D-brane Dynamics in the $c = 1$ Matrix Model

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Abstract

Recent work has shown that unstable D-branes in two dimensional string theory are represented by eigenvalues in a dual matrix model. We elaborate on this proposal by showing how to systematically include higher order effects in string perturbation theory. The full closed string state produced by a rolling open string tachyon corresponds to a sum of string amplitudes with any number of boundaries and closed string vertex operators. These contributions are easily extracted from the matrix model. As in the AdS/CFT correspondence, the sum of planar diagrams in the open string theory is directly related to the classical theory in the bulk, i.e. sphere diagrams. We also comment on the description of static D-branes in the matrix model, in terms of a solution representing a deformed Fermi sea.

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1. Introduction

The presence of a tachyon in the spectrum of a string theory signals an instability of the string background. Recently, following the seminal work of Sen [1], a great deal of progress has been made in understanding the dynamics of open string tachyon condensation, which is associated with the decay of unstable D-branes. In particular the rolling tachyon solution [2][3] provides an exact boundary CFT describing the decay of a bosonic D-brane. The rolling tachyon has been analyzed from various points of view, among them are open string particle production in the rolling tachyon solution [4,5,6], supergravity description of S-branes [7,8,9,10], time dependent solutions in string field theory [11,12,13,14,15,16,17,18], thermodynamics [19], rolling tachyons in different backgrounds [20,21,22], radiation into closed strings [23,24,25], application to of rolling tachyons to cosmology [26,27,28,29] and generalizations to closed string tachyon condensation [30,31].

In [25] it was found that to leading order in $g_s$ the energy radiated into closed strings from the decay of unstable D0 branes diverges, suggesting that the unstable brane decays completely into closed strings. An interesting question is how to reconcile this with a weak coupling open string analysis, which indicates a decay into a new form of “tachyon matter” [3].

Strings in two dimensions are interesting toy models to analyze questions which are difficult to address in critical string theory, since one can take advantage of a reformulation of the theory as a hermitian matrix model in the double scaling limit (See [32,33,34,35] and references therein). In the singlet sector the degrees of freedom reduce to the matrix eigenvalues, whose dynamics is in turn equivalent to a theory of free fermions in a potential. In the double scaling limit, the eigenvalue distribution becomes continuous and is interpreted as a spatial dimension.

The analysis of the decay of an unstable brane in two dimensional string theory was initiated in [36], where it was proposed that in the free fermion formulation of the two dimensional string an unstable brane corresponds to a free fermion which is moved from the Fermi sea to top of the inverted harmonic oscillator potential. The subsequent decay is described by the rolling of the fermion from the top to the spatial region occupied by the Fermi sea.

In [37] (see also [38][39]) this matrix model process was identified with a rolling tachyon boundary CFT which is constructed by tensoring Sen’s rolling tachyon in the time direction with a boundary state for the Liouville theory introduced by the Zamolodchikovs [40][41]. This boundary state corresponds to a Dirichlet brane localized in the Liouville direction and has a tachyon in the open string spectrum.

The nonrelativistic fermion that is rolling down the potential becomes relativistic at late times. The closed string excitations can be identified with the boson which appears in the bosonization of this fermion. This naturally leads to an identification of the rolling
fermion at late times with the coherent state produced by the closed string radiation at leading order in string perturbation theory. An important check of this proposal \cite{37} is the agreement of the outgoing radiation derived from the matrix model and the disk amplitudes in the two dimensional string theory.

The matrix model is a very convenient and powerful method for computing amplitudes in string perturbation theory. We will put to use old results on the tree level tachyon $S$-matrix to extend the study of the rolling tachyon to higher order in $g_s$. The higher order contributions to the outgoing closed string state can then be identified as coming from worldsheets with various numbers of holes and vertex operators. Using the closed string tree level $S$-matrix, we will sum up the contributions from planar diagrams with arbitrary numbers of holes, just as in AdS/CFT. Indeed, it is now appreciated that the relation between two dimensional string theory and the matrix model is perhaps the simplest realization of a holographic duality. Beyond the disk one-point function there are no continuum calculations we can compare our results against. However, we can make some consistency checks, such as the fact that introducing $N$ unstable D-branes correctly yields a factor of $N$ for each worldsheet boundary.

The plan for this note is as follows. Section two reviews the computation of the $S$-matrix for the scattering of closed string tachyons in the matrix model, following Polchinski \cite{42,34}. In section three the fermionic description of the decay of the unstable D-brane \cite{36,37} is reviewed. After bosonization, the relation between in and out bosonic oscillators is used to derive the $S$-matrix for closed strings in the decaying brane background. In section four these matrix model results are used to predict the disk $n$-point function in the rolling tachyon background. In section five the results are compared to string perturbation theory and it is found that worldsheets with multiple boundaries contribute once operator ordering is taken into account. In section six a modification of the matrix model is discussed where classically the fermion sits at the top of the potential forever. It is suggested that this state corresponds to the unstable D0 brane with the tachyon set to zero. We test this proposal by studying closed string scattering in this background. We close this note with a discussion and speculation regarding our results.

2. Review of Scattering in the Matrix Model

The $c = 1$ matrix model has a spacetime interpretation consisting of massless particles ("tachyons") propagating in an inhomogeneous 1+1 dimensional spacetime. The spacetime is effectively semi-infinite due to the presence of an exponentially rising tachyon condensate (the "Liouville wall"), and one computes an $S$-matrix describing the scattering of tachyons. While only limited results are available in the usual worldsheet CFT approach \cite{43,44} the matrix model formulation leads to explicit results to all orders in the string coupling. Tree
level amplitudes are particularly simple to extract, as we now review. We mainly follow
the discussion in [34]. Additional reviews include [32,33,35,45].

