# Pursuing The Quantum World
## Flat Family Of QFTs And Quantization Of $d$-Algebras

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*Dedicated To the Memory of Youngjai Kiem*

Exploiting the path integral approach al la Batalin and Vilkovisky, we show that any anomaly-free Quantum Field Theory (QFT) comes with a family parametrized by certain moduli space $\mathfrak{M}$, which tangent space at the point corresponding to the initial QFT is given by the space of all observables. Furthermore the tangent bundle over $\mathfrak{M}$ is equipped with flat quantum connection, which can be used to determine all correlation functions of the family of QFTs. We also argue that considering family of QFTs is an inevitable step, due to the fact that the products of quantum observables are not quantum observables in general, which leads to a new "global" perspective on quantum world. We also uncover structure of $d$-algebra in the large class of $d$-dimensional QFT. This leads to an universal quantization machine for $d$-algebras decorated by algebro-differential-topology of $(d+1)$-manifolds as well as a new perspective on differential-topology of low dimensions. This paper is a summary of a forthcoming paper of this author.

# 1 Prelude

The physical reality is supposed to be genuinely quantum such that anything classical, if any, is just an approximated or derived notion. On the other hand, classical physical theory, historically, came first and, then, certain procedure called quantization has been adapted to mimic quantum description. This idea of quantization of classical object or notion is obviously limited, though it gives us an useful guide of our journey toward genuine quantum world, if we take educated steps with constant renormalization of our viewpoints - namely by pursuing clues left out from quantum world. Only in "the" end, if any, of such journey we may have proper definition of what it meant to be Quantum. It seems also reasonable to assume that mathematical structures associated with classical physics as shadows of certain quantum mathematics. This, in particular, implies that essential properties of supposedly classical world may be understood naturally in the quantum perspective.

This article is a summary of this author's pursuit to understand Quantum Fields Theory (QFT) for the last 3 years, which details shall appear elsewhere [28].

We shall begin with, in section 2, a reflection on the idea of quantization in the path integral approach a la Batalin-Vilkovisky (BV) formalism, which will lead, in section 3, to notion of quantum flat structure on a family of QFTs. The story goes as follows; In the beginning we may start from certain classical field theory characterized by a classical action functional with certain (gauge) symmetry. The BV quantization scheme, then, suggests to build up, out of the given classical structure, a mathematical structure dubbed as *quantum weakly homotopy Lie* $(-1)$-*algebroid*, which procedure is controlled by BV algebra - the odd nilpotent and order 2 BV operator $\boldsymbol{\Delta}$ together ordinary product. A crucial property of the BV operator $\boldsymbol{\Delta}$ is that it is not a derivation of the product which leads to a problem that the products of quantum observables of a given QFT are not quantum observables in general. Then we argue that the above problem can be resolved by considering family of QFTs parametrized certain moduli space $\mathfrak{M}$ defined by set of equivalence classes of solutions of the celebrated quantum master equation. Here the initial QFT is interpreted as a base point $o$ in $\mathfrak{M}$ such that certain basis of quantum observables is regarded as a basis of tangent space to $\mathfrak{M}$ at the base point $o$. Now the problem involving products of quantum observables shall be resolved by reaching out to certain formal (beyond infinitesimal) neighborhood of $o$ in $\mathfrak{M}$, introducing certain quantum products of quantum observables. Then we shall have a system of differential equations satisfied by generating functional of path integrals for the family of QFTs.

All of these may be summarized by *quantum flat structure*, which roughly suggest to build up a kind of formal graded bundle $\mathfrak{Q} \rightarrow \mathfrak{M}$ equipped with certain graded flat connection in formal power series of the Planck constant $\hbar$ such that path integrals can be described as flat sections. This structure seems to suggest to find certain completion $\overline{\mathfrak{Q}} \rightarrow \overline{\mathfrak{M}}$ including various degenerated limits of $\mathfrak{Q} \rightarrow \mathfrak{M}$, where various perturbative QFTs may corresponds to singular points. Unfortunately this author does not have a good understanding on the above issues.

The section 4 is a sketch of a program to understand a large class of QFTs on a $(d+1)$-dimensional manifold with or without boundaries. This section is essentially an elaboration and generalization of this author's previous work. Some reflection on $(d+1)$-dimensional field theory a la BV quantization scheme suggests that one may associate pre-QFT on smooth oriented $(d+1)$-manifold with any structure of symplectic $(d+1)$-algebra. On $(d+1)$-manifold with boundary the BV quantization scheme suggest that one can associate a structure of strongly homotopy Lie $d$-algebroid on the boundary. This suggest application of $(d+1)$-dimensional QFT with

boundary to deformation theory of so-called $d$-algebra [23], generalizing the deformation quantization story of Kontsevich [22]. This section is an elaboration and generalization of this author's previous work [27], where we assumed much narrow scope. We shall also introduce possible notion of quantum cobordism.

The pre-QFT constructed from a symplectic $s$-algebra would become actual QFT if the set of equivalence class of solutions of Maurer-Cartan equation is isomorphic to the moduli space $\mathfrak{M}$ discussed in Section 3, which notion requires proper definition of the BV operator $\boldsymbol{\Delta}$. It is unfortunate that this author does not have good understanding in a proper definition of $\boldsymbol{\Delta}$, which notion is the nerve of quantum flat structure. Assuming that we have a good definition of $\boldsymbol{\Delta}$ our model shall not only provided an universal (?) quantization machine of $d$-algebras as well as new arena of differential-topological invariants of low dimensional manifolds.

For the purpose this article we didn't present any explicit example adopting the general program. This author, however, has been implanted the idea to produce many of known QFTs and numerous new examples for the last 3-years, which already appeared or shall appear in some future publications [28, 29, 30, 31, 19, 20, 21, 25, 26]. Throughout this paper we shall restrict to the case, for the sake of simplicity, that a field with even ghost number $U$ is commuting, while a field with odd ghost number $U$ is anti-commuting under the ordinary product. This restriction means, in particular, that an object with ghost number $U = 0$ is commuting and, thus, excluding physical fermions. It is straight forward to generalize the results in this paper including physical fermions.

This author has chosen a rather vague but symbolic title of this article with an intention.

I would like to dedicate this small work to the memory of Youngjai Kiem. He was a very good friend of mine and a talented young physicist who passed away by a tragic accident. His brief but enthusiastic life had been devoted to, in my humble opinion, pursuing quantum world besides from his late family and friends. In the last semester of his life he also kindly provided me a visiting position in KAIST with an excellent research environment.

## 2 From Classical Field Theory and Its Symmetry

We may view the history of understanding QFT as a long journey toward to genuine quantum world starting from a small corner near a classical world, which may be described in terms of certain classical field theory defined by an

classical action functional $\boldsymbol{s}$ with certain gauge symmetry. In the quantum theory the central object is not the action functional but Feynman Path Integral [13]. The gauge symmetry of classical action functional requires additional fields called Faddev-Popov (FP) ghosts in the path integral [12], which replace the gauge symmetry to odd global symmetry generated by BRST (Becchi-Rouet-Stora and Tuytin) charge $\boldsymbol{q}$, i.e. $\boldsymbol{qs} = 0$ [4, 39], which satisfies $\boldsymbol{q}^2 = 0$, in general, modulo classical equations of motion of $\boldsymbol{s}$. We can say that the above was the first step toward the world of QFT.

To be brief we begin with the state of arts scheme of quantization pioneered by Batalin-Vilkovisky [3], see also [40, 33, 43, 44, 46, 17, 1, 37, 22, 27] for its various aspects, and look back the history as a revisionist.

### 2.1 The BV Quantization I

Let $\{\phi^A\}$ denotes collectively all classical fields, ghosts, anti-ghost multiplets etc etc, which may be called `fields`. Here the indices $\{A\}$ are understood as both continuous and discrete, or we may say $\{\phi^A\}$ is certain coordinates on an infinite dimensional graded space $\boldsymbol{\mathcal{L}}$, where the grading are specified by an integral ghost number $U \in \mathbb{Z}$ and an $\mathbb{Z}_2$ grading called parity . BV introduced so called set of `anti-fields` $\{\phi^\bullet_A\}$ for each `fields` such that

$$U(\phi^A) + U(\phi^\bullet_A) = -1, \qquad |\phi^A| + |\phi^\bullet_A| = 1, \tag{2.1}$$

where the notation $|\star|$ means the parity (equivalently the statistics) 0 (even) or 1 (odd) of the expression $\star$ such that objects with the even parity are commuting while objects with the odd parity are anti-commuting. The space of all `fields` and `anti-fields` can be viewed as the total space $\boldsymbol{\mathcal{T}} \simeq T^*[-1]\boldsymbol{\mathcal{L}}$ of twisted by $[-1]$ cotangent bundle to $\boldsymbol{\mathcal{L}}$ - twisting by $[-1]$ simply means the convention (2.1) of ghost numbers.[1] Then we have a canonical odd symplectic structure $\boldsymbol{\omega}$ of ghost number $U = -1$ on $\boldsymbol{\mathcal{T}}$, i.e., $U(\boldsymbol{\omega}) = -1$ and $|\boldsymbol{\omega}| = 1$. We shall restrict to the case, for the sake of simplicity, that a field with even (odd) ghost number $U$ is even (odd) parity.

Let $\mathfrak{T}[[\hbar]]$ be the space of functions on $\boldsymbol{\mathcal{T}}$ formal power series in Planck constant $\hbar$ with $U(\hbar) = 0$, which is also a graded space by the ghost number $U$;

$$\mathfrak{T}[[\hbar]] = \bigoplus_{k\in\mathbb{Z}} \mathfrak{T}[[\hbar]]_k. \tag{2.2}$$

[1] In general $T^*[s]\boldsymbol{\mathcal{L}}$ shall means the conventions $U(\phi^A) + U(\phi^\bullet_A) = s$ and $|\phi^A| + |\phi^\bullet_A| = s \bmod 2$.

Note that we have ordinary product $\cdot$ carrying $U = 0$, which is graded commutative and graded associative, i.e., $\mathcal{O}_1 \cdot \mathcal{O}_2 = (-1)^{|\mathcal{O}_1| \cdot |\mathcal{O}_2|} \mathcal{O}_2 \cdot \mathcal{O}_1$ where $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{T}[[\hbar]]$.

BV introduced an order 2 odd differential $\boldsymbol{\Delta}$ operator carrying $U = 1$

$$\boldsymbol{\Delta} \text{"="} (-1)^{|\phi^A|+1} \frac{\delta_r^2}{\delta\phi^A \delta\phi^\bullet_A}, \qquad \boldsymbol{\Delta} : \mathfrak{T}[[\hbar]]_k \longrightarrow \mathfrak{T}[[\hbar]]_{k+1}, \tag{2.3}$$

satisfying

$$\boldsymbol{\Delta}^2 = 0. \tag{2.4}$$

We denote $\frac{\delta_r}{\delta\phi}$ and $\frac{\delta_l}{\delta\phi}$ the right and left differentiation, respectively. We shall also use convention that repeated up and down indices are summed (or integrated) over.

A *BV action functional* $\boldsymbol{S} \in \mathfrak{T}[[\hbar]]_0$ is an even function on $\boldsymbol{\mathcal{T}}$ in formal power series of $\hbar$ satisfying the celebrated *master equation*

$$\boldsymbol{\Delta} e^{-\boldsymbol{S}/\hbar} = 0. \tag{2.5}$$

We should not forget to emphasis that the definition of $\boldsymbol{\Delta}$ is formal or even symbolic and always requires careful regularization to ensure the crucial property $\boldsymbol{\Delta}^2 = 0$. We shall not care about it at this stage and simply assume that the space $\boldsymbol{\mathcal{T}}$ is equipped with such an operator - but QFT is largely characterized by $\boldsymbol{\Delta}$.

