Precision Spectroscopy of AdS/CFT

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Abstract

We extend recent remarkable progress in the comparison of the dynamical energy spectrum of rotating closed strings in $AdS_5 \times S^5$ and the scaling weights of the corresponding non-near-BPS operators in planar $\mathcal{N} = 4$ supersymmetric gauge theory. On the string side the computations are feasible, using semiclassical methods, if angular momentum quantum numbers are large. This results in a prediction of gauge theory anomalous dimensions to all orders in the 't Hooft coupling $\lambda$. On the gauge side the direct computation of these dimensions is feasible, using a recently discovered relation to integrable (super) spin chains, provided one considers the lowest order in $\lambda$. This one-loop computation then predicts the small-tension limit of the string spectrum for all (i.e. small or large) quantum numbers. In the overlapping window of large quantum numbers and small effective string tension, the string theory and gauge theory results are found to match in a mathematically highly non-trivial fashion. In particular, we compare energies of states with (i) two large angular momenta in $S^5$, and (ii) one large angular momentum in $AdS_5$ and $S^5$ each, and show that the solutions are related by an analytic continuation. Finally, numerical evidence is presented on the gauge side that the agreement persists also at higher (two) loop order.

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1 Introduction

It is believed that free type IIB superstring theory on the $AdS_5 \times S^5$ background is exactly dual to planar $\mathcal{N} = 4$ supersymmetric $SU(N)$ quantum gauge theory \cite{Klebanov:1998hh, Kutasov:1998cj, Witten:1998qj}. Here

On the string theory side, it was understood that in the case when some of the quantum numbers of the string states become large, the $AdS_5 \times S^5$ string sigma model can be efficiently treated by semi-classical methods \cite{Harmark:2001, Harmark:2002} (see also \cite{Harmark:2002, Harmark:2003}). It was then suggested \cite{Harmark:2001, Harmark:2002} that a novel possibility for a quantitative comparison with SYM theory in non-BPS sectors appears when one considers classical solutions describing closed strings rotating in several directions in the product space $AdS_5 \times S^5$ with the metric

$$\begin{align*}
(ds^2)_{AdS_5} &= d\rho^2 - \cosh^2 \rho \, dt^2 + \sinh^2 \rho \, (d\theta^2 + \cos^2 \theta \, d\phi_1^2 + \sin^2 \theta \, d\phi_2^2) \\
(ds^2)_{S^5} &= d\gamma^2 + \cos^2 \gamma \, d\varphi_3^2 + \sin^2 \gamma \, (d\psi^2 + \cos^2 \psi \, d\phi_1^2 + \sin^2 \psi \, d\phi_2^2)
\end{align*}$$

(1.1)

Here $t$ is the global $AdS_5$ time, which, together with the 5 angles $(\phi_1, \phi_2; \varphi_1, \varphi_2, \varphi_3)$, correspond to the obvious “linear” isometries of the metric, i.e. are related to the 3+3 Cartan generators of the $SO(2,4) \times SO(6)$ bosonic isometry group. Rotating strings can thus carry the 2+3 angular momentum charges (spins) $Q_i = (S_1, S_2; J_1, J_2, J_3)$, while $t$ is associated with the energy $E$. Once such classical solutions representing string states with several charges are found \cite{Harmark:2001, Harmark:2002}, one may evaluate the energy as a function of the spins and $\lambda$: $E = E(Q_i, \lambda)$. A remarkable feature of string solutions in $AdS_5$ is that their energy grows, for large charges, linearly with the charges \cite{Harmark:2001, Harmark:2002}. Corrections to subleading terms in the classical energy can then be computed using the standard semiclassical (inverse string tension) expansion \cite{Harmark:2001, Harmark:2002}. For certain string states with large total spin $J = J_1 + J_2 + J_3$ on $S^5$ for which

Considering string states represented by the classical solutions with several charges $(S_1, S_2; J_1, J_2, J_3)$ has the added advantage that it helps significantly in identifying the corresponding gauge theory operators. As is well known, $\mathcal{N} = 4$ supersymmetric gauge theory is superconformally invariant, and the bosonic subgroup of the full superconformal group $PSU(2,2|4)$ is $SO(2,4) \times SO(6)$. The energy $E$ of a string state is expected to correspond to the scaling dimension $\Delta$ of the associated conformal operator on the gauge theory side:

Fortunately, it was recently discovered that planar $\mathcal{N} = 4$ SYM theory is integrable at the one-loop level \cite{Faddeev:1998sh, Frolov:2001}. We can therefore make use of the Bethe ansatz for a corresponding spin chain model to obtain directly the eigenvalues of the matrix of anomalous

\footnote{Here $Q_i \sim J$ means that $Q_i = \gamma_i J$, where $\gamma_i$ are arbitrary constants which can be numerically large, small or even zero. Then we can set up a power counting scheme in $1/J$ and $\lambda/J^2$. While we keep all orders of $\lambda/J^2$, we systematically drop terms of subleading orders in $1/J$.}
dimensions. This observation proves to be especially useful in the (“thermodynamic”) limit $L \gg 1$, i.e. for a very long spin chain, where the algebraic Bethe equations are approximated by integral equations. For a large number of fields $J = L$, the dimension $\Delta$ appears to have a loop expansion equivalent to the one in (??), ²

The string semiclassical expression (??), while formally valid for $\sqrt{\lambda} \gg 1$, is actually exact, since, as was mentioned above, all sigma model corrections are suppressed by $\frac{1}{J}$. Assuming the conditions (??) are satisfied, one should be able to compare directly the classical $\mathcal{O}(J)$ term in the string energy (??) to the $\mathcal{O}(J)$ scaling dimension in gauge theory (??), and to show that

The contents of the present paper is the following. In Section 2 we review the results of the semi-classical computation of the energy of folded strings rotating in two planes on $S^5$, the “$(J_1, J_2)$” solution [?]. We also review the results of the Bethe ansatz calculations of the anomalous dimensions of the corresponding gauge theory operators [?]. We then present a full analytic proof that in the region of large quantum numbers the relevant terms in the string energy and the gauge operator dimension match, i.e that $\epsilon_1$ and $\delta_1$ are indeed the same as functions of the spin ratio. This goes beyond the previous “experimental” evidence of matching series expansions. The central part of the present paper is Section 3, where we show that the “$(J_1, J_2)$” state represented by the string rotating in two planes in $S^5$ can be analytically continued, in both string and gauge theory, to an “$(S, J)$” state represented by a string rotating in just one plane in $S^5$, but having also one large spin in $AdS_5$ [?]. On the gauge theory side, this requires the use of a recently constructed [?] supersymmetric extension of the above Bethe ansatz. As a result, we find the agreement between the string theory and gauge theory expressions of the energy/dimension also for the “$(S, J)$” solution. In Section 4 we study the possibility to check the matching (??) beyond the leading order $n = 1$. Using the expression [?] for the gauge theory two-loop dilatation operator, we present numerical evidence that the matching between the string theory and gauge theory results in the case of the $(J_1, J_2)$ state extends to at least to the $n = 2$ (two-loop) level. Section 5 contains some concluding remarks, The Appendices contain some general remarks and technical details. In particular, in Appendix B we explain the relation between the $(J_1, J_2)$ and $(S, J)$ string solutions, and in Appendix C we discuss the solution of the Bethe ansatz system of equations for the spin chain which appeared in Section 3 in connection with the $(S, J)$ case. In Appendix D we compare circular strings on $S^5$ with a different “imaginary” solution of the Bethe equations. Finally, in Appendix E we consider the dependence of string energy on the ratio of two spins.

²For the $\mathcal{O}(\lambda)$ term the $1/J$ dependence can be read off from the thermodynamic limit of the Bethe ansatz. For higher-loops, there are some numerical indications for this $1/J$ dependence, but so far there is no general proof (which, perhaps, may be given using maximal supersymmetry of the theory).
2 Strings rotating on $S^5$

