Bulk and boundary $g_2$ factorized $S$-matrices

N. J. MacKay\textsuperscript{1} and B. J. Short

\textit{Department of Mathematics, University of York, York YO10 5DD, U.K.}

ABSTRACT
We investigate the $g_2$-invariant bulk (1+1D, factorized) $S$-matrix constructed by Ogievetsky, using the bootstrap on the three-point coupling of the vector multiplet to constrain its CDD ambiguity. We then construct the corresponding boundary $S$-matrix, demonstrating it to be consistent with $Y(g_2, a_1 \times a_1)$ symmetry.

1. Introduction

As a preliminary step in the investigation using tensor methods of 1+1-dimensional factorized $S$-matrices with exceptional $g$ (and Yangian $Y(g)$) invariance, we investigate the case $g = g_2$. The factorized $S$-matrix for the seven-dimensional representation $\mathbf{7}$ of $g_2$ was constructed by Ogievetsky [1], and we use this to construct the $g_2 \times g_2$-invariant $S$-matrix, applicable in the principal chiral model (PCM). We may choose the $S$-matrix to have a bootstrap pole for the self-coupling of the $\mathbf{7}$ multiplet, and the bootstrap applied to this process constrains the CDD factor.

We then investigate the corresponding solutions of the boundary Yang-Baxter equations and the boundary $S$-matrices for the $g_2$ PCM — that is, the extension to $g_2$ of the calculations carried out for classical $g$ in [2]. The spectral decomposition is precisely that expected from the $Y(g_2, a_1 \times a_1)$ symmetry [3].

The method used is the diagrammatic technique of Cvitanovic [4]. We denote the cubic antisymmetric invariant of $g_2$ as $\lambda$, then construct the $\mathbf{7}$ of $g_2$ by taking the defining

\textsuperscript{1}email: nm15@york.ac.uk
representation of $SO(7)$ and restricting to those $\sigma \in SO(7)$ such that $\mathcal{D}\sigma = \mathcal{Y}$. Here $\mathcal{Y}$ satisfies the identities

2. The bulk $S$-matrix

The following $g_2$-invariant $S$-matrix satisfies the Yang-Baxter equation [1, 5]:

$$S_{(1,1)}(\theta) = S(\theta) = \sigma(\theta) \left( P_{27} + [2] P_{14} + [8] P_7 + [2][12] P_1 \right),$$

where $\sigma(\theta)$ is a scalar prefactor and

$$[y] = \frac{\frac{2n\pi}{12} + \theta}{\frac{2n\pi}{12} - \theta}.$$

Imposing $R$-matrix unitarity on the $S$-matrix gives $\sigma(\theta)\sigma(-\theta) = 1$, and imposing hermitian-analyticity gives $\sigma(\theta) = \sigma(-\theta^*)^*$. We rewrite the $S$-matrix as

We seek a minimal $S$-matrix, with no poles on the physical strip. The factor $\mu_0(\theta)$ has simple poles at $\theta = -2i\pi n - \frac{a_2n}{6}$, $\theta = 2i\pi n + i\pi + \frac{a_1n}{6}$ and simple zeroes at $\theta = 2i\pi n + \frac{a_1n}{6}$, $\theta = -2i\pi n - i\pi - \frac{a_1n}{6}$ for $n = 0, 1, 2, \ldots$. Thus, to cancel the poles in (??), we are led to

