Abstract

We give a simple counter-example to the “principle” proposed by Koashi and Imoto in the paper entitled “What is possible without disturbing partially known quantum states?” (quant-ph/0101144 v2). Several other papers, by the same authors, are based on this principle, so the results of those papers are also in doubt.
In Ref.[1], Koashi and Imoto (KI) consider a set of density matrices \( \{ \rho_s \} \) which is preserved by an operator \( U \). They then claim that the states \( \{ \rho_s \} \) and the operator \( U \) can be expressed in a special canonical form. Their canonical form for the states \( \rho_s \) immediately implies that the degrees of freedom of the states \( \rho_s \) can be separated into three kinds (classical, non-classical and redundant). In this letter, we will give a simple counter-example which demonstrates that their canonical form is not generally true; it is too narrow and fails to describe even some classical systems. Several other papers (see Refs. [2], [3], [4]), by the same authors, are based on this principle, so the results of those papers are also in doubt.

In this letter, we will try to adhere to the notation of Ref.[1] as much as possible. We will indicate by the symbol \( ? \) any equations which are claimed in Ref.[1] to be generally true, but which will be shown in this paper to be false in general, although they might be true in particular instances.

Eqs.(2), (85) and (87) of Ref.[1] are

\[
\rho_s = \text{tr}_E[U(\rho_s \otimes \Sigma_E)U^\dagger], \quad (1a)
\]

\[
\rho_s = \bigoplus_l p^{(s,l)} \rho_j^{(l)} \otimes \rho_k^{(l)}, \quad (1b)
\]

\[
U = \bigoplus_l 1^{(l)} \otimes U_{KE}^{(l)}, \quad (1c)
\]

Tracing both sides of Eq.(1b) shows that \( p^{(s,l)} \) is the conditional probability \( P(l|s) \). In this letter, we will consider Eqs.(1) in the classical case when all density matrices are diagonal in the same basis. Hence, \( (\rho_s)_{x,x'} \rightarrow P(x'|s) \), \( (\rho_j^{(l)})_{j,j'} \rightarrow P(j|s,l) \), \( (\rho_k^{(l)})_{k,k'} \rightarrow P(k|l) \). Classically, the transformation called \( \Gamma \) in Ref.[1] is merely a relabelling of the index \( x \) as \( x = (j,k,l) \). Thus, Eqs.(1) reduce classically to:

\[
P(x'|s) = \sum_x P(x'|x)P(x|s), \quad (2a)
\]

\[
P(x|s) = P(l|s)P(j|s,l)P(k|l), \quad (2b)
\]

\[
P(x'|x) = \delta_{j'}^j \delta_{l'}^l P(k'|k,l). \quad (2c)
\]

Eqs.(2) can be interpreted as describing a Markov process with a reducible transition matrix that has multiple, \( s \) labelled, stationary probability distributions of \( x \), \( P(x|s) \). (See Ref. [5]).

Some clarification is in order here about the ranges of the variables \( x, j, k, l \). Let \( S_x \) be the (finite) set of possible values of the random variable \( x \) and let \( N_x \) be the cardinality of \( S_x \). In general, the transition matrix \( P(x'|x) \) can be represented by a transition diagram; that is, a directed graph whose nodes represents the states of \( x \) and whose arrows represent state transitions \( x \rightarrow x' \) with non-zero transition.
probabilities \( P(x'|x) \). In general, the transition diagram will have one or more islands. By an island, we will mean a collection of nodes that are connected by arrows among themselves but are not connected to the nodes of other islands. We will call a solitary state an island consisting of a single node. We will call a quasi-island a set of nodes containing one island plus any number of solitary states. The set \( S_x \) is partitioned in Ref.[1] into the sets \( \{K^{(l)}\}_{\forall l} \), where \( l \in \{1,2,\ldots, l_{\text{max}}\} \). Each \( K^{(l)} \) is generally a collection of islands. We can add solitary states to each island within \( S_x \) so that every island grows into a quasi-island that contains the same number \( k_{\text{max}} \) of states. We can then add even more solitary states to each \( K^{(l)} \) so that it grows into a collection of \( j_{\text{max}} \) quasi-islands, where \( j_{\text{max}} \) is independent of \( l \). Let \( \tilde{S}_x \) be the extended set containing the original set \( S_x \) and all the solitary states that were added. Now \( \tilde{S}_x \) can be partitioned into \( \{\tilde{K}^{(l)}\}_{\forall l} \), where \( \tilde{K}^{(l)} \supseteq K^{(l)} \), and each \( \tilde{K}^{(l)} \) contains \( j_{\text{max}} \) quasi-islands, and each of these quasi-islands contains \( k_{\text{max}} \) nodes. We define \( P(x|s) = 0 \) for all \( x \in \tilde{S}_x - S_x \). Henceforth we will assume that \( x \in \tilde{S}_x \) and \( x = (j,k,l) \), \( j \in \{1,2,\ldots, j_{\text{max}}\} \), \( k \in \{1,2,\ldots, k_{\text{max}}\} \), \( l \in \{1,2,\ldots, l_{\text{max}}\} \), \( N_x = j_{\text{max}}k_{\text{max}}l_{\text{max}} \). According to Eq.(2c), the following selection rules hold: \( \Delta j \neq 0 \) and \( \Delta l \neq 0 \) transitions are forbidden. In what follows, we will omit the tilde from \( \tilde{S}_x \) and \( \tilde{K}^{(l)} \) except in explanations where this might lead to serious confusion.