Let us start with a discussion of a particular $(J_1, J_2)$ state corresponding to a *folded* string rotating in two planes on the five-sphere. This folded solution should have minimal value of the energy for given values of the spins. Our aim will be to demonstrate the equivalence between the leading correction to the classical string-theory energy and the one-loop gauge theory anomalous dimension at the *functional* level, i.e. going beyond particular expansions and limits considered previously in [?, ?]. The solution in question [?] has the following non-zero coordinates in (1.1): $t = \kappa \tau$, $\varphi_1 = w_1 \tau$, $\varphi_2 = w_2 \tau$, $\gamma = \frac{\pi}{2}$, $\psi = \psi(\sigma)$ and $\psi$ satisfies a 1-d sine-Gordon equation in $\sigma$. The string is stretched in $\psi$ with the maximal value $\psi_0$ (we refer to Appendices A and B for details on the string solutions). The classical energy and the angular momenta of the rotating string may be written as

Assuming that $J = J_1 + J_2$ is large, i.e. that the condition (??) is satisfied, one can expand the solution for the energy in powers of the total spin $J = \sqrt{\lambda} J$ (cf. (??))

Turning attention to the gauge theory side, the natural operators carrying the same $SO(6)$ charges $(J_1, J_2)$ are of the general form

Can we test this highly non-trivial prediction by a direct one-loop computation in the gauge theory? In the case where $J$ is large, doing this from scratch by Feynman diagram techniques is a formidable task due to the large number of possible field orderings (one needs to diagonalize the anomalous dimension matrix whose size grows exponentially with $J$). What helps is the crucial observation of ref. [?] that the one-loop anomalous dimension matrix for the operators of the two-scalar type (??) can be related to a Hamiltonian of an integrable Heisenberg spin chain (XXX+1/2 model), i.e. its eigenvalues can be found by solving the Bethe ansatz equations of the spin chain. The upshot of the Bethe ansatz procedure [?] is that the system of equations diagonalizing the one-loop anomalous dimension matrix (for any, small or large, values of $J_1, J_2$) is given by

Let us now pause and compare the string theory system (??) for the classical energy and the gauge theory system (??) for the one-loop anomalous dimension. Both systems are parametric, i.e. finding energy/dimension as a function of spins involves elimination of auxiliary parameters. The string result is valid for all $\lambda$, but restricted to large $J$, namely, $J \gg \sqrt{\lambda}$ and $J \gg 1$. The gauge result is valid for all $J_1, J_2$, but restricted to lowest order in $\lambda$. Remarkably, there is a region of joint validity: large charge $J$ and first order in $\lambda$!

Extracting the leading-order or “one-loop” term $\epsilon_1$ from the string-theory relations (??) is straightforward, as discussed in Appendix B. For large $J = J_1 + J_2$ one sets
\[ x = x_0 + x_1/J^2 + \ldots \] and solves the resulting transcendental equation for \( x_0 \). One then finds the parametric solution for \( \epsilon_1 = \epsilon_1(J) \).

We have thus demonstrated the equivalence between the string theory and gauge theory results for a particular two-spin part of the spectrum at the full functional level. Previously the equality \( \epsilon_1 = \delta_1 \) was checked \[?, \?\] only for the first few terms in an expansion around special values of \( J \).

3 Strings rotating on \( AdS_5 \) and \( S^5 \)

Recently, it was shown in \[?\] that the complete one-loop planar dilatation operator of \( \mathcal{N} = 4 \) SYM \[?\] is integrable.\(^3\) To diagonalize any matrix of anomalous dimensions, the corresponding Bethe ansatz was written down in \[?\]. This enables one to access a much wider class of states and perform similar comparisons between gauge theory and semiclassical string theory.

Here we will present a first interesting example of such a novel test: we shall consider the case of only one non-vanishing angular momentum in \( S^5 \) \( (J = J_3) \), but also one non-zero spin in \( AdS_5 \) \( (S = S_1) \). This situation is clearly different from the one discussed in the last section. However, on the string side, the two scenarios are, in fact, mathematically closely related, as we will explain below (see also Appendix B). Is this also true on the gauge side? There the relevant local operators carrying the same \( SO(2,4) \times SO(6) \) charges \( (S, J) \) have the following generic form