Starting from an action for $N \times N$ hermitian matrices,

$$S = \beta N \int dt \left\{ \frac{1}{2} \text{Tr}(\dot{M})^2 - \text{Tr}(M) \right\}$$  \hspace{1cm} (2.1)

a standard procedure (see [34]) leads to a second quantized Hamiltonian describing non-
relativistic fermions moving in an inverted harmonic oscillator potential,

$$H = \int dx \left\{ \frac{1}{2} \partial_x \psi^\dagger \partial_x \psi - \frac{x^2}{2} \psi^\dagger \psi + \mu \psi^\dagger \psi \right\}.$$  \hspace{1cm} (2.2)

$\psi$ is a fermionic field obeying the anticommutation relations

$$\{\psi(x,t), \psi^\dagger(x',t)\} = \delta(x - x').$$  \hspace{1cm} (2.3)

We will be interested in the region $x \leq 0$; the role of the second $x > 0$ region is explained
in [46,47]. In the ground state of our system all energy levels up to $E = 0$ are filled by
fermions, so the Fermi surface is a distance $\mu$ below the top of the potential. Classically,
fermions at the Fermi surface reflect off the potential at $x = -\sqrt{2\mu}$. $\mu$ is related to the
string coupling:

$$\mu \sim 1/g.$$  \hspace{1cm} (2.4)

One class of excited states consists of smooth fluctuations of the Fermi surface. Fluc-
tuations propagating in from $x = -\infty$ scatter off the potential and propagate back out
to $x = -\infty$. This is the matrix model version of the tachyon S-matrix. The tree level S-
matrix is obtained by treating the fermions classically. In the classical limit, each fermion
is described by a point in phase space, moving according to its classical equations of mo-
tion, and we think of the Fermi sea as the region of phase space occupied by fermions.
Since the single particle Hamiltonian is

$$H = \frac{p^2 - x^2}{2} + \mu,$$  \hspace{1cm} (2.5)

filling states up to $E = 0$ means that in the ground state the Fermi sea is the region
bounded by $p_{\pm}(x)_{\text{gnd}}$, where

$$p_{\pm}(x)_{\text{gnd}} = \pm \sqrt{x^2 - 2\mu}.$$  \hspace{1cm} (2.6)

Excited states are then described by other choices for $p_{\pm}(x)$, and have a total energy (we
drop the constant $N\mu$ term)

$$H = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{p_-}^{p_+} dp \frac{1}{2}(p^2 - x^2)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left\{ \frac{1}{6}(p_+^3 - p_-^3) - \frac{1}{2}x^2(p_+ - p_-) \right\}.$$  \hspace{1cm} (2.7)
It is convenient to write
\[ p_\pm(x,t) = \mp x \pm \frac{1}{x} \epsilon_\pm(x,t) \] (2.8)
and to further relate \( \epsilon_\pm \) to a massless scalar field \( S \) via
\[ \pi^{-1/2} \epsilon_\pm(q,t) = \pm \pi S(q,t) - \partial_q S(q,t) \] (2.9)
where
\[ q = -\ln(-x). \] (2.10)

The Hamiltonian (2.7) then becomes
\[ H = \frac{1}{2} \int_{-\infty}^{\infty} dq \left\{ \pi_S^2 + (\partial_q S)^2 + \pi^{1/2} e^{2q} \left[ \partial_q S \pi_S^2 + \frac{1}{3} (\partial_q S)^3 \right] \right\} \] (2.11)
and
\[ [S(q,t), \pi_S(q',t)] = i\delta(q - q'). \] (2.12)

For \( q \to -\infty \), \( S \) becomes a free massless scalar field, admitting the mode expansion
\[ S(q,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{8\pi^2 k^2}} \left[ a_k e^{-i|k|t + ikq} + a_k^\dagger e^{i|k|t - ikq} \right] \] (2.13)
with
\[ [a_k, a_k^\dagger] = 2\pi |k| \delta(k - k'). \] (2.14)

Due to reflection off the wall, only half of the operators \( a_k \) are independent. Finding the relation between the rightmovers, \( a_{k>0} \), and the lef'tmovers, \( a_{k<0} \), is equivalent to computing the S-matrix for scattering from the wall. This can be achieved as follows [34][48].

Given the classical equations of motion
\[ \dot{x} = p, \quad \dot{p} = x, \] (2.15)
we have the conserved quantities
\[ v = (-x - p)e^{-t}, \quad w = (-x + p)e^t, \] (2.16)
as well as arbitrary powers of these. Scattering amplitudes follow from equating the values of the conserved quantities at early and late times. In particular, we consider the conserved quantities
\[ v_{mn} = e^{(n-m)t} \int_{F_{-}F_{0}} \frac{dp \, dx}{2\pi} (-x - p)^m (-x + p)^n \] (2.17)
where we subtract off the static Fermi $F_0$ sea for finiteness. At early times the rightmoving fluctuations are $\epsilon_+(q, t) = \epsilon_+(t - q)$, and at late times the leftmoving fluctuations are $\epsilon_-(q, t) = \epsilon_-(t + q)$. The conserved quantities (2.17) then become, after a short calculation

$$v_{mn} = \frac{2^n}{2\pi(m+1)} \int_{-\infty}^{\infty} dt \ e^{(n-m)(t-q)} \left[ (\epsilon_+(t-q))^{m+1} - \mu^{m+1} \right]$$

After setting $m = 0$ and $n = i k$ (where $k$ is real and positive) in (2.18), and substituting in the mode expansion gives

$$a_k^\dagger = (\frac{1}{2}\mu)^{-ik} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{i}{\sqrt{2\pi\mu}} \right)^{n-1} \frac{\Gamma(1-ik)}{\Gamma(2-n-ik)}$$

$$\int_{-\infty}^{0} dk_1 \ldots dk_n (a_{k_1}^\dagger a_{k_1}) \cdots (a_{k_n}^\dagger a_{k_n}) \delta(\pm|k_1| \pm \cdots \pm |k_n| - k).$$

The notation is somewhat schematic: expanding out the string of operators, each argument $\pm|k_i|$ of the delta function comes with a plus sign if the term contains $a_{k_i}^\dagger$, and a minus sign if it contains $a_{k_i}$.