Being an order 2 differential $\boldsymbol{\Delta}$ is not a derivation of the product;

$$(-1)^{|\mathcal{O}_1|} \boldsymbol{\Delta}(\mathcal{O}_1 \cdot \mathcal{O}_2) - (-1)^{|\mathcal{O}_1|} \boldsymbol{\Delta}\mathcal{O}_1 \cdot \mathcal{O}_2 - \mathcal{O}_1 \cdot \boldsymbol{\Delta}\mathcal{O}_2 = (\mathcal{O}_1, \mathcal{O}_2), \tag{2.6}$$

where the bracket $(\star, \star)$, called BV bracket, is identical to the odd Poisson bracket carrying $U = 1$ associated with the symplectic structure $\boldsymbol{\omega}$ of ghost number $U = -1$ on $\boldsymbol{\mathcal{T}}$.;

$$(\star, \star) : \mathfrak{T}[[\hbar]]_{k_1} \otimes \mathfrak{T}[[\hbar]]_{k_2} \longrightarrow \mathfrak{T}[[\hbar]]_{k_1+k_2+1}. \tag{2.7}$$

The BV bracket operation may be expressed as

$$(\mathcal{O}_1, \mathcal{O}_2) = \left( \frac{\delta_r \mathcal{O}_1}{\delta\phi^A} \cdot \frac{\delta_l \mathcal{O}_2}{\partial\phi^\bullet_A} - \frac{\delta_r \mathcal{O}_1}{\partial\phi^\bullet_A} \cdot \frac{\delta_l \mathcal{O}_2}{\partial\phi^A} \right) \tag{2.8}$$

and satisfies

$$\begin{aligned}
(\mathcal{O}_1, \mathcal{O}_2) &= -(-1)^{(|\mathcal{O}_1|+1)(|\mathcal{O}_2|+1)} (\mathcal{O}_2, \mathcal{O}_1), \\
(\mathcal{O}_1, \mathcal{O}_2 \cdot \mathcal{O}_3) &= (\mathcal{O}_1, \mathcal{O}_2) \cdot \mathcal{O}_3 + (-1)^{(|\mathcal{O}_1|-1)|\mathcal{O}_2|} \mathcal{O}_2 \cdot (\mathcal{O}_1, \mathcal{O}_3), \\
(\mathcal{O}_1, (\mathcal{O}_2, \mathcal{O}_3)) &= ((\mathcal{O}_1, \mathcal{O}_2), \mathcal{O}_3) + (-1)^{(|\mathcal{O}_1|-1)(|\mathcal{O}_2|-1)} (\mathcal{O}_2, (\mathcal{O}_1, \mathcal{O}_3)).
\end{aligned} \tag{2.9}$$

Now the master equation (2.5) is equivalent to the following equation

$$-\hbar\boldsymbol{\Delta}\boldsymbol{S} + \frac{1}{2}(\boldsymbol{S}, \boldsymbol{S}) = 0. \tag{2.10}$$

Let's consider the odd Hamiltonian vector $\boldsymbol{Q}_{\boldsymbol{S}}$

$$\boldsymbol{Q}_{\boldsymbol{S}} = (\boldsymbol{S}, \ldots), \tag{2.11}$$

carrying $U = 1$ as well as the following odd operator $\boldsymbol{K}_{\boldsymbol{S}}$

$$\boldsymbol{K}_{\boldsymbol{S}} := -\hbar\boldsymbol{\Delta} + \boldsymbol{Q}_{\boldsymbol{S}}, \qquad \boldsymbol{K}_{\boldsymbol{S}} : \mathfrak{T}[[\hbar]]_k \longrightarrow \mathfrak{T}[[\hbar]]_{k+1}, \tag{2.12}$$

carrying $U = 1$. The master equation (2.10) implies that

$$\boldsymbol{K}_{\boldsymbol{S}}^2 = 0, \tag{2.13}$$

while $\boldsymbol{Q}_{\boldsymbol{S}}^2 \neq 0$ in general. Thus we have the BV complex

$$\left(\boldsymbol{K}_{\boldsymbol{S}}, \mathfrak{T}[[\hbar]] = \bigoplus_k \mathfrak{T}[[\hbar]]_k\right) \tag{2.14}$$

The set of *observables* of given QFT is identified with set of cohomology classes of the above BV complex, beautiful! There is another crucial property, which looks vexing initially but beautiful, that the products $\boldsymbol{K}_{\boldsymbol{S}}$-closed elements in $\mathfrak{T}[[\hbar]]$ are not $\boldsymbol{K}_{\boldsymbol{S}}$-closed in general, since $\boldsymbol{\Delta}$ is not a derivation of products. We note that both $\boldsymbol{Q}_{\boldsymbol{S}}$ and $\boldsymbol{\Delta}$ are derivations of the BV bracket.

At this stage we trace back to our starting point in the following two subsections.

## 2.2 Quantum Weakly Homotopy Lie $(-1)$-Algebroid

We consider a solution of master as the formal power series in $\hbar$ as

$$\boldsymbol{S} = \boldsymbol{S}^{(0)} + \sum_{\ell=1}^{\infty} \hbar^\ell \boldsymbol{S}^{(\ell)}, \qquad \boldsymbol{\Delta}\boldsymbol{S}^{(\ell)} = \frac{1}{2}\sum_{r+s=\ell+1}\left(\boldsymbol{S}^{(r)}, \boldsymbol{S}^{(s)}\right) \text{ for } \forall n \geq 0 \tag{2.15}$$

such that

$$\begin{aligned} 0 &= \left(\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(0)}\right), \\ \boldsymbol{\Delta}\boldsymbol{S}^{(0)} &= \left(\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)}\right), \end{aligned} \tag{2.16}$$

etc. etc. Then one may Taylor expand the each term $\boldsymbol{S}^{(\ell)}$ around the space $\boldsymbol{\mathcal{L}}$ of `fields`

$$
\begin{aligned}
\boldsymbol{S}^{(\ell)} &= \sum_{n=0}^{\infty} \boldsymbol{M}_n^{(\ell)}, \\
\boldsymbol{M}_n^{(\ell)} &= \sum\!\!\!\!\!\!\int (\boldsymbol{m}(\phi)^{(\ell)})^{A_1 \dots A_n} \phi^{\bullet}_{A_1} \dots \phi^{\bullet}_{A_n}
\end{aligned}
\tag{2.17}
$$

leading to double infinite sequence of relations

$$
\boldsymbol{\Delta M}_n^{(\ell)} = \frac{1}{2} \sum_{r+s=\ell+1} \sum_{p+q=n} \left( \boldsymbol{M}_p^{(r)}, \boldsymbol{M}_q^{(s)} \right). \tag{2.18}
$$

For each $\boldsymbol{M}_n^{(\ell)}$ one may assign $n$-poly differential operator $\boldsymbol{m}_n^{(\ell)}$ acting on the $n$-th tensor product $\mathfrak{L}^{\otimes m}$ of the space $\mathfrak{L}$ of functions on $\boldsymbol{\mathcal{L}}$ such as;

$$
\begin{aligned}
\boldsymbol{m}_0^{(\ell)} &: \boldsymbol{\mathcal{L}} \to \mathbb{k}, \\
\boldsymbol{m}_n^{(\ell)} &: \mathfrak{L}^{\otimes n} \to \mathfrak{L}, \qquad \text{for } n \geq 1
\end{aligned}
\tag{2.19}
$$

by canonically "quantization", i.e., replacing the BV bracket $(\phi^A, \phi^{\bullet}_B) = \delta^A{}_B$ to commutators of operators $[\widehat{\phi}^A, \widehat{\phi}^{\bullet}_B] = \delta^A{}_B$; naively $\widehat{\phi}^B = \phi^B$ and $\widehat{\phi}^{\bullet}_A = \frac{\delta}{\delta \phi^A}$. In this way we have following double infinite sequence

$$
\begin{array}{ccccc}
\boldsymbol{s} & \boldsymbol{q} & \boldsymbol{m}_2^{(0)} & \boldsymbol{m}_3^{(0)} & \cdots \\
\boldsymbol{m}_0^{(1)} & \boldsymbol{m}_1^{(1)} & \boldsymbol{m}_2^{(1)} & \boldsymbol{m}_3^{(1)} & \cdots \\
\boldsymbol{m}_0^{(2)} & \boldsymbol{m}_1^{(2)} & \boldsymbol{m}_2^{(2)} & \boldsymbol{m}_3^{(2)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots
\end{array}
\tag{2.20}
$$

where we denoted

$$
\begin{aligned}
\boldsymbol{m}_0^{(0)} &= \boldsymbol{s}, \\
\boldsymbol{m}_1^{(0)} &= \boldsymbol{q}.
\end{aligned}
\tag{2.21}
$$

It is not difficult to check $\boldsymbol{m}_n^{(\ell)}$ carry ghost number $U = n$ for $\forall \ell$. For instance $\boldsymbol{s}$ and $\boldsymbol{q}$ have the ghost number $U = 0$ and $U = 1$, respectively.

We consider the classical master equation;

$$
\left( \boldsymbol{S}^{(0)}, \boldsymbol{S}^{(0)} \right) = 0, \tag{2.22}
$$

which can be decomposed as $\sum_{p+q=m}\left(\boldsymbol{M}_p^{(0)},\boldsymbol{M}_q^{(0)}\right)=0$ for $\forall n\geq 0$ , i.e.,

$$\begin{aligned} 0 &= -\left(\boldsymbol{M}_0^{(0)},\boldsymbol{M}_1^{(0)}\right), \\ \frac{1}{2}\left(\boldsymbol{M}_1^{(0)},\boldsymbol{M}_1^{(0)}\right) &= -\left(\boldsymbol{M}_0^{(0)},\boldsymbol{M}_2^{(0)}\right), \\ \left(\boldsymbol{M}_1^{(0)},\boldsymbol{M}_2^{(0)}\right) &= -\left(\boldsymbol{M}_0^{(0)},\boldsymbol{M}_3^{(0)}\right), \\ \frac{1}{2}\left(\boldsymbol{M}_2^{(0)},\boldsymbol{M}_2^{(0)}\right)+\left(\boldsymbol{M}_1^{(0)},\boldsymbol{M}_3^{(0)}\right) &= -\left(\boldsymbol{M}_0^{(0)},\boldsymbol{M}_4^{(0)}\right), \\ \vdots \quad &= \quad \vdots \end{aligned} \tag{2.23}$$

We define a structure of *weakly homotopy Lie* $(-1)$*-algebroid* on $\boldsymbol{\mathcal{L}}$ by the sequence

$$\boldsymbol{s},\boldsymbol{q},\boldsymbol{m}_2^{(0)},\boldsymbol{m}_3^{(0)},\cdots \tag{2.24}$$

associated with a solution of the tree level master equation (2.23).[2] In more familiar terminology $\boldsymbol{s}$, which is a function on $\boldsymbol{\mathcal{L}}$, is nothing but the *classical action functional* from which we may have been started. We also note that

$$\boldsymbol{s}=\lim_{\hbar\to 0}\boldsymbol{S}|_{\boldsymbol{\mathcal{L}}}. \tag{2.25}$$

The first order differential operator $\boldsymbol{q}$ with ghost number $U=1$ is the BRST operator, which equivalent to an odd vector field on $\boldsymbol{\mathcal{L}}$. The first relation in RHS of (2.23) is the familiar BRST invariance of the classical action functional

$$\boldsymbol{q}\boldsymbol{s}=0. \tag{2.26}$$

The second relation in RHS of (2.23) implies that nilpotency of the BRST operator $\boldsymbol{q}$ may be violated up to the equation of motion of $\boldsymbol{s}$

$$\boldsymbol{q}^2(anything)=-\boldsymbol{m}_2^{(0)}(\boldsymbol{s},anything)\propto\delta\boldsymbol{s}. \tag{2.27}$$

Thus BV quantization suggests to find the whole sequence (2.24) of a structure of weakly homotopy Lie $(-1)$-algebroid from the classical action functional $\boldsymbol{s}$ and its symmetry. Such a procedure would produce the tree level

[2] The terminology weakly homotopy Lie $(-1)$-algebroid is a composition from the notions of (weakly or strongly) homotopy Lie algebra, $d$-algebra, and Lie algebroid. Strongly homotopy algebra has been first introduced by Stasheff [35]. For the notion of $d$-algebra with $d=2,3,\ldots$ see Kontsevich [23]. Lie algebroid may be regarded as the infinitesimal of Lie groupoid, (see Weinstein for an introduction). The relation between solution classical master equation (2.22) and homotopy algebra seems to be trace back to Witten [44], Zwiebach [46] and to Alexandrov et. al., [1].

BV action functional $\boldsymbol{S}^{(0)}$ satisfying the first equation in (2.16). Then one should check if $\boldsymbol{\Delta S}^{(0)} = 0$ and, otherwise, find $\boldsymbol{S}^{(1)}$ satisfying the second equation in (2.16), etc. etc.

We may also combine the sequence $(\boldsymbol{m}_n^{(0)}, \boldsymbol{m}_n^{(1)}, \boldsymbol{m}_n^{(2)}, \cdots)$ as

$$\boldsymbol{m}_n = \boldsymbol{m}_n^{(0)} + \sum_{\ell=1}^{\infty} \hbar^\ell \boldsymbol{m}_n^{(\ell)} \tag{2.28}$$

such that

$$\begin{aligned} \boldsymbol{m}_0 :& \boldsymbol{\mathcal{L}} \rightarrow \Bbbk[[\hbar]], \\ \boldsymbol{m}_n :& \mathfrak{L}[[\hbar]]^{\otimes n} \rightarrow \mathfrak{L}[[\hbar]], \qquad \text{for } n \geq 1 \end{aligned} \tag{2.29}$$

We may call the sequence $(\boldsymbol{m}_0, \boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3, \cdots)$ a structure of *quantum weakly homotopy Lie* $(-1)$*-algebroid* on $\boldsymbol{\mathcal{L}}$, which classical limit $\hbar = 0$ is the structure $(\boldsymbol{s}, \boldsymbol{q}, \boldsymbol{m}_2^{(0)}, \boldsymbol{m}_3^{(0)}, \cdots)$ of weakly homotopy Lie $(-1)$-algebroid on $\boldsymbol{\mathcal{L}}$. We may say that construction of quantum weakly homotopy Lie $(-1)$-algebroid is equivalent to defining a quantum field theory. Such a strategy has been first adopted by Zwiebach in his construction of string field theory [46]. Note that, for Zwiebach, non-vanishing $\boldsymbol{s}$ corresponds to non-conformal background.