$$\sigma(\theta) = \mu_0(-\theta)\mu_1(\theta)\mu_3(\theta)\mu_4(\theta),$$

so that

$$\sigma(\theta) = \frac{\Gamma\left(\frac{\theta}{2\pi} + \frac{1}{2}\right) \Gamma\left(\frac{\theta}{2\pi} + \frac{7}{12}\right) \Gamma\left(\frac{\theta}{2\pi} + \frac{1}{12}\right) \Gamma\left(\frac{\theta}{2\pi} + \frac{3}{4}\right) \Gamma\left(\frac{\theta}{2\pi} + \frac{5}{6}\right) \Gamma\left(\frac{\theta}{2\pi} + \frac{1}{3}\right)}{\Gamma\left(\frac{-\theta}{2\pi} + \frac{1}{2}\right) \Gamma\left(\frac{-\theta}{2\pi} + \frac{7}{12}\right) \Gamma\left(\frac{-\theta}{2\pi} + \frac{1}{12}\right) \Gamma\left(\frac{-\theta}{2\pi} + \frac{3}{4}\right) \Gamma\left(\frac{-\theta}{2\pi} + \frac{5}{6}\right) \Gamma\left(\frac{-\theta}{2\pi} + \frac{1}{3}\right)}$$

(in fact we may choose plus or minus this – our choice of the positive sign will not affect the $S$-matrix). Thus we have established a minimal $S$-matrix which is $g_2$ invariant.

The $g_2$ PCM $S$-matrix acts on multiplets which are representations of $g_2 \times g_2$, and is constructed from two minimal $S$-matrices together with a CDD factor $X(\theta)$:

$$S_{(1,1)}^{PCM}(\theta) = X_{(1,1)}(\theta) \left( S(\theta)_L \otimes S(\theta)_R \right).$$

In order that $S_{(1,1)}^{PCM}(\theta)$ satisfy $R$-matrix unitarity and crossing-symmetry we require

$$X_{(1,1)}(\theta)X_{(1,1)}(-\theta) = 1 \quad \text{and} \quad \frac{X_{(1,1)}(\theta)}{X_{(1,1)}(i\pi - \theta)} = 1.$$
To construct $X$ we use
\[(y) = (y)_\theta = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi}{24}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{24}\right)};\]
this satisfies
\[(y)_\theta (y)_{-\theta} = 1, \quad \frac{(y)_\theta}{(y)_{i\pi - \theta}} = (2y)_{2\theta} \quad \text{and} \quad (y) = (y + 24).\]
The natural choice might be $X = -(2)(4)(8)(10)$, where we have allowed two $7$s to fuse (via simple poles with positive residues) to form either a $7$ (at $\theta = 2i\pi/3$) or a $14 \oplus 1$ (at $\theta = i\pi/6$, yielding a multiplet of mass $2\cos(\pi/12) = 12(\sqrt{6} + \sqrt{2})$ times the mass of the $7$). We must then check that the bootstrap equations are satisfied for the scattering of a $7$ off a fused $7 \subset 7 \otimes 7$ (an intricate calculation requiring much repeated application of (?)). The minimal $S$-matrix is consistent with this, but the CDD factor requires an extra factor $(6)^2$, and we must have
\[X_{(1,1)}(\theta) = -(2)(4)(6)^2(8)(10).\]
The apparent double pole at $i\pi/2$ thus introduced is spurious: it is cancelled by a simple zero in each minimal $S$.