The result (2.20) allows us to express any collection of incoming rightmoving fluctuations in terms of outgoing leftmoving fluctuations, and so yields the S-matrix. The derivation of (2.20) treated the creation/annihilation operators as classical objects, neglecting operator ordering issues. It is not hard to check that the tree level S-matrix is independent of the choice of ordering in (2.20), and we will find it convenient to use normal ordering.

It is important to note that the modes $a_k$ of the collective field $S$ are nontrivially related to those of the tachyon as defined by the worldsheet CFT,

$$[a_k]_{ws} = -i(4\mu)^{-ik} \frac{\Gamma(ik)}{\Gamma(-ik)} a_k.$$ 

Since the operators are related by pure phases for real $k$, these terms only affect probabilities for processes involving superpositions of different $k$.

3. Rolling tachyon states

Besides smooth fluctuations of the Fermi surface, another class of states consists of exciting a single fermion. It is now understood \([36,39,38,46,47,49]\) that a single fermion on
the top of the inverted harmonic oscillator potential corresponds to a D0-brane localized in the strong coupling region. A state in which the fermion rolls down the potential corresponds to a rolling of the open string tachyon on the D0-brane. At late times, as the fermion moves into the spatial region occupied by the Fermi sea, the state is best described in closed string language as an outgoing pulse of radiation. As shown in [37], there is a very precise relation between the single fermion state and the profile of the outgoing pulse of closed string radiation.

Bosonization provides a dictionary between single fermion states and collective field states [50][51]. We now review this dictionary in the region of large negative $x$; the full formulas are found in [32]. Starting from the $\psi$ field appearing in (2.2), we change variables as

$$\psi(x,t) = \frac{1}{x} e^{-i\mu t + \frac{1}{2}x^2} \psi_L(x,t) + \frac{1}{x} e^{-i\mu t - \frac{1}{2}x^2} \psi_R(x,t)$$

which amounts to stripping off the WKB part of the wavefunction for large negative $x$.

Define $q$ as in (2.10) and substitute into the Hamiltonian, keeping only terms which survive for large negative $q$, to obtain

$$H = \int_{-\infty}^{\infty} dq \left[ i\psi_R^\dagger \partial_q \psi_R - i\psi_L^\dagger \partial_q \psi_L \right].$$

This is the Hamiltonian of a relativistic fermion. Now bosonize as

$$\psi_R = \frac{1}{\sqrt{2\pi}} \exp \left[ i\sqrt{\pi} \int^q (\pi_S - \partial_q S) dq' \right]:$$

$$\psi_L = \frac{1}{\sqrt{2\pi}} \exp \left[ i\sqrt{\pi} \int^q (\pi_S + \partial_q S) dq' \right]:$$

The bosonic field $S$ is the same as the field which appeared earlier in (2.9).

Single fermion states are obtained by acting with $\psi_{L,R}$ on the vacuum $|0\rangle$. Of course, in our context the relevant ground state consists of the filled Fermi sea. However, if we consider wavepackets which have very small overlap with states of the Fermi sea, we can effectively consider acting on the zero particle ground state $|0\rangle$. Such states are

$$\psi_R(q,t)|0\rangle = \frac{1}{\sqrt{2\pi}} \exp \left[ -2i\sqrt{\pi} \int_0^{\infty} \frac{dk}{\sqrt{8\pi^2 k^2}} a_k^\dagger e^{-i(kq - |k|t)} \right] |0\rangle$$

$$\psi_L(q,t)|0\rangle = \frac{1}{\sqrt{2\pi}} \exp \left[ 2i\sqrt{\pi} \int_{-\infty}^0 \frac{dk}{\sqrt{8\pi^2 k^2}} a_k^\dagger e^{-i(kq - |k|t)} \right] |0\rangle$$

Classically, the fermions move along trajectories obeying the equations of motion. Trajectories with a turning point at $t = 0$ are

$$x(t) = -\hat{\lambda} \cosh t, \quad \hat{\lambda} = \sin \pi \lambda.$$
At early and late times, the trajectories become relativistic

\[ t \to \pm \infty : \quad q(t) = \mp t - \ln \frac{\lambda}{2}. \]  \hspace{1cm} (3.6)

Therefore, at early and late times we can write

\[ \psi_R = \psi_R(t - q), \quad \psi_L = \psi_L(t + q). \]  \hspace{1cm} (3.7)

Asymptotic states can be obtained by acting with either \( \psi_R \) or \( \psi_L \) in the region of large negative \( q \). We can choose to work in terms of the \( \psi_R \) states, since the \( \psi_L \) states are related to these by reflecting off the potential. It turns out to be convenient to write the incoming state as

\[ \psi_R(t - q + \ln \frac{\mu}{2}) |0\rangle, \quad t - q = -\ln \frac{\lambda}{2}. \]  \hspace{1cm} (3.8)

The \( \ln \frac{\mu}{2} \) shift is convenient since it will cancel the prefactor in (2.20).