## 2.3 Symmetry and Anomaly: Resolutions and Obstructions

### 2.3.1 Gauge Symmetry and Resolutions

Now we bring out an important issue, that we ignored so far, which is actually related with the historical introduction of FP ghosts and the notion of consistent anomaly.

Let's return to a solution $\boldsymbol{S}^{(0)} \in \mathfrak{T}_0$ to the classical master equation. We know that another solution $\boldsymbol{S}'^{(0)} \in \mathfrak{T}_0$ the classical master equation gives rise to the equivalent physical theory if $\boldsymbol{S}'^{(0)}$ is related with $\boldsymbol{S}^{(0)} \in \mathfrak{T}_0$ by the ghost number and the parity preserving canonical transformation. Such a canonical transformation would be generated by an odd element $\boldsymbol{\Psi} \in \mathfrak{T}_{-1}$. We also note that, since the BV bracket has $U = 1$,

$$(\star, \star) : \mathfrak{T}_{-1} \otimes \mathfrak{T}_{-1} \longrightarrow \mathfrak{T}_{-1}, \tag{2.30}$$

which means, together with the 1st and the 3rd relations in (2.9), that the BV bracket endows a structure of Lie algebra on $\mathfrak{T}_{-1}$. Related to the above the following, ghost number preserving, adjoint action by $\boldsymbol{\Psi} \in \mathfrak{T}_{-1}$

$$e^{ad_{\boldsymbol{\Psi}}} \circ (\star) = \star + (\boldsymbol{\Psi}, \star) + \frac{1}{2!} (\boldsymbol{\Psi}, (\boldsymbol{\Psi}, \star)) + \ldots \tag{2.31}$$

is equivalent to the canonical transformation. Now we may define moduli space $\mathfrak{N}$ of classical BV action functional by

$$\mathfrak{N} = \left\{ \boldsymbol{S}^{(0)} \in \mathfrak{T}_0 | \left( \boldsymbol{S}^{(0)}, \boldsymbol{S}^{(0)} \right) = 0 / \sim \right\} \tag{2.32}$$

where the equivalence $\sim$ is defined by the adjoint action by $\boldsymbol{\Psi} \in \mathfrak{T}_{-1}$. Then a given QFT, modulo equivalence, with classical BV action functional $\boldsymbol{S}_o^{(0)}$, corresponds to a point $o \in \mathfrak{N}$. Note that a tangent vector to $o \in \mathfrak{N}$ is Ker $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$ modulo the infinitesimal version of the adjoint action (2.31). It follows that the ghost number $U = 0$ part of $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$-cohomology corresponds to tangent space $T_o\mathfrak{N}$ to the point $o \in \mathfrak{N}$, provided that $\mathfrak{N}$ is smooth (around $o$).

We observe that the adjoint action $ad_\Psi(\boldsymbol{S}_o^{(0)})$ fixes $\boldsymbol{S}_o^{(0)}$ if $\Psi \in$ Ker $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \cap \mathfrak{T}_{-1}$, i.e,

$$\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \Psi \equiv \left( \boldsymbol{S}_o^{(0)}, \Psi \right) = 0. \tag{2.33}$$

In other words the canonical transformation generated by an element of $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$-cohomology $H^{-1}_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ in $\mathfrak{T}_{-1}$ is gauge symmetry of the classical BV action functional $\boldsymbol{S}_o^{(0)}$.[3] Also the nontrivial $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$-cohomology classes in $\mathfrak{T}_{-1}$ are related with sigularities the moduli space $\mathfrak{N}$ (around $o$), since the the correponding adjoint action (2.31) fix $\boldsymbol{S}_o^{(0)}$.

In our presentation so far we assumed implicitly that $H^{-1}_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ is trivial. It is, however, certainly true that $H^{-1}_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ may be non-trivial in general. It is, thus, more precise to say that we assumed suitable trivialization or *resolution*, which notion shall be discussed briefly.

Let $\{\boldsymbol{\Upsilon}_a^{(0)}\}$ be a basis of $H^{-1}_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$. Then we introduce dual basis $\{C^a\}$ with ghost number $U = 1$ and regard as a set of new `fields` called *ghosts*. We also introduce a set $\{C_a^\bullet\}$ of `anti-fields` with ghost number $U = -2$ for the ghosts $\{C^a\}$. Then extend $\boldsymbol{\mathcal{T}}$ to $\widetilde{\boldsymbol{\mathcal{T}}} = \boldsymbol{\mathcal{T}} \times T^*[-1] \left( H^{-1}_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}[1] \right)$ and the BV bracket $(\star, \star)$ to $\widetilde{(\star, \star)}$. We note that $\{\boldsymbol{\Upsilon}_a^{(0)}\}$ are, in general, functional of both the original `fields` and `anti-fields` $(\phi^A, \phi_A^\bullet)$. Since the BV bracket endows a structure of Lie algebra on $\mathfrak{T}_{-1}$ and since $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$ is a derivation of

[3]Note that the obvious invariance generated by $\boldsymbol{\Psi} \in \mathrm{Im}\, \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \cap \mathfrak{T}_{-1}$ is so called *fake symmetry.*

the bracket, we have a structure of Lie algebra on $H^{-1}_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$ such that

$$\left(\boldsymbol{\Upsilon}^{(0)}_a, \boldsymbol{\Upsilon}^{(0)}_b\right) = f^c_{ab}\boldsymbol{\Upsilon}^{(0)}_c \text{ mod } \mathrm{Im}\,\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}, \tag{2.34}$$

with certain structure "constants" $f^c_{ab} = -f^c_{ba}$.[4] Then we may extend $\boldsymbol{S}^{(0)}_o$ to $\widetilde{\boldsymbol{S}}^{(0)}_o$ as follows

$$\widetilde{\boldsymbol{S}}^{(0)}_o = \boldsymbol{S}^{(0)}_o + C^a_a\boldsymbol{\Upsilon}^{(0)} + \frac{1}{2}C^aC^bf^c_{ab}C^{\bullet}_c, \tag{2.35}$$

which satisfies $\left(\widetilde{\boldsymbol{S}}^{(0)}_o, \widetilde{\boldsymbol{S}}^{(0)}_o\right) = 0$ from the relation (2.34) and its Jacobi identity as the BV bracket is Lie bracket on $\mathfrak{T}_{-1}$. Now we have odd nilpotent Hamiltonian vector field $\boldsymbol{Q}_{\widetilde{\boldsymbol{S}}^{(0)}_o} = \left(\widetilde{\boldsymbol{S}}^{(0)}_o, \ldots\right)$ on $\widetilde{\boldsymbol{\mathcal{T}}}$. Then we need to check if $H^{-1}_{\widetilde{\boldsymbol{Q}}_{\widetilde{\boldsymbol{S}}^{(0)}_o}}$ is trivial. Otherwise we need to repeat the above procedure until we get trivial $U = -1$ cohomology.

We may call the above procedure small resolution $(\widetilde{\boldsymbol{\mathcal{T}}}, \widetilde{\boldsymbol{S}}^{(0)}_o)$ of the pairs $(\boldsymbol{\mathcal{T}}, \boldsymbol{S}^{(0)}_o)$. Now we turn to the another source of obstruction in $\mathfrak{N}$.

### 2.3.2 Constent Anomaly and Obstruction

For a given solution $\boldsymbol{S}^{(0)}_o$ we consider nearby solution $\boldsymbol{S}^{(0)}_o + \delta\boldsymbol{S}^{(0)}$, where $\delta\boldsymbol{S}^{(0)} \in H^0_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$. It is obvious not every $\delta\boldsymbol{S}^{(0)}$ can be the actual tangent vector of $\mathfrak{N}$ at $o$ since $\left(\delta\boldsymbol{S}^{(0)}, \delta\boldsymbol{S}^{(0)}\right) \neq 0$, in general. Let $\{\boldsymbol{O}^{(0)}_a\}$ be a basis of $H^0_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$ and set $\delta\boldsymbol{S}^{(0)} = t^a\boldsymbol{O}^{(0)}_a$, where $\{t^a\}$ be the dual basis with $\{|t^a|\} = \{0\}$. We note that we may have

$$\left(\boldsymbol{O}^{(0)}_a, \boldsymbol{O}^{(0)}_b\right) = c^c_{ab}\boldsymbol{O}^{(0)}_{1c} + \boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}\boldsymbol{O}^{(0)}_{ab} \in \mathrm{Ker}\,\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o} \cap \mathfrak{T}_1 \tag{2.36}$$

where $\boldsymbol{O}^{(0)}_{1c} \in H^1_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$. It is obvious that there are no second order (in $\{t^a\}$) corrections to $\boldsymbol{S}^{(0)}_o + t^a\boldsymbol{O}^{(0)}_a$ to satisfy the classical master equation modulo

[4] Each component of $\{f^c_{ab}\}$ can be even function, in general, with ghost number $U = 0$. The Jacobi identity of BV bracket suggest that one may regard the pairs $(\{f^c_{ab}\}, \{(\boldsymbol{\Upsilon}^{(0)}_a\})$ as structure of Lie algebroid over graded space. Remark that we are talking about minimal model here such that we also need to introduce *anti-ghost multiplets* for each ghost $C^a$ and their `anti-fields`.

the third order terms in $\{t^a\}$ unless $\left(\boldsymbol{O}_a^{(0)}, \boldsymbol{O}_b^{(0)}\right)$ are $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$-exact. Thus $H^1_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}} \cap \left(H^0_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}, H^0_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}\right)$ is obstruction of $\mathfrak{N}$.

Let's assume, temporarily, that $H^1_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ is trivial. Then $\boldsymbol{S}_o^{(0)} + t^a\boldsymbol{O}_a^{(0)} + \frac{1}{2}t^a t^b \boldsymbol{O}_{ab}^{(0)}$ solves the classical master equation modulo 3rd order terms in $\{t^a\}$;

$$t^a t^b t^c \left(\boldsymbol{O}_a^{(0)}, \boldsymbol{O}_{bc}^{(0)}\right). \tag{2.37}$$

To find the next order the above term should be $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$-exact. It is suffice to show that the above is $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$-closed as $H^1_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ is trivial. Note that

$$\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}\left(\boldsymbol{O}_a^{(0)}, \boldsymbol{O}_{bc}^{(0)}\right) = \left(\boldsymbol{O}_a^{(0)}, \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}\boldsymbol{O}_{bc}^{(0)}\right) = \left(\boldsymbol{O}_a^{(0)}, \left(\boldsymbol{O}_b^{(0)}, \boldsymbol{O}_c^{(0)}\right)\right) \tag{2.38}$$

It follows that

$$t^a t^b t^c \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}\left(\boldsymbol{O}_a^{(0)}, \boldsymbol{O}_{bc}^{(0)}\right) = t^a t^b t^c \left(\boldsymbol{O}_a^{(0)}, \left(\boldsymbol{O}_b^{(0)}, \boldsymbol{O}_c^{(0)}\right)\right) = 0, \tag{2.39}$$

where we used, in the last equality, the Jacobi-identity of BV bracket on $\mathfrak{T}_0 \otimes \mathfrak{T}_0 \otimes \mathfrak{T}_0$. Now by repeating the similar procedure iteratively one may establish the existence of solution of classical master equation in the form

$$\boldsymbol{S}^{(0)}(t) = \boldsymbol{S}_o^{(0)} + t^a\boldsymbol{O}_a^{(0)} + \sum_{n=2}^{\infty}\frac{1}{n!}t^{a_1}\ldots t^{a_n}\boldsymbol{O}^{(0)}_{a_1\ldots a_n}. \tag{2.40}$$

We recall that $H^1_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ is known to be also related with 1-loop anomaly $\mathfrak{A}_o^{(1)}$, defined in the BV language by the formula;

$$\mathfrak{A}_o^{(1)} = \boldsymbol{\Delta}\boldsymbol{S}_o^{(0)} - \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}\boldsymbol{S}_o^{(1)}, \tag{2.41}$$

namely the 1st obstruction for the classical BV action functional $\boldsymbol{S}_o^{(0)}$ to be extended to quantum BV action functional (see (2.16)). We note that $\left(\boldsymbol{S}_o^{(0)}, \boldsymbol{\Delta}\boldsymbol{S}_o^{(0)}\right) \equiv \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}\boldsymbol{\Delta}\boldsymbol{S}_o^{(0)} = 0$, since $\boldsymbol{\Delta}$ is a derivation of BV bracket and $\left(\boldsymbol{S}_o^{(0)}, \boldsymbol{S}_o^{(0)}\right) = 0$. It follows, from (2.41), that

$$\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}\mathfrak{A}_o^{(1)} = 0, \tag{2.42}$$

which is equivalent to so called Wess-Zumino consistency condition for anomaly [45]. It follows that, if $H^1_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ is trivial, the 1-loop anomaly vanishes -

$\mathfrak{A}_o^{(1)}$ is a $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}$-exact expression, if non-vanishing, and we may simply modify $\boldsymbol{S}_o^{(1)}$ in (2.41). It is shown by Troost et. al. [40] that under the same condition the all loop anomaly $\mathfrak{A}_o = \sum_{\ell=1}^{\infty}\hbar^\ell \mathfrak{A}_o^{(\ell)}$ also vanish, or, equivalently, for the given solution $\boldsymbol{S}_o^{(0)}$ of classical master equation there exist solution

$$\boldsymbol{S}_o = \boldsymbol{S}_o^{(0)} + \sum_{\ell=1}^{\infty}\hbar^\ell \boldsymbol{S}_o^{(\ell)}, \tag{2.43}$$

of quantum master equation if $H^1_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ is trivial.