3. The boundary $S$-matrix

We now consider the half-line case. Following [2], we try a minimal boundary $S$-matrix of the form
\[K(\theta) = \frac{\tau(\theta)}{(1 - c(\frac{i\pi}{2} - \theta))} (\ldots + c(\frac{i\pi}{2} - \theta)\mathcal{O}).\]
The conditions of boundary $R$-matrix unitarity and hermitian analyticity impose the constraints
\[(-\mathcal{O})^\dagger = -\mathcal{O}, \quad -\mathcal{O} = -\mathcal{O}, \quad c \in i\mathbb{R}, \quad \tau(\theta) = \tau(-\theta)^* \quad \text{and} \quad \tau(\theta)\tau(-\theta) = 1.\]
We must also impose crossing-unitarity,
\[\frac{\tau(\frac{i\pi}{2} - \theta)}{(1 - c(\frac{i\pi}{2} - \theta))} \left(\langle + c\left(\frac{i\pi}{2} - \theta\right)\mathcal{O}\right) = \frac{\omega(i\pi - 2\theta)\tau(\frac{i\pi}{2} + \theta)}{(1 - c(\frac{i\pi}{2} + \theta))} \left(\langle - \frac{6(i\pi - 2\theta)}{i\pi} \times + \frac{(i\pi - 2\theta)}{\theta} \mathcal{O} \right)}{(2\theta - \frac{i\pi}{3})} \left(\langle + c\left(\frac{i\pi}{2} + \theta\right)\mathcal{O}\right) + \frac{\omega(i\pi - 2\theta)(1 - c(\frac{i\pi}{2} - \theta))(\theta - \frac{i\pi}{3})}{(1 - c(\frac{i\pi}{2} + \theta))(2\theta - \frac{i\pi}{3})} \left(14 + 2i\pi c\mathcal{O} + \frac{i\pi}{\theta} + 4 \left(c\mathcal{O} + \frac{6}{i\pi}\right)\theta\right),\]
After applying (??) we find that this implies
\[\frac{\tau(\frac{i\pi}{2} - \theta)}{\tau(\frac{i\pi}{2} + \theta)} = \frac{\omega(i\pi - 2\theta)(1 - c(\frac{i\pi}{2} - \theta))(\theta - \frac{i\pi}{3})}{(1 - c(\frac{i\pi}{2} + \theta))(2\theta - \frac{i\pi}{3})} \left(14 + 2i\pi c\mathcal{O} + \frac{i\pi}{\theta} + 4 \left(c\mathcal{O} + \frac{6}{i\pi}\right)\theta\right),\]
\[
\tau(\frac{i\pi}{2} - \theta) \over \tau(\frac{i\pi}{2} + \theta) = \frac{\omega(i\pi - 2\theta)(1 - c(\frac{i\pi}{2} - \theta))(\theta - \frac{i\pi}{3})(i\pi + 2\theta)}{(1 - c(\frac{i\pi}{2} + \theta))(2\theta - \frac{i\pi}{3})} \left( -\frac{\alpha}{\theta - \frac{i\pi}{3}} + \frac{1}{\theta} + \frac{12}{i\pi} \right),
\]

Together with (for non-trivial \(\sigma\)) \(\bigotimes\bigodot = \pm \bigotimes\) and \(\bigodot = \alpha \bigotimes\) for some constant \(\alpha\). Comparing the two expressions we find \(\alpha = 0\) and

\[c \bigodot = -\frac{6(1 \pm 1)}{i\pi}.\]

However, \(\bigotimes = 0\) together with \(\bigotimes \bigotimes = \bigotimes\) has no solutions in odd dimensions (the eigenvalues of such a matrix are \(\pm 1\), an odd number of which cannot sum to zero). We thus have \((\bigotimes)^t = \bigotimes\) and

\[
\tau(\frac{i\pi}{2} - \theta) \over \tau(\frac{i\pi}{2} + \theta) = \frac{\omega(i\pi - 2\theta)(1 - c(\frac{i\pi}{2} - \theta))(\theta - \frac{i\pi}{3})(i\pi + 2\theta)}{(1 - c(\frac{i\pi}{2} + \theta))(2\theta - \frac{i\pi}{3})} \left( \frac{1}{\theta} - \frac{12}{i\pi} \right),
\]

or

\[
\tau(\frac{i\pi}{2} - \theta) \over \tau(\frac{i\pi}{2} + \theta) = [6] \left[ \frac{12}{ci\pi} - 6 \right] \sigma(2\theta), \quad (\bigotimes)^t = \bigotimes \quad \text{and} \quad c \bigotimes = -\frac{12}{i\pi}.
\]

Last we have to impose the boundary Yang-Baxter equation (bYBe). After some algebra we find that this is satisfied if

\[
\bigotimes + \bigotimes = \frac{ci\pi}{12} \bigotimes \bigotimes.
\]