According to the bosonization (3.4), the state (3.8) is

\[ \sqrt{2\pi}\psi_R(t - q + \ln \frac{\mu}{2}) |0\rangle = \exp \left[ -2i\sqrt{\pi} \int_0^\infty \frac{dk}{\sqrt{8\pi^2 k^2}} a_k^\dagger e^{ik(t-q+\ln \frac{\mu}{2})} \right] |0\rangle \]

\[ = \exp \left[ -2i\sqrt{\pi} \int_0^\infty \frac{dk}{\sqrt{8\pi^2 k^2}} a_k^\dagger \mu^i e^{-ik\ln \lambda} \right] |0\rangle \]  \hspace{1cm} (3.9)

The state (3.9) can be viewed either as a single incoming fermion (the “open string interpretation”), or as an incoming pulse of tachyons (the “closed string interpretation”).

We can now use our tree level S-matrix result (2.20) to reexpress it in terms of outgoing states as

\[ \exp \left[ 2i\sqrt{\pi} \sum_{n=1}^\infty \frac{1}{n!} \int_0^\infty \frac{dk}{\sqrt{8\pi^2 k^2}} \int_{-\infty}^0 dk_1 \ldots dk_n e^{i\theta(k)} \left( \frac{i}{\sqrt{2\pi \mu}} \right)^{n-1} \frac{\Gamma(1 - ik)}{\Gamma(2 - n - ik)} \right] : (a_{k_1}^\dagger - a_{k_1}) \ldots (a_{k_n}^\dagger - a_{k_n}) : \delta(\pm |k_1| \pm \cdots \pm |k_n| - k) \]  \hspace{1cm} (3.10)

with

\[ e^{i\theta(k)} = -2^i e^{-ik \ln \lambda}. \]  \hspace{1cm} (3.11)

(3.10) represents a particular state of outgoing tachyons, and whose expansion can be matched against contributions in string perturbation theory.

As an expansion in \( g \sim 1/\mu \), if we keep just the lowest \( n = 1 \) term we find the state

\[ \exp \left[ 2i\sqrt{\pi} \int_{-\infty}^0 \frac{dk_1}{\sqrt{8\pi^2 k_1^2}} e^{i\theta(|k_1|)} a_{k_1}^\dagger \right] |0\rangle. \]  \hspace{1cm} (3.12)
As was shown in [37], this state agrees with that produced by the disk one-point function, or more precisely, by the sum over any number of disks with one vertex operator inserted on each. To get agreement, the time part of the CFT should include a boundary interaction $\lambda \cosh t$, and the zero mode should be integrated using the Hartle-Hawking contour extending to $t = +i\infty$.\(^3\) One feature of the Hartle-Hawking contour is that it restricts us to computing just the production of closed string states and not their absorption, since convergence of the zero mode integral requires vertex operators to behave as $e^{i\omega t}$ with $\omega > 0$.

If we only keep terms with all creation operators we get

$$
\exp \left[ 2i\sqrt{\pi} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{0} dk_1 \cdots dk_n e^{i\theta(|k|)} \left( \frac{i}{\sqrt{2\pi\mu}} \right)^{n-1} \frac{\Gamma(1-ik)}{\Gamma(2-n-ik)} a_1^\dagger \cdots a_n^\dagger \right] |0\rangle.
$$

(3.13)

This has the correct form to arise from the sum of disk diagrams with any number of tachyon vertex operators. The disk amplitudes should be evaluated using the same Hartle-Hawking contour as above.

As will be shown in more detail later, the remaining terms in the expansion of (3.10) come from worldsheets with multiple boundaries — the annulus and so on. We should emphasize that the only approximation we have made was to treat the closed strings (the collective field) classically, which means that we are correctly including quantum effects due to open strings. Of course, this is very familiar from the AdS/CFT correspondence, where we are used to saying that classical nonlinear closed string effects are dual to quantum open string effects.

4. Field coupled to classical source

To make a precise connection between string amplitudes and the state (3.10), we need to recall a few basic facts regarding the states produced by classical sources. Start from

$$
S = \frac{1}{2} \int d^2x \left[ \dot{\phi}^2 - (\phi')^2 + J\phi \right].
$$

(4.1)

The field equations are solved in terms of the retarded propagator as

$$
\phi_{out} = \phi_{in} + \int d^2x' G_{ret}(x-x')J(x')
$$

(4.2)

$$
G_{ret}(x) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i(kx-\omega t)}}{-(\omega + i\epsilon)^2 + k^2}
$$

(4.3)

\(^3\) Alternatively, one can use the boundary interaction $\lambda e^t$. We will think in terms of the $\lambda \cosh t$ interaction since it seems to correspond more naturally to our setup at early times.
We define the mode expansions as
\[
\phi_{in} = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{8\pi^2 k^2}} \left[ b_k e^{i(kx - |k|t)} + b_k^\dagger e^{-i(kx - |k|t)} \right]
\]
\[
\phi_{out} = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{8\pi^2 k^2}} \left[ a_k e^{i(kx - |k|t)} + a_k^\dagger e^{-i(kx - |k|t)} \right]
\] (4.4)

It then follows that the in vacuum is a coherent state when expressed in terms of out operators,
\[
|0_{in}\rangle = \exp \left( i \int \frac{dk}{\sqrt{8\pi^2 k^2}} \tilde{J}(|k|, k) a_k^\dagger \right) |0_{out}\rangle
\] (4.5)

with
\[
\tilde{J}(\omega, k) = \int d^2 x J(x) e^{-i(kx - \omega t)}.
\] (4.6)