### 2.3.3 From classical observables to quantum observables

Here we take a brief look at the condition for $\boldsymbol{K}_{\boldsymbol{S}}$-cohomology classes (that of the BV complex (2.14)). Consider

$$\boldsymbol{Q}_{\boldsymbol{S}} = (\boldsymbol{S}, \ldots) = \sum_{n=0}^{\infty}\hbar^n \boldsymbol{Q}_{\boldsymbol{S}^{(n)}} = \sum_{n=0}^{\infty}\hbar^n \left(\boldsymbol{S}^{(n)}, \ldots\right), \tag{2.44}$$

so that the condition $\boldsymbol{K}_{\boldsymbol{S}}\boldsymbol{O}$ has the following decompositions

$$\boldsymbol{K}_{\boldsymbol{S}}\boldsymbol{O} = 0 \quad \Longrightarrow \begin{cases} \boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\boldsymbol{O}^{(0)} = 0, \\ -\boldsymbol{\Delta}\boldsymbol{O}^{(0)} + \boldsymbol{Q}_{\boldsymbol{S}^{(1)}}\boldsymbol{O}^{(0)} + \boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\boldsymbol{O}^{(1)} = 0, \\ \vdots \end{cases} \tag{2.45}$$

We note that $\boldsymbol{Q}^2_{\boldsymbol{S}^{(0)}} = 0$ due to the tree level master equation in (2.16). Thus, to find a solution $\boldsymbol{K}_{\boldsymbol{S}}\boldsymbol{O} = 0$, we may start from $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\boldsymbol{O}^{(0)} = 0$. Note that $\boldsymbol{O}'$ would be $\boldsymbol{K}_{\boldsymbol{S}}$-exact if there exists $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{(0)} + \sum_{n=1}^{\infty}\hbar^n \boldsymbol{\Lambda}^{(n)}$ such that

$$\boldsymbol{O}' = \boldsymbol{K}_{\boldsymbol{S}}\boldsymbol{\Lambda} \quad \Longrightarrow \begin{cases} \boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\boldsymbol{\Lambda}^{(0)} = \boldsymbol{O}'^{(0)}, \\ -\boldsymbol{\Delta}\boldsymbol{\Lambda}^{(0)} + \boldsymbol{Q}_{\boldsymbol{S}^{(1)}}\boldsymbol{\Lambda}^{(0)} + \boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\boldsymbol{\Lambda}^{(1)} = \boldsymbol{O}'^{(1)}, \\ \vdots \end{cases} \tag{2.46}$$

Thus to find $\boldsymbol{K}_{\boldsymbol{S}}$ cohomology classes we may start from cohomology of the following complex

$$\left(\boldsymbol{Q}_{\boldsymbol{S}^{(0)}}, \mathfrak{T} = \bigoplus_k \mathfrak{T}_k\right). \tag{2.47}$$

But we see that building a representative $\boldsymbol{K_S}$ cohomology class from a $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}}$ cohomology class is an elaborated as well as complicated procedure. We may ask why QFT instruct us to do such procedure, which an answer would be given, implicitly, in a later part of this notes.

We shall call an element of cohomology of the complex (2.47) a classical observable, while an element of cohomology of the complex (2.14) quantum observables. We may also call a solution $\boldsymbol{S}^{(0)}$ of the classical master equation (2.22) classical (BV) action functional while a solution $\boldsymbol{S}$ of the master equation (2.5) quantum (BV) action functional. We may say that quantizablity of a classical theory means that (assuming the existence of $\boldsymbol{\Delta}$), the existence of quantum (BV) action functional as well as a *quasi-isomorphism* between the quantum BV complex (2.14) to the classical complex (2.47).

Now we shall clarify the last statement. Consider a classical observable $\boldsymbol{O}^{(0)}$. The 1st obstruction $\mathfrak{B}^{(1)}$ to have the corresponding quantum observable is, from (2.45);

$$\mathfrak{B}^{(1)} = -\boldsymbol{\Delta}\boldsymbol{O}^{(0)} + \boldsymbol{Q}_{\boldsymbol{S}^{(1)}}\boldsymbol{O}^{(0)} + \boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\boldsymbol{O}^{(1)}. \tag{2.48}$$

We can show that $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\mathfrak{B}^{(1)} = 0$ as follows; to begin with we have $\left(\boldsymbol{S}^{(0)}, \boldsymbol{O}^{(0)}\right) = 0$ implying

$$\left(\boldsymbol{\Delta}\boldsymbol{S}^{(0)}, \boldsymbol{O}^{(0)}\right) + \left(\boldsymbol{S}^{(0)}, \boldsymbol{\Delta}\boldsymbol{O}^{(0)}\right) = 0, \tag{2.49}$$

as $\boldsymbol{\Delta}$ is a derivation of the BV bracket. From the quantum master equation $\boldsymbol{\Delta}\boldsymbol{S}^{(0)} = \left(\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)}\right)$, we have

$$\left(\left(\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)}\right), \boldsymbol{O}^{(0)}\right) + \left(\boldsymbol{S}^{(0)}, \boldsymbol{\Delta}\boldsymbol{O}^{(0)}\right) = 0. \tag{2.50}$$

Now we note that

$$\boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\mathfrak{B}^{(1)} \equiv \left(\boldsymbol{S}^{(0)}, \mathfrak{B}^{(1)}\right) = -\left(\boldsymbol{S}^{(0)}, \boldsymbol{\Delta}\boldsymbol{O}^{(0)}\right) + \left(\boldsymbol{S}^{(0)}, \left(\boldsymbol{S}^{(1)}, \boldsymbol{O}^{(0)}\right)\right) \tag{2.51}$$

From $\left(\boldsymbol{S}^{(0)}, \boldsymbol{O}^{(0)}\right) = 0$ and the Jacobi-identity of BV barcket, we have

$$\left(\boldsymbol{S}^{(0)}, \left(\boldsymbol{S}^{(1)}, \boldsymbol{O}^{(0)}\right)\right) = -\left(\left(\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)}\right), \boldsymbol{O}^{(0)}\right). \tag{2.52}$$

Thus we have $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}}\mathfrak{B}^{(1)} = 0$. Then the situation is exactly like the 1-loop anomaly we discussed before, and the absense of 1-loop anomaly implies that $\mathfrak{B}^{(1)} = 0$. The similar argument can be extended to all the higher order terms in $\hbar$. Consequently the existence of quantum BV action functional also implies existence of quasi-isomorphism between the classical and quantum BV complexes as well.

### 2.4 Family of QFTs

Now we are going to show that the deformed solution $\boldsymbol{S}^{(0)}(t)$, defined by (2.40), of the classical master equation can be also quantum corrected as follows

$$\begin{aligned}\boldsymbol{S}(t) =&\boldsymbol{S}^{(0)}(t) + \sum_{\ell=1}^{\infty} \hbar^{\ell} \boldsymbol{S}^{(\ell)}(t)\\ =&\boldsymbol{S}_o^{(0)} + t^a \boldsymbol{O}_a^{(0)} + \sum_{n=2}^{\infty} \frac{1}{n!} t^{a_1} \dots t^{a_n} \boldsymbol{O}^{(0)}_{a_1 \dots a_n}\\ &+ \sum_{\ell=1}^{\infty} \hbar^{\ell} \left( \boldsymbol{S}_o^{(\ell)} + t^a \boldsymbol{O}_a^{(\ell)} + \sum_{n=2}^{\infty} \frac{1}{n!} t^{a_1} \dots t^{a_n} \boldsymbol{O}^{(\ell)}_{a_1 \dots a_n} \right)\end{aligned} \tag{2.53}$$

to satisfy quantum master equation if $H^1_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}}$ is trivial.

The 1-loop anomaly for the deformed theory is

$$\mathfrak{A}^{(1)}(t) = \boldsymbol{\Delta} \boldsymbol{S}^{(0)}(t) - \boldsymbol{Q}_{\boldsymbol{S}^{(0)}(t)} \boldsymbol{S}^{(1)}(t), \tag{2.54}$$

where $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}(t)} = \left( \boldsymbol{S}^{(0)}(t), \dots \right)$ and $\mathfrak{A}^{(1)}(t) = \mathfrak{A}_o^{(1)} + \sum_{n=1}^{\infty} \frac{1}{n!} t^{a_1} \dots t^{a_n} \mathfrak{A}^{(1)}_{a_1 \dots a_n}$. We have

$$\begin{aligned}\mathfrak{A}_o^{(1)} &= \boldsymbol{\Delta} \boldsymbol{S}_o^{(0)} - \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \boldsymbol{S}_o^{(1)},\\ t^a \mathfrak{A}_a^{(1)} &= t^a \boldsymbol{\Delta} \boldsymbol{O}_a^{(0)} - t^a \left( \boldsymbol{O}_a^{(0)}, \boldsymbol{S}_o^{(1)} \right) - t^a \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \boldsymbol{O}_a^{(1)},\\ t^a t^b \mathfrak{A}_{ab}^{(1)} &= t^a t^b \boldsymbol{\Delta} \boldsymbol{O}_{ab}^{(0)} - t^a t^b \left( \boldsymbol{O}_a^{(0)}, \boldsymbol{O}_b^{(1)} \right) - t^a t^b \left( \boldsymbol{O}_{ab}^{(0)}, \boldsymbol{S}_o^{(1)} \right) - t^a t^b \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \boldsymbol{O}_{ab}^{(1)},\end{aligned} \tag{2.55}$$

etc. Based on the relation $\left( \boldsymbol{S}^{(0)}(t), \boldsymbol{S}^{(0)}(t) \right) = 0$ and $\left( \boldsymbol{S}^{(0)}(t), \boldsymbol{\Delta} \boldsymbol{S}^{(0)}(t) \right) = 0$, it can be shown that

$$\begin{aligned}\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \mathfrak{A}_o^{(1)} &= 0,\\ \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \mathfrak{A}_a^{(1)} &= (\boldsymbol{O}_a^{(0)}, \mathfrak{A}_o),\\ \boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \mathfrak{A}_{ab}^{(1)} &= \left( \boldsymbol{O}_{ab}^{(0)}, \mathfrak{A}_o^{(1)} \right) + \left( \boldsymbol{O}_a^{(0)}, \mathfrak{A}_b^{(1)} \right) + \left( \mathfrak{A}_a^{(1)}, \boldsymbol{O}_b^{(0)} \right),\end{aligned} \tag{2.56}$$

etc. Since $H^1_{\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}}} = 0$ leading to $\mathfrak{A}_o^{(1)} = 0$, we have $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \mathfrak{A}_a^{(1)} = 0$ leading to $\mathfrak{A}_a^{(1)} = 0$. Using $\mathfrak{A}_o^{(1)} = \mathfrak{A}_a^{(1)} = 0$ we have $\boldsymbol{Q}_{\boldsymbol{S}_o^{(0)}} \mathfrak{A}_{ab}^{(1)} = 0$ leading to $\mathfrak{A}_{ab}^{(1)} = 0$. Using induction it can be shown that $\mathfrak{A}^{(1)}(t) = 0$. Adopting the similar

procedure it can be shown that the all-loop anomaly $\mathfrak{A}(t) = \sum_{\ell=1}^{\infty} \mathfrak{A}^{(\ell)}(t)$ also vanishes.