\[ \text{Let us now turn to the rotating folded} \ (S, J) \ \text{string solution} \ [?] \ \text{which would be expected to correspond to the just discussed gauge theory operators} \ (\left[?\right]). \ \text{This rotating string is stretched in the radial direction of} \ AdS_5 \ \text{while its center of mass rotates in} \ S^5, \ \text{it has the following non-zero coordinates in} \ (1.1): \ (t = \kappa \tau, \ \rho = \rho(\sigma), \ \phi_1 = \omega_1 \tau, \ \phi_3 = \omega_3 \tau) \ \text{(see Appendices A and B for details). Now the energy} E \ \text{and the spin} S \ \text{can be viewed as two “charges” in} \ AdS_5 \ \text{while} \ J - \ \text{as the charge in} \ S^5. \ \text{This is clearly reminiscent of the previous example where we had two charges} \ (J_1, J_2) \ \text{in} \ S^5 \ \text{and one charge} \ (E) \ \text{in} \ AdS_5, \ \text{and we have just found evidence on the gauge side that one should actually expect the two solutions to be related by an analytic continuation. Indeed, as explained in Appendices A and B, a beautiful way to see this connection on the string side stems from the close relation between the} \ AdS_5 \ \text{and} \ S^5 \ \text{metrics in} \ (1.1). \]

On the level of the final expressions for the string charges the relation is as follows. The analogue of the parametric system of equations for the energy in the \( (J_1, J_2) \) case here is easily found, using the relations in \[?] \ (see Appendix B). We have again \( E = \)

\(^3\)Integrability is related to Yangians. A Yangian structure in the bosonic coset sigma model was recently shown \[?\] to have a generalization to (classical) supercoset sigma model of \[?\]. Very recently \[?\], this structure was “mapped” to planar gauge theory. Possibly, this line of thought will lead to a deeper understanding of the matching of energies/anomalous dimensions.
$\sqrt{\lambda} \mathcal{E}, \; S_1 \equiv S = \sqrt{\lambda} S, \; J_3 \equiv J = \sqrt{\lambda} J$, where $\mathcal{E}, S, J$ depend only on the classical parameters $\kappa, \omega_1, w_3$ and satisfy

\[ \sqrt{\lambda} \mathcal{E}, S_1 \equiv S = \sqrt{\lambda} S, \quad J_3 \equiv J = \sqrt{\lambda} J, \]

where $\mathcal{E}, S, J$ depend only on the classical parameters $\kappa, \omega_1, w_3$ and satisfy

\[ \sqrt{\lambda} \mathcal{E}, S_1 \equiv S = \sqrt{\lambda} S, \quad J_3 \equiv J = \sqrt{\lambda} J, \]

4 Higher loop corrections

Let us now comment on a generalization of the above results to higher orders in $\lambda$ ("higher loops"). First, let us note that on the string side, we have a complete expression for the energies to all orders in $\lambda$ which follows from the systems (??) and (??). In the interaction picture of perturbation theory, the only non-trivial system of equations is the one determining the leading order contribution; all higher-loop terms can be expressed through the leading order modulus $x_0$. The two-loop energies $\epsilon_2$ for the $(J_1, J_2)$ case and $\tilde{\epsilon}_2$ for the $(S, J)$ case are given by (see Appendix B)

\[
\epsilon_2 = \frac{2}{\pi^4} (K(x_0))^3 \left( (1 - 2x_0)E(x_0) - (1 - x_0)^2 K(x_0) \right),
\]

\[
\tilde{\epsilon}_2 = -\frac{2}{\pi^4} (K(x_0))^3 \left( E(x_0) - (1 - x_0^2)K(x_0) \right).
\]

(4.1)

As implied by the relation (??), the two expressions are not expected to (and do not) look similar.

Given that integrability and the Bethe ansatz allow us to obtain the exact one-loop energies for infinite length operators, while string theory gives us an all-loop prediction, it would be interesting to find higher-loop energies in gauge theory to compare to string theory. Although the integrability property of the dilatation operator acting on the states (??) seems to be maintained (at least) at the two-loop level [?], the corresponding extension of the Bethe ansatz is not yet known.\(^4\) Therefore, in order to find higher-loop anomalous dimensions of the operators (??) we have to rely on numerical methods of diagonalization of the matrix of anomalous dimensions. For the states (??), this matrix is generated by the planar dilatation operator [?]

\[
D(\lambda) = J + \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} (1 - P_{k,k+1}) \quad (4.2)
\]

\[ + \frac{\lambda^2}{128\pi^4} \sum_{k=1}^{J} (-4 + 6P_{k,k+1} - P_{k,k+1}P_{k+1,k+2} - P_{k+1,k+2}P_{k,k+1}) + \mathcal{O}(\lambda^3), \]


\(^4\)In principal agreement with the string theory result, one might express higher-loop energies in terms of the one-loop Bethe roots. However, this would require calculating matrix elements of the higher-loop dilatation operator between Bethe states – presently a very non-trivial issue.
The numerical one-loop results for finite $J$ are already reasonably close to the string theory prediction. As proposed in \[\text{[?]}\], we can improve the results by extrapolating to $J = \infty$. This is done fitting to the first two terms in the series expansion in $1/J$.