Now using (??) we find

\[
\bigotimes \bigodot = \bigotimes + \frac{12}{ci\pi} \bigotimes.
\]

Thus, putting these two results together,

\[
\bigotimes = \frac{ci\pi}{12} \bigotimes \bigotimes \Leftrightarrow \bigotimes = \frac{ci\pi}{12} \bigotimes \bigodot.
\]

Consequently we must have \(c = \pm \frac{12}{i\pi}\), with \(\bigotimes \bigodot = \pm \bigotimes\) and \(\bigotimes = \mp 1\).

In summary, we have shown that the conditions of \(R\)-matrix unitarity, hermitian analyticity, crossing unitarity and the bYBe are satisfied by a minimal boundary ‘\(K\)’-matrix

\[
\frac{\tau(\theta)}{(1 \mp \frac{12\theta}{i\pi})} = \tau(\theta)\left( P_- - [\pm 1]P_+ \right), \quad \left( P_\pm = \frac{1}{2}(\bigotimes \pm \bigotimes) \right),
\]

where

\[\bigotimes \bigotimes = \bigotimes, \quad \bigotimes \bigotimes = \bigotimes, \quad \bigotimes \bigotimes = \bigotimes, \quad \bigotimes \bigotimes = \pm \bigotimes, \quad \bigotimes = \mp 1,\]

\[\tau(\theta)\tau(-\theta) = 1, \quad \tau(\theta) = \tau(-\theta^*), \quad \frac{\tau(\frac{i\pi}{2} - \theta)}{\tau(\frac{i\pi}{2} + \theta)} = [6] \left[ \pm 1 - 6 \right] \sigma(2\theta).\]
In fact, since \([1]_{\pi/2}^\theta/\lbrack 1 \rbrack_{\pi/2+\theta} = [-7]/[-5]\), the choice of sign is redundant – both choices give the same minimal \(K\)-matrix. We can write it as
\[
\frac{\tau(\theta)}{\lbrack 1 \rbrack_{\pi/2}^\theta} \left( I + \frac{i\theta}{\pi} E \right) = \tau(\theta) \left( P_- - \lbrack 1 \rbrack P_+ \right), \quad \left( P_\pm = \frac{1}{2} (I \pm E) \right),
\]
where \(E = QXQ^{-1}, \ Q \in \mathbb{G}_2, \ X = \text{diag}(1,1,1,-1,-1,-1,-1).\) This is clearly a subspace of the symmetric space \(SO(7)/SO(3) \times SO(4)\); in fact we have
\[
E \in \frac{\mathbb{G}_2}{SU(2) \times SU(2)},
\]
the space of quaternionic subalgebras of the octonions, as may be seen by considering the action of \(\mathbb{G}_2\) on a basic triple of octonions \([6]\).

The following constraints are imposed on \(\tau(\theta)\):
\[
\tau(\theta)\tau(-\theta) = 1, \quad \tau(\theta) = \tau(-\theta^*)^*, \quad \frac{\tau(\frac{i\pi}{2} - \theta)}{\tau(\frac{i\pi}{2} + \theta)} = [6] [-5] \sigma(2\theta).
\]

To solve these we note that
\[
\frac{\mu_a(\frac{i\pi}{2} - \theta)}{\mu_a(\frac{i\pi}{2} + \theta)} = [2a - 6],
\]
and we define
\[
\eta_a(\theta) = \frac{\Gamma \left( \frac{-\theta}{2\pi} + \frac{a}{12} \right) \Gamma \left( \frac{\theta}{2\pi} + \frac{a}{12} + \frac{1}{4} \right)}{\Gamma \left( \frac{\theta}{2\pi} + \frac{a}{12} \right) \Gamma \left( \frac{\theta}{2\pi} + \frac{a}{12} + \frac{1}{4} \right)}, \quad \text{so that} \quad \frac{\eta_a(\frac{i\pi}{2} - \theta)}{\eta_a(\frac{i\pi}{2} + \theta)} = \mu_{2a - 6}(2\theta).
\]