In summary we see that two possible obstruction of the moduli space $\mathfrak{N}$ in the neighborhood of $o \in \mathfrak{N}$ are $H^{-1}_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$ and $H^{1}_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$, which are related with symmetry and anomaly, respectively. Assuming the small resolution the moduli space $\mathfrak{N}$ in the neighborhood of $o \in \mathfrak{N}$ is unobstructed and the tangent space $T_o\mathfrak{N}$ is isomorphic to $H^{0}_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$ provided that there is no anomaly. In other words the absence of anomaly in QFT implies that a QFT come with family, which property shall be crucial in the next section.

### 2.4.1 Generalization

Now we consider an natural generalization the above picture. We may decompose the space $\mathfrak{T} = \mathfrak{T}_{even} \oplus \mathfrak{T}_{odd}$ into the $U = even$ and $U = odd$ subspaces. We have

$$(\star, \star) : \mathfrak{T}_{odd} \otimes \mathfrak{T}_{odd} \longrightarrow \mathfrak{T}_{odd}, \tag{2.57}$$

which means that the BV bracket endows a structure of graded Lie algebra on $\mathfrak{T}_{odd}$. Then we have the parity preserving adjoint action generated by elements of $\mathfrak{T}_{odd}$. Now we allow classical BV action functional to be an element of $\mathfrak{T}_{even}$ in general and define extended moduli space $\mathfrak{M}$ by

$$\mathfrak{M} = \left\{ \boldsymbol{S}^{(0)} \in \mathfrak{T}_{even} \middle| \left( \boldsymbol{S}^{(0)}, \boldsymbol{S}^{(0)} \right) = 0 / \sim \right\} \tag{2.58}$$

where the equivalence $\sim$ is defined by the adjoint action by elements in $\mathfrak{T}_{odd}$.

We choose a base point $o \in \mathfrak{N} \subset \mathfrak{M}$ and the corresponding classical BV action functional $\boldsymbol{S}^{(0)}_o \in \mathfrak{T}_0 \subset \mathfrak{T}_{even}$. Let $\{\boldsymbol{\Upsilon}^{(0)}_m\}$ be a basis of $H^{odd}_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$. Then we have the structure of Lie algebra on $H^{odd}_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$ such that

$$\left( \boldsymbol{\Upsilon}^{(0)}_m, \boldsymbol{\Upsilon}^{(0)}_n \right) = f_{mn}^{\ r} \boldsymbol{\Upsilon}^{(0)}_r \ \mathrm{mod}\ \mathrm{Im}\, \boldsymbol{Q}_{\boldsymbol{S}^{(0)}}, \tag{2.59}$$

with certain structure "constants" $f_{mn}^{\ r} = -f_{nm}^{\ r}$..[5] We extend $\boldsymbol{\mathcal{T}}$ to

$$\widetilde{\boldsymbol{\mathcal{T}}} = \boldsymbol{\mathcal{T}} \times T^*[-1] \left( \prod_{k \in \mathbb{Z}} H^{-(2k+1)}_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}[2k+1] \right) \tag{2.60}$$

[5] Each component of $\{f_{mn}^{\ r}\}$ can be even function, in general, with ghost number $U = even$. The Jacobi identity of BV bracket suggest that one may regard the pairs $(\{f_{mn}^{\ r}\}, \{(\boldsymbol{\Upsilon}^{(0)}_m\})$ as structure of graded Lie algebroid over graded space.

by introducing generalized ghosts `fields` $\{C^m\}$ such that

$$U(C^m) + U(\boldsymbol{\Upsilon}^{(0)}_m) = 0, \qquad |C^m| + |\boldsymbol{\Upsilon}^{(0)}_m| = 0, \tag{2.61}$$

and their `antifields` $\{C^\bullet_m\}$. Then we extend $\boldsymbol{S}^{(0)}_o$ to $\widetilde{\boldsymbol{S}}^{(0)}_o$;

$$\widetilde{\boldsymbol{S}}^{(0)}_o = \boldsymbol{S}^{(0)}_o + C^m \boldsymbol{\Upsilon}^{(0)}_m + \frac{1}{2} C^m C^n f^r_{mn} C^\bullet_r \tag{2.62}$$

which satisfies the classical master equation. Then we need to check if $H^{odd}_{\widetilde{\boldsymbol{Q}}_{\widetilde{\boldsymbol{S}}^{(0)}_o}}$ is trivial. Otherwise we need to repeat the above procedure until we get trivial cohomology for odd elements.

We may call the above procedure an extended resolution. From now on we assume that $(\boldsymbol{\mathcal{T}}, \boldsymbol{S}^{(0)})$ had been already resolved in the extended sense. Let $\{\boldsymbol{O}^{(0)}_\alpha\}$ be a basis of $\oplus H^\bullet_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}} = H^{even}_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$ and $\{t^a\}$ be the dual basis with $U(t^\alpha) + U((O)^{(0)}_\alpha) = 0$ and $|t^\alpha| + |\boldsymbol{O}^{(0)}_\alpha| = 0$. Then the tangent space $T_o\mathfrak{M}$ is isomorphic to $\oplus_\bullet H^\bullet_{\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_o}}$ and there exist another solution of classical master equation given by

$$\mathcal{S}^{(0)} = \boldsymbol{S}^{(0)}_o + t^\alpha \boldsymbol{O}^{(0)}_\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} t^{\alpha_1} \ldots t^{\alpha_n} \boldsymbol{O}^{(0)}_{\alpha_1 \ldots \alpha_n}, \tag{2.63}$$

which can be found by following the similar procedure described before. Applying the same reasoning as before we conclude that there exist solution $\mathcal{S} = \mathcal{S}^{(0)} + \hbar \mathcal{S}^{(1)} + \mathcal{O}(\hbar^2)$ of quantum master equation. Thus the extended moduli space $\mathfrak{M}$ is smooth in the neighborhood of $o$.

### 2.5 The BV Quantization II

End of digression and we turn to the partition function of the theory is defined by

$$\boldsymbol{Z}(\hbar)_{[\boldsymbol{\mathcal{L}}]} = \int_{\boldsymbol{\mathcal{L}}} d\mu \, e^{-\boldsymbol{S}/\hbar}, \tag{2.64}$$

where $\boldsymbol{\mathcal{L}}$ is a Lagrangian subspace and $d\mu$ is its "measure". The master equation is the condition that the $Z(\hbar)$ depends only on the homology class $[\boldsymbol{\mathcal{L}}]$ of $\boldsymbol{\mathcal{L}}$.[6] The master equation for $\boldsymbol{S}$ also implies that the partition function

[6] On might argue that the "measure" $d\mu$ is ambiguous and ill defined. We should, however, note that $\boldsymbol{\mathcal{L}}$, the space of all `fields` that we start with, is typically affine (linear) graded space, which has "standard measure", $\prod d\phi^A$, though infinite product. Furthermore the actual measure is $d\mu e^{-\boldsymbol{S}/\hbar}\Big|_{\boldsymbol{\mathcal{L}}}$, which may be viewed as a top form on $\boldsymbol{\mathcal{L}}$. In other words the BV quantization scheme want to achieve good notion of cohomology class by carefully defining $\boldsymbol{S}$.

is independent to *canonical transformation* connected to identity generated by so called *gauge fermion*;

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(0)} + \sum_{\ell=1}^{\infty} \hbar^{\ell} \boldsymbol{\Gamma}^{(\ell)} \in \mathfrak{L}[[\hbar]]_{-1}, \tag{2.65}$$

which is an odd function, with ghost number $U = -1$, on $\boldsymbol{\mathcal{L}}$ in formal power series in $\hbar$. Then we may choose suitable representative in the middle dimensional cohomology class $\left[d\mu\, e^{-\boldsymbol{S}/\hbar}\right]$ in $\boldsymbol{\mathcal{T}}$ such that the path integral has good behavior. Equivalently one can choose suitable representative in the middle dimension homology class $[\boldsymbol{\mathcal{L}}]$. The above procedure is called gauge fixing. We note that the two inequivalent QFTs may share the same or equivalent BV action functional and $\boldsymbol{\mathcal{T}}$, while correspond to choosing two inequivalent Lagrangian subspaces. We note that the problem we are dealing with has close analogue in defining pairings between homology and cohomology classes in the finite dimensional manifold.

Thus, in practice, we may choose a simple Lagrangian subspace $\boldsymbol{\mathcal{L}}$, like the space of all initial `fields`, and a good gauge fermion $\boldsymbol{\Gamma} \in \mathfrak{L}[[\hbar]]_{-1}$. After the canonical transformation we have so called gauge fixed action functional

$$\begin{aligned} S_{[\boldsymbol{\mathcal{L}}]} =& \boldsymbol{m}_0 + \boldsymbol{m}_1(\boldsymbol{\Gamma}) + \sum_{n=2}^{\infty} \frac{1}{n!} \boldsymbol{m}_n \left(\boldsymbol{\Gamma}, \cdots, \boldsymbol{\Gamma}\right) \\ =& \boldsymbol{s} + \boldsymbol{q}\boldsymbol{\Gamma}^{(0)} + \sum_{n=3}^{\infty} \frac{1}{n!} \boldsymbol{m}_n^{(0)} \left(\boldsymbol{\Gamma}^{(0)}, \cdots, \boldsymbol{\Gamma}^{(0)}\right) \\ & + \mathcal{O}(\hbar), \end{aligned} \tag{2.66}$$

where $S_{[\boldsymbol{\mathcal{L}}]} := \boldsymbol{S}^{\boldsymbol{\Gamma}}|_{\boldsymbol{\mathcal{L}}} \in \mathfrak{L}[[\hbar]]_0$, so that $Z(\hbar)_{[\boldsymbol{\mathcal{L}}]} = \int_{\boldsymbol{\mathcal{L}}} d\mu\, e^{-S_{[\boldsymbol{\mathcal{L}}]}/\hbar}$.

Remark that in the semi-classical limit $\hbar \to 0$ and the case that $\boldsymbol{m}_n^{(0)} = 0$ for $\forall n \geq 2$ the gauge fixed action functional becomes $\boldsymbol{s} + \boldsymbol{q}\boldsymbol{\Gamma}^{(0)}$ exactly as in the familiar BRST quantization. In such a case we have $\boldsymbol{q}\boldsymbol{s} = \boldsymbol{q}^2 = 0$ by construction. The above also show that the path integral is independent to gauge choice in the BRST quantization, which proof seems to be the original motivation of Batalin-Vilkovisky.

We also note that it may also possible that the gauge fixed action functional (2.66) can be zero;

$$\boldsymbol{m}_0 + \boldsymbol{m}_1(\boldsymbol{\Gamma}) + \sum_{n=2}^{\infty} \frac{1}{n!} \boldsymbol{m}_n \left(\boldsymbol{\Gamma}, \cdots, \boldsymbol{\Gamma}\right) = 0, \tag{2.67}$$

i.e., the Maurer-Cartan equation of quantum weakly homotopy Lie $(-1)$-algebroid. The above condition implies that the operator $(\boldsymbol{m}_1)_{\boldsymbol{\Gamma}}$ defined by

$$(\boldsymbol{m}_1)_{\boldsymbol{\Gamma}} := \boldsymbol{m}_1 + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \boldsymbol{m}_n (\boldsymbol{\Gamma}, \cdots, \boldsymbol{\Gamma}, \ \ ) \tag{2.68}$$

satisfies that

$$(\boldsymbol{m}_1)_{\boldsymbol{\Gamma}}^2 = 0. \tag{2.69}$$

Then $(\boldsymbol{m}_1)_{\boldsymbol{\Gamma}}$ may be used to define physical states by its cohomology.[7]

The similar argument can be applied to expectation value of observable. An observable is $\boldsymbol{K_S} = -\hbar\boldsymbol{\Delta} + \boldsymbol{Q_S}$ closed function $\boldsymbol{O} = \boldsymbol{O}^{(0)} + \sum_{n=1}^{\infty} \hbar^n \boldsymbol{O}^{(n)} \in \mathfrak{T}[[\hbar]]$ on $\boldsymbol{\mathcal{T}}$ in the formal power series of $\hbar$;

$$\boldsymbol{K_S O} = 0. \tag{2.70}$$

The *expectation value* of such an $\boldsymbol{O}$ is denoted by

$$\langle \boldsymbol{O} \rangle_{[\boldsymbol{\mathcal{L}}]} = \int_{\boldsymbol{\mathcal{L}}} d\mu \, \boldsymbol{O} e^{-\boldsymbol{S}/\hbar} \tag{2.71}$$

Then we have a fundamental identity

$$\langle \boldsymbol{O} + \boldsymbol{K_S \Lambda} \rangle_{[\boldsymbol{\mathcal{L}}]} = \langle \boldsymbol{O} \rangle_{[\boldsymbol{\mathcal{L}}]} \tag{2.72}$$

namely the expectation value depends only on the $\boldsymbol{K_S}$ cohomology class (and on the homology class of $[\boldsymbol{\mathcal{L}}]$ of $\boldsymbol{\mathcal{L}}$. The above identity of BV is generalization of identities in QFT like those of Slavanov-Taylor, Ward-Takasaki and Zinjustin. We may take an alternative view such that the above identity as a necessary condition to have a proper definition of path integral, namely a representative of homology class $[\boldsymbol{\mathcal{L}}]$ should be chosen to ensure $\langle \boldsymbol{K_S \Lambda} \rangle_{[\boldsymbol{\mathcal{L}}]} = 0$. It is clear that two inequivalent QFTs may have the equivalent BV model but correspond to two inequivalent Lagrangian subspaces as the space of `fields`. More precisely path integrals of a BV model define certain "periods"[8] matrix of pairing between homology and cohomology classes, while each of its matrix element has been, traditionally, referred as path integral of a certain QFT.