Comparing (4.5) with (3.12), we can read off the source corresponding to the disk one-point function; we find
\[
\tilde{J}(|k|, k) = 2\sqrt{\pi} e^{i\theta(|k|)}, \quad (k < 0).
\] (4.7)

4.1. Generalization

Now consider
\[
S = \frac{1}{2} \int d^2 x \left[ \dot{\phi}^2 - (\phi')^2 \right] + \int d^2 x_1 \cdots d^2 x_n J(x_1, \ldots, x_n) \phi(x_1) \cdots \phi(x_n).
\] (4.8)

In first order perturbation theory we have
\[
|0_{in}\rangle = e^{-i \int H(t) dt} |0_{out}\rangle = |0\rangle + i \int d^2 x_1 \cdots d^2 x_n J(x_1, \ldots, x_n) \phi(x_1) \cdots \phi(x_n) |0_{out}\rangle
\]
\[
= |0_{out}\rangle + i \int \frac{dk_1}{\sqrt{8\pi^2 k_1^2}} \cdots \frac{dk_n}{\sqrt{8\pi^2 k_n^2}} \tilde{J}(|k_1|, k_1; \cdots; |k_n|, k_n) a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0_{out}\rangle
\] (4.9)

where
\[
\tilde{J}(k_1, \ldots, k_n) = \int d^2 x_1 \cdots d^2 x_n J(x_1, \ldots, x_n) e^{-i(k_1 \cdot x_1 + \cdots + k_n \cdot x_n)}.
\] (4.10)

Comparing this to (3.13) we find
\[
\tilde{J}(|k_1|, k_1; \cdots; |k_n|, k_n) = 2\sqrt{\pi} e^{i\theta(k)} \prod_{i=1}^{n} \frac{1}{\sqrt{8\pi^2 k_i^2}} \frac{1}{n!} \left( \frac{1}{\sqrt{2\mu}} \right)^{n-1} \frac{\Gamma(1 - ik)}{\Gamma(2 - n - ik)}.
\] (4.11)

Here $k_i < 0$ and $k = \sum_i |k_i|$. So (4.11) is our prediction for the disk n-point functions.
5. Comparison with string perturbation theory

As was shown in [37], the result (4.7) agrees with the disk one-point function, after taking into account the additional leg-pole factors (2.21). In particular, we consider the one-point functions of the normalizable vertex operators

\[ V_k = e^{(2+ik)\phi-i|k|t}. \]  

The Liouville part is described by the Zamolodchikov boundary state [40],

\[ \langle e^{(2+ik)\phi} \rangle = \frac{2}{\sqrt{\pi}} i \sinh(\pi k) \mu^{-i\frac{k}{2}} \frac{\Gamma(i k)}{\Gamma(-i k)}, \]  

and the time part is described by the boundary interaction \( \lambda \cosh t \) with Hartle-Hawking contour,

\[ \langle e^{i|k|t} \rangle = \frac{\pi e^{-i|k|\log \lambda}}{\sinh(\pi |k|)}. \]  

Combining these gives

\[ \langle V_k \rangle = 2\sqrt{\pi} e^{-i|k|\log \lambda} e^{i\delta(k)}. \]  

with

\[ e^{i\delta(k)} = i \text{sgn}(k) \mu^{-i\frac{k}{2}} \frac{\Gamma(i k)}{\Gamma(-i k)}. \]

(5.4) agrees with (4.7) modulo a \( \lambda \) independent phase, which is attributed to the leg pole factor (2.21).

5.1. Generalization

By the same logic, we can relate our general result (4.11) to the general disk amplitude with any number of outgoing tachyon vertex operators. Modulo the leg pole factors, we then get a prediction for the disk amplitudes of the rolling tachyon times Liouville boundary state:

\[ \langle V_{k_1,\ldots,k_n} \rangle = 2\sqrt{\pi} \prod_{i=1}^{n} \frac{\sqrt{8\pi^2 k_i^2}}{\sqrt{8\pi^2 k_i^2} n!} \frac{1}{\Gamma(1-i k)} \left( \frac{1}{\sqrt{2\pi\mu}} \right)^{n-1} \frac{\Gamma(1-i k)}{\Gamma(2-n-i k)}. \] 

The precise statement is that these are the amplitudes for outgoing particles evaluated using the Hartle-Hawking contour.

5.2. Amplitudes with multiple boundaries

Now we return to the issue of the remaining states in the expansion of (3.10). For illustration, consider working to first order in \( g \sim 1/\mu \). At this order two amplitudes
contribute: the disk with two vertex operators, and the annulus with one vertex operator. The claim is that the annulus amplitude arises upon normal ordering (3.10). In particular, we can write

\[ e^{A+gB} = e^A \left\{ 1 + g \left( B + \frac{1}{2!} [B, A] + \frac{1}{3!} [[B, A], A] + \cdots \right) \right\} + O(g^2). \] (5.7)

In our case

\[ A = 2i \sqrt{\pi} \int_0^0 \frac{dk}{\sqrt{8 \pi^2 k^2}} e^{i \theta(|k|)} a_k^\dagger \]

\[ gB = \frac{i}{\sqrt{2 \mu}} \int_{-\infty}^0 \frac{dk_1 dk_2}{\sqrt{8 \pi^2}} \left( e^{i \theta(|k_1|+|k_2|)} a_{k_1}^\dagger a_{k_2}^\dagger + 2e^{i \theta(|k_1|-|k_2|)} \Theta(|k_1| - |k_2|) a_{k_1}^\dagger a_{k_2} \right) \] (5.8)

We then find

\[ g[B, A] = -i \left( \frac{2\pi}{\mu} \right)^{3/2} \int_{-\infty}^0 dk_1 k_1^2 e^{i \theta(|k_1|)} a_{k_1}^\dagger, \] (5.9)

and the remaining nested commutator terms in (5.7) vanish. From this we read off the annulus one-point function to be \( \frac{1}{2} (2\pi)^{3/2} k^2 \) (modulo a phase factor).