The extrapolated values of the dimensions at one-loop and two-loop orders \[\text{[?]}\] are found to be about 1\% off the string theory prediction. These results agree very well and we can clearly confirm that the correspondence works at $O(\lambda^2)$! For the three-loop conjecture of \[\text{[?]}\] (see also \[\text{[?]}\]) the results are somewhat inconclusive. On the one hand, using the vertex that was constructed assuming that integrability holds at $O(\lambda^3)$, we get an extrapolation, $\delta_3$, which is 17\% off the string prediction. On the other hand, the vertex that was matched to near plane-wave string theory results \[\text{[?]}\] gives an extrapolation, $\delta'_3$, that is only 2\% away. Nevertheless, we expect the three-loop energies to converge rather slowly and the three values of $J$ used to extrapolate are clearly not sufficient: In the spin chain picture the three-loop interaction already extends over four lattice sites, and finite size effects, due to the relatively small chain lengths, become more pronounced. An indication for this is that the extrapolation is still 50\% away from the input values. The $1/J^2$ terms which were neglected are expected to have a much stronger influence on the finite $J$ values as compared to $O(\lambda)$ term. Therefore, a 17\% mismatch seems reasonable and we can neither confirm nor rule out any of the conjectured three-loop vertices \[\text{[?]}\] (or the correspondence at $O(\lambda^3)$).

### 5 Conclusions and Outlook

In this paper we demonstrated that spectroscopy is becoming a very precise and versatile tool for establishing the validity of the AdS/CFT duality conjecture on a quantitative, dynamical level. Following the suggestion of \[\text{[?], [?]}\] and extending the earlier break-through work of \[\text{[?], [?], [?]}\] we have shown that in the non-BPS sector of two large charges, as in the near-BPS BMN sector with single large charge \[\text{[?]}\], the duality between the SYM theory and $AdS_5 \times S^5$ string theory relates perturbative results on both sides of the correspondence and thus can be tested using existing tools.
It should be fairly evident that our derivation of mathematically highly involved energy expressions, such as eqs. (??), (??), from both string theory and gauge theory constitutes a “physicist’s proof” of the correspondence. We believe that the present work is just the beginning of a much wider unraveling of dynamical details of the AdS/CFT duality. At the end, we expect to gain much insight into superstring theory on curved backgrounds, and into gauge theory at finite coupling.

Our work suggests a large number of further inquiries. The precise interpretation of the circular versus folded string solutions remains somewhat obscure in the Bethe ansatz picture. In particular, it would be important to understand the analog of the string solution for $J_2 > J/2$ in the Bethe ansatz and thus complete the picture outlined in Appendix E. Furthermore, it seems that the Bethe ansatz allows for very complicated distributions of “Bethe root strings”, involving multi-cut solutions, the role of which is unclear so far on the string theory side.

Another obvious problem is to extend the comparison to include $1/J$ terms by computing (as in [?, ?]) the 1-loop string sigma model correction to the $(J_1, J_2)$ string energy and comparing the result to the leading correction to the “thermodynamic” limit of the $XXX_{+1/2}$ Bethe system. It would be interesting also to compute energies of excited string states by expanding the superstring action near the ground-state two-spin $(J_1, J_2)$ solution. In contrast to the BMN case [?], here one expects (from experience with special circular solutions [?]) that there will be many nearby states with the same charges and with energies differing from the ground state energy by order $\frac{1}{J^2} = \frac{\lambda}{J^2}$ terms (these are of course negligible as compared to similar terms in the classical ground-state energy in the limit $J \gg 1$).