[7]In Zwiebach's string field theory [46] non-zero $\boldsymbol{s} = \boldsymbol{m}_0^{(0)}$ in $\boldsymbol{m}_0 = \sum_{\ell=0}^{\infty} \hbar^\ell \boldsymbol{m}_0^{(\ell)}$ corresponds to non-conformal background. Then the condition (2.67) may be interpreted as the equation for moduli space of conformal background. We also remark that an equation similar to the semi-classical version of the equation (2.67) also appears in the Floer homology of Lagrangian intersection in the work of Fukaya et. als. [14] as condition for vanishing obstruction for Floer homology, where the semi-classical version of $(\boldsymbol{m}_1)_{\boldsymbol{\Gamma}}$ is interpreted as correct Floer boundary operator. We also remark that the semi-classical in the above contexts correpond to genus zero in string theory.

[8]It may be worth to remark that path integrals of QFT with finite dimensional field

# 3 Reaching Out To One's Family

The BV quantization scheme suggests us to work out all $\boldsymbol{K_S}$ cohomology classes (quantum observables) and study their expectation values via path integral. One of the most crucial purpose of QFT is to understand correlation functions, which are, naively, the expectation values of products of observables. Here, the BV quantization scheme introduce a vexing problem that products of quantum observables are not quantum observables in general (comments below (2.14) that products of $\boldsymbol{K_S}$ cohomology classes are not, in general, Kernel of $\boldsymbol{K_S}$[9] ), meaning that the expectation value of products of quantum observables may depend on continuous variations of Lagrangian subspace (or gauge choice). We shall see the above problem is actually a clue suggesting that we need to find family QFTs. The upshot is that QFTs come with a family parametrized by certain moduli space and every path integrals of a given QFT belonging to such a family means reaching out to its neighborhood. If so QFT(s) should eventually be understood in its totality.

Let $\{\boldsymbol{O}_\alpha\}$ be a basis of all $\boldsymbol{K_S}$ cohomology classes among elements in $\mathfrak{T}[[\hbar]]$. Being an observable, $\boldsymbol{K_S}\boldsymbol{O}_\alpha = 0$, we have

$$\boldsymbol{\Delta}\left(\boldsymbol{O}_\alpha e^{-\boldsymbol{S}/\hbar}\right) = 0, \quad \text{since} \quad \boldsymbol{\Delta} e^{-\boldsymbol{S}/\hbar} = 0. \tag{3.1}$$

The above relation strongly suggests that evaluation the expectation values of observables corresponds to infinitesimal deformation of given QFT, characterized by given $\boldsymbol{S}$ and $\boldsymbol{\mathcal{T}}$. Let's consider the following would be "generating functional"

$$Z(\hbar)_{\vec{t}} = \int_{\boldsymbol{\mathcal{L}}} d\mu e^{-(\boldsymbol{S}+t^\alpha \boldsymbol{O}_\alpha)/\hbar}, \tag{3.2}$$

where $\{t^\alpha\}$ be the dual basis of $\{\boldsymbol{O}_\alpha\}$ such that $|t^\alpha| = -|\boldsymbol{O}_\alpha|$. Then the expectation value of observable is

$$\langle \boldsymbol{O}_\alpha \rangle = -\hbar \frac{\partial Z_t}{\partial t^\alpha}\bigg|_{\{\vec{t}\}=\{\vec{0}\}} \tag{3.3}$$

space (or $0+0$ dimensional QFT) is closely related to the exponential period considered briefly in the exposition of Kontsevich-Zagier [24]. The quantum flat structure we will talk about in the next section, then, may be translated into theory of (extended) variations of exponential periods.

[9]This also implies that one should work at the level of quantum BV complex rather than its cohomology, which view shall not be elaborated in this notes.

Thus the "generating functional" might contains all the information of expectation values. Now we consider the expectation value of products of two observables, which may be written as

$$\langle \boldsymbol{O}_\alpha \boldsymbol{O}_\beta \rangle = \hbar^2 \frac{\partial^2 Z_t}{\partial t^\beta \partial t^\alpha}\bigg|_{\{\vec{t}\}=\{\vec{0}\}}. \tag{3.4}$$

We recall, however, that the product of two observables is not an observable in general as $\boldsymbol{K_S}$ is not a derivation of product, which obstruction is measured by the BV bracket $(\boldsymbol{O}_\alpha, \boldsymbol{O}_\beta)$. Thus the naive definition (3.2) of generating functional is *not correct*. The problem is that the path integral (3.2) is ill defined as the $\boldsymbol{S} + t^\alpha \boldsymbol{O}_\alpha$ satisfies the master equation, in general, only up to the first order in $\{t^\alpha\}$.

Instead we seek for solution $\boldsymbol{S}(\{t^\alpha\})$ of the master equation

$$-\hbar \boldsymbol{\Delta} \boldsymbol{S}(\{t^a\}) + \frac{1}{2}\Big(\boldsymbol{S}(\{t^a\}), \boldsymbol{S}(\{t^\alpha\})\Big) = 0. \tag{3.5}$$

such that

$$\boldsymbol{S}(\{t^a\}) = \boldsymbol{S} + t^\alpha \boldsymbol{O}_\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} t^{\alpha_1} \cdots t^{\alpha_n} \boldsymbol{O}_{\alpha_1 \cdots \alpha_n}, \qquad |\boldsymbol{S}(\{t^a\})| = 0, \tag{3.6}$$

where $\boldsymbol{S}$ may be regarded the initial condition and $\{\boldsymbol{O}_\alpha\}$ as infinitesimal deformations. Then we have an well-defined generating functional

$$\boldsymbol{Z}(\{t^\alpha\}) = \int_{\boldsymbol{\mathcal{L}}} d\mu \, e^{-\boldsymbol{S}(\{t^\alpha\})/\hbar}, \tag{3.7}$$

which contains all information of QFT defined by the BV action functional $\boldsymbol{S} = \boldsymbol{S}(\vec{t} = \vec{0})$.[10]

For simplicity in notation we shall denote $\boldsymbol{\mathcal{S}} = \boldsymbol{S}(\{t^\alpha\})$ and $\boldsymbol{\mathcal{Z}} = \boldsymbol{Z}(\{t^\alpha\})$, such that

$$\boldsymbol{\mathcal{Z}} = \int_{\boldsymbol{\mathcal{L}}} d\mu \, e^{-\boldsymbol{\mathcal{S}}/\hbar}, \qquad -\hbar \boldsymbol{\Delta} \boldsymbol{\mathcal{S}} + \frac{1}{2}(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}) = 0, \qquad \boldsymbol{\mathcal{S}}|_{\vec{t}=\vec{0}} = \boldsymbol{S}. \tag{3.8}$$

[10] At this point one might ask if solutions of master equation (3.5) always exists for a given field theory such that the first order is spanned by a basis of all $\boldsymbol{K_S}$-cohomology classes. The answer would be certainly "No" in general, but the answer should be "Yes" for a QFT that, if otherwise, our starting point is wrong or we need some modifications and extensions until the answer is Yes (see Section 2.4). This is part of our demand of "renormalized" quantizablity.

Note that the master equation implies that

$$\mathcal{K}_{\mathcal{S}} := -\hbar \boldsymbol{\Delta} + \mathcal{Q}_{\mathcal{S}}, \quad \text{where } \mathcal{Q}_{\mathcal{S}} := (\mathcal{S}, \ldots) \tag{3.9}$$

satisfies

$$\mathcal{K}_{\mathcal{S}}^2 = 0. \tag{3.10}$$

Then we have the corresponding identity;

$$\int_{\boldsymbol{\mathcal{L}}} d\mu \ (\mathcal{K}_{\mathcal{S}}(\textit{anything}))\, e^{-\mathcal{S}/\hbar} = 0. \tag{3.11}$$

Now let's apply derivative $\frac{\partial}{\partial t^\alpha}$ to the master equation (the middle equation in (3.8)) to obtain

$$-\hbar \boldsymbol{\Delta} \mathcal{S}_\alpha + (\mathcal{S}, \mathcal{S}_\alpha) = 0 \tag{3.12}$$

where $\mathcal{S}_\alpha := \frac{\partial \mathcal{S}}{\partial t^\alpha}$. Thus $\{\mathcal{S}_\alpha\}$ span, at least, subspace of $\mathcal{K}_{\mathcal{S}}$ cohomology group. It is obvious that $\mathcal{K}_{\mathcal{S}}\mathcal{S}_\alpha\Big|_{\vec{t}=\vec{0}} = 0$, leading to $\boldsymbol{K_S O}_\alpha = 0$, where $\boldsymbol{O}_\alpha = \mathcal{S}_\alpha|_{\vec{t}=\vec{0}}$.

From the definition of $\mathcal{Z}$ we have

$$\hbar \mathcal{Z}_\alpha = -\int_{\boldsymbol{\mathcal{L}}} d\mu\, \mathcal{S}_\alpha e^{-\mathcal{S}/\hbar} \tag{3.13}$$

where

$$\mathcal{Z}_\alpha := \frac{\partial \mathcal{Z}}{\partial t^\alpha}. \tag{3.14}$$

From (3.13) we also have

$$\hbar^2 \frac{\partial \mathcal{Z}_\alpha}{\partial t^\beta} = \int_{\boldsymbol{\mathcal{L}}} d\mu \ \left(-\hbar \frac{\partial \mathcal{S}_\alpha}{\partial t^\beta} + \mathcal{S}_\beta \mathcal{S}_\alpha\right) e^{-\mathcal{S}/\hbar} \tag{3.15}$$

Then we check if the expression $\left(-\hbar \frac{\partial \mathcal{S}_\alpha}{\partial t^\beta} + \mathcal{S}_\beta \mathcal{S}_\alpha\right)$ is also $\mathcal{K}_{\mathcal{S}}$-closed. From (3.12), after taking derivative $\frac{\partial}{\partial t^\beta}$ we have

$$-\hbar \boldsymbol{\Delta} \frac{\partial \mathcal{S}_\alpha}{\partial t^\beta} + \left(\mathcal{S}, \frac{\partial \mathcal{S}_\alpha}{\partial t^\beta}\right) + (\mathcal{S}_\beta, \mathcal{S}_\alpha) = 0. \tag{3.16}$$

It follows, further using (2.6) and (3.12), that

$$\mathcal{K}_{\mathcal{S}} \left(-\hbar \frac{\partial \mathcal{S}_\alpha}{\partial t^\beta} + \mathcal{S}_\beta \mathcal{S}_\alpha\right) = 0. \tag{3.17}$$

### 3.1 Quantum Algebra of Observables at Base Point

We may regard the master equation (3.5) defines formal neighborhood of the initial QFT defined by $\boldsymbol{S}$ in certain moduli space $\mathfrak{M}$ of family of QFTs such that the initial QFT is a (classical) base point $o$ and the given basis $\{\boldsymbol{O}_a\}$ of $\boldsymbol{K_S}$-cohomology is a basis of tangent space $T_o\mathfrak{M}$ at $o \in \mathfrak{M}$. Then the dual basis $\{t^\alpha\}$ of $\boldsymbol{K_S}$-cohomology can be regarded as a local coordinate system around the base point $o$. Now we want to characterize expectation values of all observables and their correlation functions of the initial QFT.