The annulus amplitude grows rapidly in \( k \), and the corresponding state obtained from (5.9) is an infinite energy, highly non-normalizable, state. This is not a surprise: this behavior is obtained already at the level of the disk, and the two boundaries of the annulus essentially squares this divergence. As discussed in [37], these divergences have a simple origin: our incoming state was taken to be a single fermion with a definite position, but such a state has infinite energy quantum mechanically. A more accurate treatment replaces the fermion state by a localized wavepacket; the wavepacket then cuts off the large \( k \) divergences. Note that this also makes it clear that the divergence in the energy of the outgoing state is not somehow cured by including all terms in the expansion of (3.10), since it is obvious that our single, perfectly localized, fermion state has an infinite energy at early times, and energy is conserved.

To further illustrate that (5.9) yields an annulus amplitude, consider replacing the single rolling fermion by \( N \) of them. Since this corresponds to \( N \) unstable D-branes, we should find that the disk with two vertex operators is proportional to \( N \), while the annulus with one vertex operator is proportional to \( N^2 \). At our level of approximation, the \( N \) fermion state is obtained by simply inserting an \( N \) in the exponent of (3.9). This then modifies (5.7) to

\[ e^{N(A+gB)} = e^{NA} \left\{ 1 + g \left( NB + \frac{1}{2} N^2 [B, A] \right) \right\} + O(g^2), \] (5.10)

\(^{4}\) Amplitudes with no vertex operators of course do not contribute to the closed string state.
which immediately leads to the correct $N$ behavior.

Amplitudes with more than two boundaries are similarly obtained by considering higher order terms in the exponent of (3.10); i.e. expanding an expression of the form $e^{N(A+g^{n-1}B)}$ to first order in $g^{n-1}B$. Using (5.7), the series starts with a term representing $n$ vertex operators on the disk. Each addition of a commmutator with $A$ removes a vertex operator (since one less creation operator appears) and adds a boundary (since the amplitude is proportional to one higher power of $N$.) If we hold $N$ fixed, then we cannot really justify keeping amplitudes with arbitrary number of boundaries without also including worldsheets with handles. In particular, the torus amplitude appears at the same order in $g$ as the annulus. In principle, the contribution of handles could be incorporated by replacing (3.10) by the full perturbative tachyon S-matrix. Alternatively, as in AdS/CFT, if we take $g \to 0$ and $N \to \infty$ with $gN$ fixed, then the effect of handles is suppressed compared to adding any number of boundaries.

The complete expansion of (3.10) also includes the contributions from disconnected worldsheets, since it describes the complete closed string state.

6. Static unstable D-branes in the matrix model

Classically, a fermion placed at the top of the inverted harmonic oscillator potential can stay there forever. In the two dimensional string theory this corresponds to the fact that one can construct a boundary state corresponding to an eternal unstable D-brane. The boundary state is constructed by tensoring the $(m,n) = (1,1)$ boundary state of Zamolodchikov and Zamolodchikov with a Neumann boundary state for the free time directions $X_0$. Quantum mechanically, a localized wavefunction will spread and the unstable brane will have a finite lifetime. In the string theory the instability of the static unstable D0 brane manifests itself in the appearance of an imaginary part in the annulus partition function [52][53].

It is an interesting question to ask whether such an unstable brane will modify the classical closed string scattering. From the worldsheet perspective one again expects to find corrections from disk amplitudes. But on the matrix model side our preceding analysis does not directly apply since it heavily used the bosonization of the fermion in the asymptotic region, whereas here we want to keep the fermion at the origin. In this section we attempt to address this question using a modified version of collective field theory. Surprisingly, we will in fact find vanishing corrections corresponding to disk amplitudes.

6.1. Modified collective field theory

In the singlet sector the matrix model action (2.1) reduces to the following action for the eigenvalues
\[ L = \sum_i \left( \frac{1}{2} (\partial_t \lambda_i)^2 - \frac{1}{2} \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} - V(\lambda_i) \right). \] (6.1)

In (6.1) the eigenvalues are treated as bosonic with a repulsive potential. Das and Jevicki [54] introduced a collective field to describe the dynamics of the eigenvalues in the large \( N \) limit:

\[ \partial_x \phi(x, t) = \sum_i \delta(x - \lambda_i(t)). \] (6.2)

The dynamics of the collective field \( \phi \) is described by the following Lagrangian

\[ L = \int dx \left( \frac{1}{2} \frac{\partial_t \phi \partial_t \phi}{\partial_x \phi} - \frac{\pi^2}{6} (\partial_x \phi)^3 - (V(x) - \mu_F) \partial_x \phi \right). \] (6.3)

In the double scaling limit the potential is given by \( V(x) = \frac{1}{2}(V_0 - x^2) \) and one takes \( N \to \infty, \tilde{\mu} = V_0 - \mu_F \to 0, \) keeping \( N \tilde{\mu} = \mu \) fixed. The Lagrangian (6.3) becomes then

\[ L = \int dx \left( \frac{1}{2} \frac{\partial_t \phi \partial_t \phi}{\partial_x \phi} - \frac{\pi^2}{6} (\partial_x \phi)^3 + \left( \frac{1}{2} x^2 - \mu \right) \partial_x \phi \right). \] (6.4)