Now consider Ref.[1], page 5, first paragraph. Let

\[
\alpha(s,l) = \sum_{x \in K^{(l)}} P(x|s) = P(x \in K^{(l)}|s). \tag{3}
\]

The refinement process described in Ref.[1], page 5, first paragraph, will come to an end when, for all \( x \in K^{(l)} \), \( \frac{P(x|s)}{\alpha(s,l)} \) is independent of \( s \). Let

\[
\frac{P(x|s)}{\alpha(s,l)} = g(x,l). \tag{4}
\]

Thus,

\[
P(x|s) = g(x,l)P(x \in K^{(l)}|s). \tag{5}
\]

Multiplying both sides of the previous equation by \( P(s) \) and summing over \( s \) yields

\[
P(x) = g(x,l)P(\bar{x} \in K^{(l)}). \tag{6}
\]

Hence, for all \( x \in K^{(l)} \),

\[
\frac{P(x|s)}{P(x)} = \frac{P(\bar{x} \in K^{(l)}|s)}{P(\bar{x} \in K^{(l)})}. \tag{7}
\]

Equivalently, for all \( x_1 \) and \( x_2 \) in \( K^{(l)} \),

\[
\frac{P(x_1|s)}{P(x_1)} = \frac{P(x_2|s)}{P(x_2)}. \tag{8}
\]
Let $\xi = (j, k)$ and $x = (\xi, l)$. Then

$$P(\xi, l|s) = P(\xi|l, s)P(l|s), \quad (9a)$$

and

$$P(\xi, l) = P(\xi|l)P(l). \quad (9b)$$

Let $x_1 = (\xi_1, l)$ and $x_2 = (\xi_2, l)$. Combining Eqs. (8) and (9) yields, for all $\xi_1$ and $\xi_2$,

$$\frac{P(\xi_1|l, s)}{P(\xi_1|l)} = \frac{P(\xi_2|l, s)}{P(\xi_2|l)}. \quad (10)$$

Summing both sides of $P(\xi_1|l, s)P(\xi_2|l) = P(\xi_1|l)P(\xi_2|l, s)$ over $\xi_2$ yields

$$P(\xi|l, s) = P(\xi|l). \quad (11)$$

Thus

$$P(x|s) = P(\xi|l)P(l|s). \quad (12)$$

Eq. (12) implies that

$$P(x|s) = P(j|k, l)P(k|l)P(l|s). \quad (13)$$

Eq. (2b) can be rewritten as

$$P(x|s) = P(j|l, s)P(k|l)P(l|s). \quad (14)$$

If both Eqs. (13) and (14) are assumed to be true, then

$$P(j|k, l) = P(j|l, s) = P(j|l). \quad (15)$$

$$P(j, k|l) = P(j|l)P(k|l). \quad (16)$$

$$P(x|s) = P(j|l)P(k|l)P(l|s). \quad (17)$$

We are finally ready to give our numerical counter-example of the KI principle.