It should be relatively straightforward, if laborious to extend the analysis to more than two spins. In string theory this has largely been accomplished for three non-vanishing angular momenta on $S^5$ in [?], but one could try to also include concurrently the two $AdS_5$ charges. For gauge theory, the corresponding Bethe equations are known [?], but have not yet been analyzed in any generality. Ideally, one would like to understand how to prove these equivalences directly, i.e. without actually solving the classical string sigma model equations and the Bethe equations in the thermodynamic limit.

The biggest challenge clearly is to find out how to extend the calculational power on either side of the correspondence in a way that would allow one to derive results that are not in the overlapping window of large quantum numbers and small effective string tension. On the string theory side, this would require to include quantum (inverse string tension) corrections in the Green-Schwarz supercoset sigma model of [?]. For gauge theory, we would need to understand the proper extension of the Bethe ansatz so as to make it applicable to all orders in Yang-Mills perturbation theory. Maybe integrability will lead the way.

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A Rotating string solutions

Let us make some general observations on 5-spin string solutions in $AdS_5 \times S^5$ pointing out some relations between different types of solutions via an analytic continuation. The general rotating strings carrying 2+3 charges $(S_1, S_2; J_1, J_2, J_3)$ and the energy $E$ are described by the following ansatz [?] (see (1.1))

One can find also other transformations that map solutions into solutions by combining (??) with special (discrete) $SO(2, 4) \times SO(6)$ isometries that do not induce other components of the rotation generators except the above Cartan ones (e.g., interchanging the angular coordinates induces interchanging of the charges in (??), etc.). Below we shall consider such an example.

B Relation between two-spin solutions

Let us now show that the two previously known two-spin folded string solutions are, in fact, related by the above analytic continuation. Firstly, there is the “$(S, J)$” solution [?] Similarly, for the $(J_1, J_2)$ solution (??) one finds from the expressions given in [?] (we assume $w_2^2 > w_1^2$)

$$\kappa^2 - w_2^2 = \sin^2 \psi_0 \equiv x > 0 , \quad 1 = \frac{J_1}{w_1} + \frac{J_2}{w_2}, \quad \mathcal{E} = \kappa,$$

$$J_1 = \frac{2w_1}{\pi \sqrt{w_2^2 - w_1^2}} E(x), \quad \sqrt{w_2^2 - w_1^2} = \frac{2}{\pi} K(x). \quad (B.1)$$

Solving for $w_1, w_2$ in terms of $J_1$ and $x$

Depending on the region of the parameter space (or values of the integrals of motion) one finds different functional form of dependence of the energy on the two spins. We discuss some aspects of this dependence in Appendix E below. A direct comparison with gauge theory we are interested in here is possible in the case when the two spins $S$ and $J$ are both large compared to $\sqrt{\lambda}$, i.e. $S \gg 1$, $J \gg 1$. The analogous limit [?] for the $(J_1, J_2)$ solution is when $J_1 \gg 1$, $J_2 \gg 1$. In the two cases we can then expand
the energies, e.g., in powers of the total $S^5$ spin $J$. This amounts to an expansion in powers of $J \equiv J_3$ in the $(S,J)$ case and in powers of $J \equiv J_1 + J_2$ in the $(J_1,J_2)$ case, respectively,

$$E = S + J + \frac{\lambda}{J} \tilde{\epsilon}_1(S/J) + \frac{\lambda^2}{J^3} \tilde{\epsilon}_2(S/J) + \ldots, \quad J \equiv J_3, \quad S \gg \sqrt{\lambda},$$

$$E = J + \frac{\lambda}{J} \epsilon_1(J_2/J) + \frac{\lambda}{J^3} \epsilon_2(J_2/J) + \ldots, \quad J \equiv J_1 + J_2, \quad J_2 \gg \sqrt{\lambda}, \quad (B.2)$$

where we introduced tildes on the correction functions $\epsilon_n$ in the first solution case. One may wonder if the coefficients $\tilde{\epsilon}_1$ and $\epsilon_1$ in (B.2) are related in some way, given that the two solutions are related by the analytic continuation. Applying formally the substitution (??) in (B.2) we get, to the leading order,

Let us now demonstrate that (??) follows also from the string-theory equations (??) and (B.1) or the systems (??) and (??). Expanding the parameter $x$ for large $J$ as (with $J$ being $J_1 + J_2$)