The condition (3.17) implies that

$$\left(\mathcal{K}_{\mathcal{S}}\left(-\hbar\frac{\partial \mathcal{S}_\alpha}{\partial t^\beta} + \mathcal{S}_\beta\mathcal{S}_\alpha\right)\right)\Bigg|_{\vec{t}=0} = 0. \quad \Longrightarrow \boldsymbol{K_S}\left(-\hbar\boldsymbol{O}_{\alpha\beta} + \boldsymbol{O}_\beta\boldsymbol{O}_\alpha\right) = 0. \tag{3.18}$$

That is $(-\hbar\boldsymbol{O}_{\alpha\beta} + \boldsymbol{O}_\beta\boldsymbol{O}_\alpha)$ is $\boldsymbol{K_S}$-closed while $(\boldsymbol{O}_\beta\boldsymbol{O}_\alpha)$ may not. Since $\{\boldsymbol{O}_\alpha\}$ is a (complete) basis of $\boldsymbol{K_S}$-cohomology we have

$$-\hbar\boldsymbol{O}_{\alpha\beta} + \boldsymbol{O}_\beta\boldsymbol{O}_\alpha = \boldsymbol{A}_{\alpha\beta}^{\gamma}\boldsymbol{O}_\gamma + \boldsymbol{K_S}\boldsymbol{\Lambda}_{\alpha\beta} \tag{3.19}$$

for some structure constants $\{\boldsymbol{A}_{\alpha\beta}^{\gamma}\}$, in formal power series of $\hbar$, and for some $\boldsymbol{\Lambda}_{\alpha\beta} \in \mathfrak{T}[[\hbar]]$. It follows that

$$\langle \boldsymbol{O}_\beta\boldsymbol{O}_\alpha\rangle_{\vec{t}=0} = \boldsymbol{A}_{\alpha\beta}^{\gamma}\langle \boldsymbol{O}_\gamma\rangle_{\vec{t}=0} + \hbar\,\langle \boldsymbol{O}_{\alpha\beta}\rangle_{\vec{t}=0}\,. \tag{3.20}$$

We note that the structure constants $\{\boldsymbol{A}_{\alpha\beta}^{\gamma}\}$ are independent to any choice of Lagrangian subspace. The expectation value $\langle \boldsymbol{O}_\gamma\rangle_{\vec{t}=0}$ depends on homology class of Lagrangian subspace, where we integrated over. One the other hand both $\langle \boldsymbol{O}_\beta\boldsymbol{O}_\alpha\rangle_{\vec{t}=0}$ and $\hbar\,\langle \boldsymbol{O}_{\alpha\beta}\rangle_{\vec{t}=0}$ depends on variation of Lagrangian subspace even within the same homology class, which dependence cancel with each others. The formula (3.20) is very suggestive as the correlation function (LHS of (3.20)) of two observables $\boldsymbol{O}_\alpha$ and $\boldsymbol{O}_\beta$ involve expectation values of all observables and something shared only by two of them.[11]

In general, a solution (3.6) of the master equation (3.5) leads to "quantum" products of observables defined by

$$\boldsymbol{a}_n\left(\boldsymbol{O}_{\alpha_1}, \ldots, \boldsymbol{O}_{\alpha_n}\right) = e^{\mathcal{S}/\hbar}\left(\frac{\hbar^n\partial^n}{\partial t^{\alpha_n}\ldots\partial t^{\alpha_1}}\right)e^{-\mathcal{S}/\hbar}\Bigg|_{\vec{t}=0} \tag{3.21}$$

such that $\boldsymbol{K_S}\boldsymbol{a}_n(\boldsymbol{O}_{\alpha_1}, \ldots, \boldsymbol{O}_{\alpha_n}) = 0$, which implies that

$$\boldsymbol{a}_n\left(\boldsymbol{O}_{\alpha_1}, \ldots, \boldsymbol{O}_{\alpha_n}\right) = \boldsymbol{A}_{\alpha_1\ldots\alpha_n}^{\gamma}\boldsymbol{O}_\gamma + \boldsymbol{K_S}\boldsymbol{\Lambda}_{\alpha_1\ldots\alpha_n}. \tag{3.22}$$

[11]Compare with the classical limit $\hbar = 0$; $\left\langle \boldsymbol{O}_\beta^{(0)}\boldsymbol{O}_\alpha^{(0)}\right\rangle = \boldsymbol{A}_{\alpha\beta}^{(0)\gamma}\left\langle \boldsymbol{O}_\gamma^{(0)}\right\rangle$, where $\boldsymbol{O} = \boldsymbol{O}^{(0)} + \hbar\boldsymbol{O}^{(1)} + \ldots$ and $\boldsymbol{A} = \boldsymbol{A}^{(0)} + \hbar\boldsymbol{A}^{(1)} + \ldots$.

For examples we have

$$
\begin{aligned}
\boldsymbol{a}_2(\boldsymbol{O}_\alpha, \boldsymbol{O}_\beta) =& \boldsymbol{O}_\alpha \boldsymbol{O}_\beta - \hbar \boldsymbol{O}_{\beta\alpha}, \\
\boldsymbol{a}_3\left(\boldsymbol{O}_\alpha, \boldsymbol{O}_\beta, \boldsymbol{O}_\gamma\right) =& \boldsymbol{O}_\alpha \boldsymbol{O}_\beta \boldsymbol{O}_\gamma + \hbar(\boldsymbol{O}_{\alpha\gamma}\boldsymbol{O}_\beta + \boldsymbol{O}_\alpha \boldsymbol{O}_{\beta\gamma} + \boldsymbol{O}_{\gamma\beta}\boldsymbol{O}_\alpha) \\
&- \hbar^2 \boldsymbol{O}_{\gamma\beta\alpha}
\end{aligned}
\tag{3.23}
$$

We note that all the structure constants $\{\boldsymbol{A}_{\alpha_1\ldots\alpha_n}\}$, $n \geq 2$, are independent to any choice of Lagrangian subspace.

It follows that we have a sequence of multiplication maps from $\boldsymbol{K}_{\boldsymbol{S}}$-cohomology to Ker $\boldsymbol{K}_{\boldsymbol{S}}$. Let $\boldsymbol{H}_{\boldsymbol{K}_{\boldsymbol{S}}}$ be space of $\boldsymbol{K}_{\boldsymbol{S}}$-cohomology classes. Then we have the following multi-linear maps, $n \geq 2$

$$
\boldsymbol{a}_n : \boldsymbol{H}_{\boldsymbol{K}_{\boldsymbol{S}}}^{\otimes n} \longrightarrow \mathrm{Ker}\ \boldsymbol{K}_{\boldsymbol{S}} \tag{3.24}
$$

and associated $[\boldsymbol{a}_n]_{\vec{t}=0}$

$$
\begin{aligned}
[\boldsymbol{a}_n]_{\vec{t}=0} : \boldsymbol{H}_{\boldsymbol{K}_{\boldsymbol{S}}}^{\otimes n} &\longrightarrow \boldsymbol{H}_{\boldsymbol{K}_{\boldsymbol{S}}}, \\
[\boldsymbol{a}_n\left(\boldsymbol{O}_{\alpha_1}, \ldots, \boldsymbol{O}_{\alpha_n}\right)]_{\vec{t}=0} &= \boldsymbol{A}^\gamma_{\alpha_1\ldots\alpha_n}[\boldsymbol{O}_\gamma]_{\vec{t}=0}.
\end{aligned}
\tag{3.25}
$$

It follows that

$$
\Big\langle \boldsymbol{a}_n\left(\boldsymbol{O}_{\alpha_1}, \ldots, \boldsymbol{O}_{\alpha_n}\right)\Big\rangle_{\vec{t}=0} = \boldsymbol{A}^\gamma_{\alpha_1\ldots\alpha_n}\Big\langle \boldsymbol{O}_\gamma \Big\rangle_{\vec{t}=0}, \tag{3.26}
$$

Symbolically

$$
< \boldsymbol{a}_n >_{\vec{t}=0} : \boldsymbol{H}_{\boldsymbol{K}_{\boldsymbol{S}}}^{\otimes n} \longrightarrow \Bbbk[[\hbar]] \tag{3.27}
$$

which map depends on homology class of Lagrangian subspace that we integrate over.

Consequently the path integrals can be determined completely by the expectation values $\{\langle \boldsymbol{O}_\alpha\rangle_{\vec{t}=0}\}$ and all the structure constants $\{\boldsymbol{A}^\gamma_{\alpha_1\ldots\alpha_n}\}$, $n \geq 2$. The former data depends only homology class of Lagrangian subspace, while the later data can be determined completely by working out algebra of $\boldsymbol{K}_{\boldsymbol{S}}$-cohomology classes without doing path integrals. As we mentioned before inequivalent choices of Lagrangian subspaces as the spaces of `fields` correspond to different QFTs in the traditional sense.

In this subsection we used a deformed solution master equation with infinitesimals given by $\boldsymbol{K}_{\boldsymbol{S}}$-cohomology classes rather passively. We shall see that one can do much better than that.

Now we are interested in the path integrals of the theory defined by the deformed BV action functional $\mathfrak{S}$ in (3.6). It is obvious that the deformed partition function $\mathfrak{Z}$ in (3.7) contains all the information about the initial QFT.

### 3.2 Quantum Flat Structures

Now we assume that $\{\mathcal{S}_\alpha\}$ form a *complete* basis of $\mathcal{K}_{\mathcal{S}}$-cohomology group, which condition, in general, may not be true. Then, from (3.17), there should be structure functions $\boldsymbol{A}(\{t\})$ in formal power series of $\hbar$ such that

$$-\hbar\frac{\partial \mathcal{S}_\alpha}{\partial t^\beta} + \mathcal{S}_\beta\mathcal{S}_\alpha = \boldsymbol{A}(\{t\})_{\beta\alpha}^{\gamma}\mathcal{S}_\gamma + \mathcal{K}_{\mathcal{S}}\boldsymbol{\Lambda}(\{t\})_{\beta\alpha} \tag{3.28}$$

for some $\boldsymbol{\Lambda}(\{t\})_{\beta\alpha}$ For simplicity in notation we shall denote $\boldsymbol{\mathcal{A}} = \boldsymbol{A}(\{t^\alpha\})$ (note that $\boldsymbol{\mathcal{A}}|_{\vec{t}=\vec{0}} = \boldsymbol{A}$, which appeared in (3.19).)

Using the identity (3.11) and from (3.13), (3.15) and (3.28) we arrive at a fundamental equation

$$\left(\frac{\partial^2}{\partial t^\alpha \partial t^\beta} + \frac{1}{\hbar}\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\gamma}\frac{\partial}{\partial t^\gamma}\right)\boldsymbol{\mathcal{Z}} = 0, \quad \Longrightarrow \quad \frac{\partial \boldsymbol{\mathcal{Z}}_\beta}{\partial t^\alpha} + \frac{1}{\hbar}\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\gamma}\boldsymbol{\mathcal{Z}}_\gamma = 0. \tag{3.29}$$

We should emphasis that the equations (3.29) valid universally for any quantum field theory under our present assumption.

Note that the following relation is obvious by definition;

$$\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\gamma} = \boldsymbol{\mathcal{A}}_{\beta\alpha}^{\gamma}. \tag{3.30}$$

Now starting from (3.29) we have

$$\begin{aligned}\frac{\partial^3 \boldsymbol{\mathcal{Z}}}{\partial t^\gamma \partial t^\alpha \partial t^\beta} &= -\frac{1}{\hbar}\left(\frac{\partial \boldsymbol{\mathcal{A}}_{\alpha\beta}^{\sigma}}{\partial t^\gamma}\right)\boldsymbol{\mathcal{Z}}_\sigma - \frac{1}{\hbar}\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\rho}\frac{\partial \boldsymbol{\mathcal{Z}}_\rho}{\partial t^\gamma}\\ &= -\frac{1}{\hbar}\left(\frac{\partial \boldsymbol{\mathcal{A}}_{\alpha\beta}^{\sigma}}{\partial t^\gamma}\right)\boldsymbol{\mathcal{Z}}_\sigma + \frac{1}{\hbar^2}\left(\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\rho}\boldsymbol{\mathcal{A}}_{\rho\gamma}^{\sigma}\right)\boldsymbol{\mathcal{Z}}_\sigma.\end{aligned} \tag{3.31}$$

and the similar manipulation gives

$$\frac{\partial^3 \boldsymbol{\mathcal{Z}}}{\partial t^\alpha \partial t^\gamma \partial t^\beta} = -\frac{1}{\hbar}\left(\frac{\partial \boldsymbol{\mathcal{A}}_{\gamma\beta}^{\sigma}}{\partial t^\alpha}\right)\boldsymbol{\mathcal{Z}}_\sigma + \frac{1}{\hbar^2}\left(\boldsymbol{\mathcal{A}}_{\gamma\beta}^{\rho}\boldsymbol{\mathcal{A}}_{\rho\alpha}^{\sigma}\right)\boldsymbol{\mathcal{Z}}_\sigma. \tag{3.32}$$

On the other hand we have the following obvious identity

$$\frac{\partial^3 \boldsymbol{\mathcal{Z}}}{\partial t^\alpha \partial t^\gamma \partial t^\beta} = \frac{\partial^3 \boldsymbol{\mathcal{Z}}}{\partial t^\gamma \partial t^\alpha \partial t^\beta}. \tag{3.33}$$

Thus we obtain another universal equation

$$\frac{\partial \boldsymbol{\mathcal{A}}_{\gamma\beta}^{\sigma}}{\partial t^\alpha} - \frac{\partial \boldsymbol{\mathcal{A}}_{\alpha\beta}^{\sigma}}{\partial t^\gamma} + \frac{1}{\hbar}\left(\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\rho}\boldsymbol{\mathcal{A}}_{\rho\gamma}^{\sigma} - \boldsymbol{\mathcal{A}}_{\beta\gamma}^{\rho}\boldsymbol{\mathcal{A}}_{\rho\alpha}^{\sigma}\right) = 0. \tag{3.34}$$