This leads us to
\[
\tau(\theta) = \mu_{1/2}(\theta)\mu_6(\theta)\eta_{1/2}(\theta)\eta_{1/2}(\theta)\eta_5(\theta)\eta_6(\theta).
\]
The simple poles of \(\eta_a(\theta)\) are at \(\theta = 2i\pi n + \frac{a\pi}{6}\) and \(\theta = -2i\pi n - \frac{i\pi}{2} - \frac{a\pi}{6}\), while the simple zeroes are at \(\theta = -2i\pi n - \frac{a\pi}{6}\) and \(\theta = 2i\pi n + \frac{i\pi}{2} + \frac{a\pi}{6}\), and so the \(K\)-matrix is minimal.

The final piece we require for the complete PCM \(K\)-matrix is the factor \(Y_1(\theta)\), which must satisfy
\[
\frac{Y_1(\frac{i\pi}{2} - \theta)}{Y_1(\frac{i\pi}{2} + \theta)} = X_{(1,1)}(i\pi - 2\theta) = X_{(1,1)}(2\theta).
\]

We make use of the fact that
\[
\frac{\langle y \rangle \frac{i\pi}{2} - \theta}{\langle y \rangle \frac{i\pi}{2} + \theta} = (2y)_{i\pi - 2\theta} = (2y + 24)_{i\pi - 2\theta}.
\]

Thus the most natural choice is
\[
Y_1(\theta) = (1)(2)(-9)^2(-8)(-7)(-6).
\]

This has a physical strip simple pole at \(\theta = \frac{i\pi}{12}\) at which the minimal \(K\)-matrix projects onto the subspace associated with \(P_+\) (the smaller one, and the \((3,1)\) of \(a_1 \times a_1\) as found
in [3]). The simple pole at $\theta = \frac{i\pi}{6}$ corresponds to an on-shell diagram which is possible precisely when the bulk three-point coupling of 7s exists.

We should also check the simpler trial solution of [2] for a minimal $K$-matrix, namely

$$K(\theta) = \rho(\theta)\cdot.$$ 

Imposing crossing-unitarity gives

$$\frac{\rho\left(\frac{i\pi}{2} - \theta\right)}{\rho\left(\frac{i\pi}{2} + \theta\right)} = \omega(i\pi - 2\theta) \left(\frac{4(\theta - \frac{i\pi}{3})}{2(\theta - \frac{i\pi}{3})} + \frac{4(i\pi - 2\theta)(i\pi - 3\theta)}{i\pi(2\theta - \frac{i\pi}{3})} \times + \frac{(i\pi - 2\theta)(\theta - \frac{i\pi}{3})}{\theta(2\theta - \frac{i\pi}{3})}\right),$$

which implies

$$\frac{\rho\left(\frac{i\pi}{2} - \theta\right)}{\rho\left(\frac{i\pi}{2} + \theta\right)} = \omega(i\pi - 2\theta) \left(\frac{4(\theta - \frac{i\pi}{3})}{2(\theta - \frac{i\pi}{3})} \bigotimes + \frac{4(i\pi - 2\theta)(i\pi - 3\theta)}{i\pi(2\theta - \frac{i\pi}{3})} \bigotimes + \frac{(i\pi - 2\theta)(\theta - \frac{i\pi}{3})}{\theta(2\theta - \frac{i\pi}{3})}\right).$$

For non-trivial $\circ$ we must have $\bigotimes = 0$, $(\circ)^t = \pm \circ$ and $\bigotimes = \alpha \gamma$. But, as pointed out earlier, the constraint $\bigotimes = 0$ is inconsistent with $\circ \circ = \circ$. Thus there are no non-trivial solutions of this form.

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**References**