Note that this Lagrangian can be derived from the Hamiltonian (2.7) by defining \( p_\pm = -P_\phi \pm \pi \partial_x \phi \) and eliminating the momentum \( P_\phi \) via a Legendre transformation. The string coupling constant is related to the height of the double scaled potential by \( \mu = 1/g. \)

A variation on the collective field theory of Das and Jevicki was developed by Brustein et al. [55][56]. One splits a single eigenvalue \( \lambda_0(t) \) from the collective field \( \phi(x, t) \) and treats it separately. This is justified for eigenvalue distributions where the single eigenvalue is away from the dense region of the Fermi sea, i.e. \( |\lambda| \ll \frac{1}{g}. \) The dynamics of the filled Fermi sea is again described by the collective field \( \phi(x, t). \) The action for the coupled system is

\[ L = \frac{1}{2} (\partial_t \lambda_0)^2 + \frac{1}{2} \lambda_0^2 - \int dx \frac{\partial_x \phi}{(x - \lambda_0)^2} + \int dx \left( \frac{1}{2} \frac{\partial_t \phi \partial_t \phi}{\partial_x \phi} - \frac{\pi^2}{6} (\partial_x \phi)^3 + \left( \frac{1}{2} x^2 - \mu \right) \partial_x \phi \right). \] (6.5)

In order to disentangle the dynamics of the single eigenvalue and the collective field it is useful to perform the following rescaling \( \phi = g^{-\frac{1}{2}} \hat{\phi}, \quad x = g^{-\frac{1}{2}} \hat{x} \) and \( \lambda = g^{-\frac{1}{2}} \hat{\lambda}. \) The action (6.5) becomes

\[ L = \frac{1}{g} \left( \frac{1}{2} (\partial_t \hat{\lambda}_0)^2 + \frac{1}{2} \hat{\lambda}_0^2 \right) - \int d\hat{x} \frac{\partial_x \hat{\phi}}{(\hat{x} - \hat{\lambda}_0)^2} + \frac{1}{g^2} \int d\hat{x} \left( \frac{1}{2} \frac{\partial_t \hat{\phi} \partial_t \hat{\phi}}{\partial_x \hat{\phi}} - \frac{\pi^2}{6} (\partial_x \hat{\phi})^3 + \left( \frac{1}{2} \hat{x}^2 - 1 \right) \partial_x \hat{\phi} \right). \] (6.6)

\[ ^5 \text{An alternative description of single eigenvalue tunneling was developed in [57], based on the formalism of [58].} \]
The form of (6.6) suggests an interpretation as an open-closed string field theory action. The part which is of order $1/g$ can be interpreted as the action for the open string degree of freedom associated with the unstable D-brane. The part which is of order $1/g^2$ is the Das-Jevicki collective field action and corresponds to the action for the closed strings. The coupling between open and closed strings comes at order $g^0$ and in the limit $g \to 0$ becomes unimportant. Furthermore the $g \to 0$ limit corresponds to the limit where $\lambda_0$ and $\phi$ can be treated as classical fields.

A simple solution of the decoupled equation is given by

$$
\lambda_0(t) = a_1 \cosh(t) + a_2 \sinh(t), \quad \partial_x \phi = \frac{1}{\pi} \sqrt{x^2 - \frac{2}{g}},
$$

(6.7)
corresponding to a rolling eigenvalue and a static Fermi sea. It is interesting to analyze what happens if $g$ is small but nonzero. When the eigenvalue $\lambda$ is of order $1/g^{1/2}$ the interaction term becomes important. For the rolling tachyon (6.7) this happens at a time $t = -\log g^{1/2}$. One might be tempted to argue that this implies that there is a strong interaction between the eigenvalue and the Fermi sea, which starts at $x^2 = 2/g$. However this is not clear, since the derivation of the action (6.5) assumed that the single eigenvalue is well separated from the Fermi sea, so the action (6.5) might not be a good description of the actual dynamics in this case.

### 6.2. Scattering from a static D-brane

Instead of the rolling tachyon, in the classical limit one can consider a solution where the eigenvalue sits on top of the inverse harmonic oscillator potential $\lambda(t) = 0$ for all $t$. The equation of motion for the collective field then becomes

$$
\partial_t \left( \frac{\partial_t \phi}{\partial_x \phi} \right) - \partial_x \left\{ \frac{1}{2} \left( \frac{\partial_t \phi}{\partial_x \phi} \right)^2 + \frac{\pi^2}{2} (\partial_x \phi)^2 - \frac{1}{2} x^2 + \frac{1}{g} + \frac{1}{x^2} \right\} = 0.
$$

(6.8)
The interaction term modifies the static solution

$$
\partial_x \phi = \frac{1}{\pi} \sqrt{x^2 - \frac{2}{g} - \frac{2}{x^2}},
$$

(6.9)
which is valid for $x^2 > 2/g$. Note that at large $x$ the corrections to the standard static solution (2.6) are of order $1/x^3$ and subleading. A possible interpretation is that a localized D-brane near $x = 0$ only has a weak backreaction on the fields in the weak coupling region at $x = -\infty$.