Suppose $s \in \{1, 2\}$ and

$$P(\mathfrak{s} = 1) = P(\mathfrak{s} = 2) = \frac{1}{2}. \quad (18)$$

Suppose $x \in \{1, 2, 3, 4\}$ and
\[ \vec{p}^{(1)} = [P(x|s = 1)] = \begin{bmatrix} \frac{1}{10} \\ \frac{2}{10} \\ \frac{3}{10} \\ \frac{4}{10} \end{bmatrix}, \quad \vec{p}^{(2)} = [P(x|s = 2)] = \begin{bmatrix} \frac{1}{7} \\ \frac{1}{6} \end{bmatrix}, \quad (19) \]

\[ T = [P(x'|x)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \left(\frac{5}{6}\right) & \left(\frac{1}{8}\right) \\ 0 & 0 & \left(\frac{1}{6}\right) & \left(\frac{7}{8}\right) \end{bmatrix}. \quad (20) \]

Fig. 1 shows the transition diagram for this Markov process. Note that in Fig. 1 we have added four solitary states \( x = 5, 6, 7, 8 \) to the original states \( x = 1, 2, 3, 4 \). Fig. 1 also gives a table that translates between \( x \) and \( jkl \) labels.

It is easy to check that

\[ TP^{(1)} = \vec{p}^{(1)}, \quad TP^{(2)} = \vec{p}^{(2)}. \quad (21) \]

Using \[ \frac{P(x|s)}{P(x)} = 2 \sum_{s} \frac{P(x|s)}{P(x|s)} \], we get
\[
\begin{array}{|c|c|c|}
\hline
\frac{P(x|s)}{P(x)} & s = 1 & s = 2 \\
\hline
x = 1 & \left(\frac{1}{9}\right) & \left(\frac{2}{9}\right) \\
x = 2 & \left(\frac{2}{9}\right) & \left(\frac{2}{9}\right) \\
x = 3 & \left(\frac{9}{7}\right) & \left(\frac{9}{7}\right) \\
x = 4 & \left(\frac{9}{7}\right) & \left(\frac{9}{7}\right) \\
\hline
\end{array}
\]

(22)

Using \(P(l|s) = \sum_{j,k} P(x|s)\), we get

\[
\begin{array}{|c|c|c|}
\hline
\frac{P(l|s)}{P(x)} & s = 1 & s = 2 \\
\hline
l = 1 & \left(\frac{1}{10}\right) & \left(\frac{1}{2}\right) \\
l = 2 & \left(\frac{9}{10}\right) & \left(\frac{1}{2}\right) \\
\hline
\end{array}
\]

(23)

From Eq. (12), \(\frac{P(x|s)}{P(l|s)} = P(j, k|l)\), so

\[
\begin{array}{|c|c|c|}
\hline
P(j, k|l) & s = 1 & s = 2 \\
\hline
jkl = 111 & 1 & 1 \\
jkl = 112 & \left(\frac{2}{9}\right) & \left(\frac{2}{9}\right) \\
jkl = 212 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right) \\
jkl = 222 & \left(\frac{4}{9}\right) & \left(\frac{4}{9}\right) \\
\hline
\end{array}
\]

(24)

Define \(\alpha_j\) and \(\beta_k\) by

\[
\alpha_j = P(j|l = 2) , \quad \beta_k = P(k|l = 2) . \quad (25)
\]

According to Eq.(16), \(j\) and \(k\) are independent at fixed \(l\). Thus,

\[
P(j = 1, k = 1|l = 2) = \frac{2}{9} = \alpha_1 \beta_1 \quad (26a)
\]

\[
P(j = 2, k = 1|l = 2) = \frac{1}{3} = \alpha_2 \beta_1 \quad (26b)
\]

\[
P(j = 2, k = 2|l = 2) = \frac{4}{9} = \alpha_2 \beta_2 \quad (26c)
\]

Eqs.(26b) and (26c) imply \(\alpha_2 = \frac{7}{9}\) and \(\alpha_1 = \frac{2}{9}\). Eq.(26a) then becomes \(\frac{2}{9} = \frac{2}{9} \beta_1\), so \(\beta_1 = 1\) and \(\beta_2 = 0\). But \(\beta_2 = 0\) contradicts Eq.(26c).

References


