] or cross-correlated non-Gaussian adiabatic and isocurvature modes [21]. Other interesting possibilities include the “curvaton” model, where the late time decay of a scalar field other than the inflaton induces curvature perturbations [22], and the so-called “inhomogeneous reheating” mechanism where the curvature perturbations are generated by spatial variations of the inflaton decay rate [23]. Our findings indicate that a positive future detection of non-linearity in the CMB anisotropy pattern will not rule out single field models as responsible for seeding structure formation in our Universe.

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References


[17] One can recognize the second-order gauge-invariant curvature perturbation (up to a gradient term) recently introduced in Ref. [13]

\[
\zeta^{(2)} = -\psi^{(2)} - \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0^\prime} + 2\mathcal{H}\frac{\delta^{(1)}\rho}{\rho_0^\prime} \frac{\delta^{(1)}\rho}{\rho_0^\prime} + 2\frac{\delta^{(1)}\rho}{\rho_0^\prime} \left(\psi^{(1)} + 2\mathcal{H}\psi^{(1)}\right) - \left(\frac{\delta^{(1)}\rho}{\rho_0^\prime}\right)^2 \left(\mathcal{H}\frac{\rho_0^\prime}{\rho_0^\prime} - \mathcal{H}' - 2\mathcal{H}^2\right)
\]

\[
= -\psi^{(2)} - \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0^\prime} - (1 + 3w)\mathcal{H}^2 \left(\frac{\delta\rho^{(1)}}{\rho_0^\prime}\right)^2 + 4\mathcal{H} \left(\frac{\delta\rho^{(1)}}{\rho_0^\prime}\right) \psi^{(1)},
\]

where in the last passage we have made use of the first-order continuity equation (3.4). The quantity \(\zeta^{(2)}\) satisfies the conservation equation \(\zeta^{(2)} = 4\zeta^{(1)}\zeta^{(1)}\) leading to
\[ \zeta^{(2)} - 2 \left( \zeta^{(1)} \right)^2 = \text{constant}. \]  Since \( \zeta^{(1)} \) is conserved on large-scales, this implies that \( \zeta^{(2)} \) is conserved as well. Incidentally, we note that the combination \( \zeta^{(2)} - 2 \left( \zeta^{(1)} \right)^2 \) is equal to the conserved quantity defined in Ref. [12]. Of course, every quantity differing from \( \zeta^{(2)} \) by \( c \left( \zeta^{(1)} \right)^2 \) with \( c \) an arbitrary constant, is constant on super-horizon scales.