The Das-Jevicki collective field is related to the description of the dynamics of the Fermi sea of Polchinski by $\partial_x \phi = \frac{1}{2\pi} (p_+ - p_-)$. This implies that the equation of motion for $p_\pm$ get modified to

$$
\partial_t p_\pm = x + \frac{2}{x^3} - p_\pm \partial_x p_\pm.
$$

(6.10)
In the classical limit the motion of the fermions is described by an incompressible fluid moving in a potential. From (6.10) it is clear that the eigenvalue at $\lambda_0 = 0$ produces a small modification of the Hamiltonian which governs the motion of points in the phase space,

$$H = \frac{1}{2} p^2 - \frac{1}{2} x^2 + \frac{1}{x^2} + \mu. \quad (6.11)$$

As closed string excitations are represented by small ripples in the Fermi sea whose turning point is at $x^2 \sim 1/g$, the extra term in (6.11) is only a small perturbation. The equations of motion following from (6.11) are hence

$$\frac{d^2}{dt^2} x(t) - x(t) - \frac{2}{x(t)^3} = 0. \quad (6.12)$$

It is interesting that the equation of motion (6.12) is one of the few modifications of the (inverted) harmonic oscillator which can be solved exactly:

$$x(t) = a \sqrt{\cosh^2(t - \sigma) + b}, \quad b = -\frac{1}{2} (1 - \sqrt{1 - \frac{8}{a^4}}). \quad (6.13)$$

We have chosen the root for which (6.13) goes over to the solution for the inverted harmonic oscillator in the limit $g \to 0$. We now follow the arguments of Polchinski to derive the classical scattering from the time delay. The time delay for the motion from a given $x$ and back is calculated using (6.13),

$$t' - q = t + q + \ln\left(\frac{a^2}{4}\right), \quad (6.14)$$

up to terms which vanish exponentially at late and early times respectively. Using the relation $\epsilon_\pm = \pm (p \pm x)x$ one finds

$$\epsilon_-(t + q) = \frac{a^2}{2} \sqrt{1 - \frac{8}{a^4}}. \quad (6.15)$$

The relation $\epsilon_-(t + q) = \epsilon_+(t' - q)$ can be reexpressed using (6.14) and (6.15) to produce a nonlinear relation between incoming and outgoing waves.

$$\epsilon_-(t + q) = \epsilon_-(t + q + \ln\left(\frac{1}{2} \sqrt{\epsilon_-(t + q)^2 + 2}\right)). \quad (6.16)$$

Expanding $\epsilon_+(t - q) = \frac{1}{g} + \delta_+(t - q)$, $\epsilon_-(t + q) = \frac{1}{g} + \delta_-(t + q)$ as in (2.19) and using the formulas (2.9) and (2.13) one can calculate the S-matrix. It is easy to see that the formula for the time delay (6.16) and hence for the S-matrix are modified at order $g^2$.

This is a puzzling feature since from string perturbation theory one would have expected that the corrections are of order $g$, coming from the disk versus sphere diagrams.
This result can be traced back to the fact that the interaction term in (6.6) comes at order $g^0$ instead of order $1/g$. It would be interesting to understand this fact better from the second quantized fermion point of view. The basic puzzle is that a fermion on the top of the potential has only an exponentially small overlap with states of the Fermi sea, and so would not seem to affect the perturbative scattering amplitudes.

A possible interpretation is that the backreaction on the closed string background caused by the presence of the fermion on the top of the potential is weaker than one might have expected. An indication that this is the case comes from considering the boundary state representing a static D-brane in 2 dimensions. The vertex operator for on shell 'massless' tachyon $V_k$ is given by (5.1). The one point function on the disk of the Liouville primary $U_k = e^{(2+i)k\phi}$ is given by (5.2). Note that the Neumann boundary conditions on the $X_0$ enforce $k = 0$ by momentum conservation. Hence in contrast to the rolling tachyon boundary state discussed in section five, the only physical state appearing in the boundary state is $V_k$ with $k = 0$. It follows that the one point function of this state is zero because of the sinh$(k)$ factor in (5.2).

Our results further seem to imply the vanishing of all disk amplitudes with on-shell closed string vertex operators. This conclusion is in fact consistent with the T-dual version of our setup, studied in [59]. It would be nice to gain a better understanding of these vanishing amplitudes from the worldsheet point of view.

7. Discussion

We have discussed closed string amplitudes in the presence of unstable D0-branes. In the case of the rolling tachyon background, we were able to relate the known sphere amplitudes for closed string tachyons to the sum of planar amplitudes with any number of boundaries. This represents a relation of the type one is familiar with from AdS/CFT duality, but here we are able to explicitly compute quantities on both sides of the duality. Of course, it would be interesting to reproduce these results using continuum methods.

Since the matrix model results correspond to worldsheet amplitudes using the Hartle-Hawking time contour, we were restricted to considering only outgoing vertex operators. It would be interesting to generalize this, as well as to be able to treat the real time contour.

For black holes in $AdS_5$ there is a phase transition at high temperatures which liberates the underlying open string degrees of freedom [60]. It is interesting that we found a hint of the non Abelian structure of the open strings in the combinatorics of boundaries in the matrix model scattering. This suggests the possibility to create a two dimensional black hole by bringing many unstable D0-branes together. It would be interesting to relate this idea to other proposals for a matrix model for the two dimensional black hole [61].

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6 We thank David Kutasov for explaining this to us.
Scattering amplitudes for the static D-brane are more surprising. We employed a modified version of the collective field formalism in which one separates out a single eigenvalue from the continuum. This results in a deformed Fermi sea, and corresponding corrections to tachyon amplitudes. But since these corrections arise at order $g^2$, whereas disks contribute at order $g$, we find that the disk amplitudes vanish, in agreement with previous results [59].

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References