\section{C Gauge theory details}

Here we will outline the solution of the Bethe ansatz system of equations (??) for the novel case of the XXX\textsubscript{−1/2} Heisenberg spin chain. We expect that the positions of the roots are of order $O(J)$, where $J$ is the length of our non-compact magnetic chain, as explained in Section 3. We then take the logarithm of the equations (??) and obtain for large $J$

As in the case of the XXX\textsubscript{+1/2} system, we shall start with assuming that in the large $J$ limit the Bethe roots accumulate on smooth contours. It is reasonable, therefore, to replace the discrete root positions $u_j$ by a (rescaled) smooth continuum variable $u$ and introduce a density $\rho(u)$ describing the distribution of the roots in the complex $u$-plane:

As opposed to the XXX\textsubscript{+1/2} case, we expect the roots for the ground state to lie on the real axis (this may be verified by explicit solution of the exact Bethe equations for small values of $J$). Furthermore, we assume the distribution of roots to be symmetric w.r.t. the imaginary axis, $\rho(-u) = \rho(u)$. We therefore expect the support of the root density to split into (at least) two disjoint intervals $C = C^- + C^+$ with $C^- = [-b,-a]$ and $C^+ = [a,b]$, where $a < b$ are both real.\footnote{After the analytical continuation to the spin +\textsubscript{1/2} case, the points $a, b$ become a complex conjugate pair.} For the ground state we expect just two contours, and the mode numbers should be $n = \mp 1$ on $C^\pm$. For this distribution of roots, the Bethe equations (??) become
D The circular vs. imaginary solution

In [?] a solution different from the type discussed in Appendix C was found. The resulting anomalous dimension matched the energy of a circular string [?] at one point of the parameter space, \( J_2 = J_1 = J/2 \). Recently the circular string solution was extended to all values of \( J_2 \) [?] where it was also shown that the agreement with gauge theory persists up to a few orders in a perturbative expansion around \( J_2 - J/2 \). Here, we will complete the analysis and prove the correspondence at the analytic level. We are grateful to Gleb Arutyunov for his collaboration on this Appendix. Without further details of the derivation, we present the final results starting with gauge theory.

There are two conditions on the endpoints \( s, it, s < t \), of the Bethe strings that arise in the solution [?] (\( \alpha = J_2/J \)):

The circular string is obtained by the same ansatz (??) as for the folded string (see Appendices A,B). The only difference is that the function \( \psi(\sigma) \) is now assumed to be periodic modulo \( 2\pi \)

E Energy as a function of the spins

In this Appendix we shall discuss the behavior of the leading term in the classical energy for the two-spin string solutions in different regions of the parameter space \( J_2/J \) or \( S/J \), respectively. For the \((J_1, J_2)\) solution the function \( \epsilon_1(J_2/J) \) is defined in the region \( 0 \leq J_2/J < 1 \) whereas for the \((S, J)\) solution \( \tilde{\epsilon}_1(S/J) \) is naturally defined for \( 0 \leq S/J < \infty \). When analytically continued, these functions are related by \( \epsilon_1(j) = -\tilde{\epsilon}_1(-j) \). In Figure 1, we therefore plot the function \( \epsilon_1(j) \) in the region \( j \in (-\infty, 1) \).

Depending on the region of the parameter space one finds different functional form of dependence of the energy on the two spins. For example, in the case of the \((S, J)\) solution there are two different asymptotics that were considered in [?]: The string can become very long and approach the boundary of AdS\(_5\), i.e. \( \rho_0 \to \infty \); the energy of this configuration is

At \( S/J = 0 \) we make the “Wick-rotation” to the \((J_1, J_2)\) solution. The energy of a short string rotating on \( S^5 \) is given by

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Figure 1: The leading order correction to the energy $\epsilon_1$ for the folded string solution. The region $J_2/J > 0$ correspond to the $(J_1, J_2)$ case, whereas $J_2/J < 0$ correspond to the $(S, J)$ case with $S/J = -J_2/J$ and energy $\tilde{\epsilon}_1 = -\epsilon_1$. We also plot a mirror image under the symmetry $J_1 \leftrightarrow J_2$ (dashed) and the energy of the circular string $\hat{\epsilon}_1$ (dotted).