Without loss of generality we may assume that there is an identity among $\{\boldsymbol{O}_\alpha\}$, say $\boldsymbol{O}_0 = 1$, as $\boldsymbol{\Delta}1 = \boldsymbol{Q}1 = 0$ and the identity can not be exact. It follows that $\boldsymbol{\mathcal{S}}_0 = \boldsymbol{O}_0$ leading to

$$\boldsymbol{\mathcal{Z}}_0 := \frac{\partial \boldsymbol{\mathcal{Z}}}{\partial t^0} = -\frac{1}{\hbar}\boldsymbol{\mathcal{Z}}, \tag{3.35}$$

Now the relation (3.35) implies another relation

$$\frac{\partial \boldsymbol{\mathcal{Z}}_\alpha}{\partial t^0} = -\frac{1}{\hbar}\boldsymbol{\mathcal{Z}}_\alpha. \tag{3.36}$$

Using (3.29), the equation (3.36) implies that

$$\boldsymbol{\mathcal{Z}}_\alpha = \boldsymbol{\mathcal{A}}_{0\alpha}^{\sigma}\boldsymbol{\mathcal{Z}}_\sigma \quad \Longrightarrow \quad \frac{\partial \boldsymbol{\mathcal{Z}}_\alpha}{\partial t^\beta} = \frac{\partial}{\partial t^\beta}\left(\boldsymbol{\mathcal{A}}_{0\alpha}^{\sigma}\boldsymbol{\mathcal{Z}}_\sigma\right). \tag{3.37}$$

Using (3.29) one more time we have another identity

$$\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\sigma} = \boldsymbol{\mathcal{A}}_{0\alpha}^{\rho}\boldsymbol{\mathcal{A}}_{\rho\beta}^{\sigma} - \hbar\frac{\partial \boldsymbol{\mathcal{A}}_{0\alpha}^{\sigma}}{\partial t^\beta} \tag{3.38}$$

From the second equation of (3.37), and from (3.36), we have $\frac{\partial \boldsymbol{\mathcal{Z}}_\alpha}{\partial t^0} = -\frac{1}{\hbar}\boldsymbol{\mathcal{Z}}_\alpha = \frac{\partial}{\partial t^0}\left(\boldsymbol{\mathcal{A}}_{0\alpha}^{\sigma}\boldsymbol{\mathcal{Z}}_\sigma\right)$, which implies that

$$\boldsymbol{\mathcal{Z}}_\alpha = \left(\boldsymbol{\mathcal{A}}_{0\alpha}^{\rho}\boldsymbol{\mathcal{A}}_{\rho 0}^{\sigma} - \hbar\frac{\partial \boldsymbol{\mathcal{A}}_{0\alpha}^{\sigma}}{\partial t^0}\right)\boldsymbol{\mathcal{Z}}_\sigma \tag{3.39}$$

We shall call above structure *Quantum Flat Structure*.

For a QFT which formal neighborhoods has quantum flat structure we may, if one wants to, forget about path integral representation of $\boldsymbol{\mathcal{Z}}$ and work with the system of differential equations (3.29). The set of all independent solutions of the system may be interpreted as path integrals of inequivalent choice of Lagrangian subspaces.

We note that

$$\mathfrak{a}_n\left(\boldsymbol{\mathcal{S}}_{\alpha_1}, \ldots, \boldsymbol{\mathcal{S}}_{\alpha_n}\right) := e^{\boldsymbol{\mathcal{S}}/\hbar}\left(\frac{\hbar^n \partial^n}{\partial t^{\alpha_1}\ldots\partial t^{\alpha_n}}\right)e^{-\boldsymbol{\mathcal{S}}/\hbar} \tag{3.40}$$

define multi-linear maps

$$\mathfrak{a}_n : \boldsymbol{H}_{\boldsymbol{\mathcal{K}}_{\boldsymbol{\mathcal{S}}}}^{\otimes n} \longrightarrow \operatorname{Ker}\boldsymbol{\mathcal{K}}_{\boldsymbol{\mathcal{S}}}. \tag{3.41}$$

Explicitly

$$
\begin{aligned}
\mathfrak{a}_2(\mathcal{S}_\alpha, \mathcal{S}_\beta) =& -\hbar \frac{\partial \mathcal{S}_\beta}{\partial t^\alpha} + \mathcal{S}_\alpha \mathcal{S}_\beta, \\
\mathfrak{a}_3(\mathcal{S}_\alpha, \mathcal{S}_\beta, \mathcal{S}_\gamma) =& -\hbar^2 \frac{\partial^2 \mathcal{S}_\gamma}{\partial t^\alpha \partial t^\beta} + \hbar \left( \frac{\partial \mathcal{S}_\beta}{\partial t^\alpha} \mathcal{S}_\gamma + \mathcal{S}_\beta \frac{\partial \mathcal{S}_\gamma}{\partial t^\alpha} + \mathcal{S}_\alpha \frac{\partial \mathcal{S}_\gamma}{\partial t^\beta} \right) \\
& - \mathcal{S}_\alpha \mathcal{S}_\beta \mathcal{S}_\gamma,
\end{aligned}
\tag{3.42}
$$

etc. etc. By taking $\mathcal{K}_\mathcal{S}$-cohomology class we have

$$
\begin{aligned}
[\mathfrak{a}_n] : \boldsymbol{H}_{\mathcal{K}_\mathcal{S}}^{\otimes n} &\longrightarrow \boldsymbol{H}_{\mathcal{K}_\mathcal{S}}, \\
\Big[ \mathfrak{a}_n \left( \mathcal{S}_{\alpha_1}, \ldots, \mathcal{S}_{\alpha_n} \right) \Big] &= \boldsymbol{\mathcal{A}}_{\alpha_1 \ldots \alpha_n}^{\gamma} [\mathcal{S}_\gamma],
\end{aligned}
\tag{3.43}
$$

leading to via path integral

$$
\begin{aligned}
\langle \mathfrak{a}_n \rangle : \boldsymbol{H}_{\mathcal{K}_\mathcal{S}}^{\otimes n} &\longrightarrow \Bbbk[[\hbar, (\{t^\alpha\})]], \\
\Big\langle \mathfrak{a}_n \left( \mathcal{S}_{\alpha_1}, \ldots, \mathcal{S}_{\alpha_n} \right) \Big\rangle &= -\hbar \boldsymbol{\mathcal{A}}_{\alpha_1 \ldots \alpha_n}^{\gamma} \mathcal{Z}_\gamma,
\end{aligned}
\tag{3.44}
$$

by path integrals, i.e.,

$$
\Big\langle \mathfrak{a}_n \left( \mathcal{S}_{\alpha_1}, \ldots, \mathcal{S}_{\alpha_n} \right) \Big\rangle \equiv \frac{\hbar^n \partial^n \mathcal{Z}}{\partial t^{\alpha_1} \ldots \partial t^{\alpha_n}} \equiv \boldsymbol{\mathcal{A}}_{\alpha_1 \ldots \alpha_n}^{\gamma} \left\langle \mathcal{S}_\gamma \right\rangle \equiv -\hbar \boldsymbol{\mathcal{A}}_{\alpha_1 \ldots \alpha_n}^{\gamma} \mathcal{Z}_\gamma.
\tag{3.45}
$$

It follows that the higher structure functions $\boldsymbol{\mathcal{A}}_{\alpha_1 \ldots \alpha_n}^{\gamma}$ can be determined uniquely in terms of compositions of $\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\gamma}$ and their derivatives; for an example we have

$$
\begin{aligned}
\boldsymbol{\mathcal{A}}_{\alpha\beta\gamma}^{\sigma} &= - \boldsymbol{\mathcal{A}}_{\beta\gamma}^{\rho} \boldsymbol{\mathcal{A}}_{\alpha\rho}^{\sigma} + \hbar \frac{\partial \boldsymbol{\mathcal{A}}_{\beta\gamma}^{\sigma}}{\partial t^\alpha}, \\
\boldsymbol{\mathcal{A}}_{\alpha\beta\gamma\rho}^{\sigma} &= - \boldsymbol{\mathcal{A}}_{\beta\gamma\rho}^{\mu} \boldsymbol{\mathcal{A}}_{\alpha\mu}^{\sigma} + \hbar \frac{\partial \boldsymbol{\mathcal{A}}_{\beta\gamma\rho}^{\sigma}}{\partial t^\alpha},
\end{aligned}
\tag{3.46}
$$

It is obvious that $\{\boldsymbol{A}_{\alpha_1 \ldots \alpha_n}^{\gamma}\} := \{\boldsymbol{\mathcal{A}}_{\alpha_1 \ldots \alpha_n}^{\gamma}|_{\vec{t}=\vec{0}}\}$ are the structure constants of the initial QFT in the previous subsection.

## 3.3 What's Next?

We recall that all those relations described so far direct consequences of master equation and the definition of path integrals with certain mild assumptions. Note, however, that our crucial $\boldsymbol{\Delta}$ is ill defined and and consequently the definition of $\mathcal{Z}$ is not immune. We may regard all of our endeavor as

an effort to find certain graded moduli space $\mathfrak{M}$ parametrizing family of QFTs. Then the "given" QFT is regarded as a a base point $o$ in the moduli space $\mathfrak{M}$, where we have a quantum flat structure. A basis $\{\mathcal{S}_\alpha\}$ of tangent space $T\mathfrak{M}$ corresponds to a basis of all observables and the dual basis $\{t^\alpha\}$ correspond to local coordinates, respectively, both in the neighborhood of $o \in \mathfrak{M}$. Over $\mathfrak{M}$ we consider certain quantum bundle $\mathfrak{Q} \rightarrow \mathfrak{M}$ with a formal power series graded-connection $\boldsymbol{\mathcal{D}}$

$$\boldsymbol{\mathcal{D}} = \frac{1}{\hbar}\boldsymbol{\mathcal{D}}^{(-1)} + \sum_{n=0}^{\infty} \hbar^n \boldsymbol{\mathcal{D}}^{(n)} \tag{3.47}$$

which may be written in terms of local coordinates,

$$\boldsymbol{\mathcal{D}} = dt^\alpha \frac{\partial}{\partial t^\alpha}\mathbb{I} + \frac{1}{\hbar}\boldsymbol{\mathcal{A}}_\alpha dt^a \tag{3.48}$$

where $\boldsymbol{\mathcal{A}}_\alpha = \sum_{n=0}^{\infty} \hbar^n \boldsymbol{\mathcal{A}}_\alpha^{(n)}$, regarding as matrices $(\boldsymbol{\mathcal{A}}_\alpha)^\beta{}_\gamma$. Then the relation (3.34) implies that $\boldsymbol{\mathcal{D}}$ is *flat*;

$$\boldsymbol{\mathcal{D}}^2 = 0, \tag{3.49}$$

and the equation (3.29) implies that $\{\mathcal{Z}_\alpha\}$, for a choice of homology class of Lagrangian subspace, is a flat section.

Now it becomes obvious that understanding "global" properties of the quantum flat bundle $\mathfrak{Q} \rightarrow \mathfrak{M}$ would be crucial. We may regard the base point $o \in \mathfrak{M}$, where we started our journey, corresponds to a point of regular singularity at the origin for the system of differential equations (3.29). The flat connection above may be degenerated at certain limiting points in $\mathfrak{M}$. Thus we compactify $\mathfrak{M} \subset \overline{\mathfrak{M}}$ by adding all the bad points and consider associated completion $\overline{\mathfrak{Q}} \rightarrow \overline{\mathfrak{M}}$. In Sect. 3.2 we assumed that $\{\mathcal{S}_\alpha\}$ form a complete basis of $\mathcal{K}_{\mathcal{S}}$-cohomology, which assumption leds to the flatness (3.49). We expect that for the generic value of $\{t^\alpha\}$ the above assumption remains valid, while for certain degenerated limits it would fail. In such a degenerated point some odd $\mathcal{K}_{\mathcal{S}}$-cohomology classes would appear. Such cohomology classes correpond to new (extended) gauge symmetry, which should be resolved as in Sect. 2.3.2. After such a resolution we need to repeat the procedure to find new family of QFTs from the degenerated point and try to build up "global" moduli space. Assuming such a procedure can be done for a given (perturbative) QFT we may interpret other degenerated points as different perturbative QFTs.[12]

[12] It may not be a far fetched idea that the conjectured M-theory moduli space may be studied by, if possible, searching family of (covariant) superstrings.

### 3.3.1 Semi-Classical Limit of Quantum Flat Structure

Recall that the structure functions $\boldsymbol{\mathcal{A}}_{\alpha\beta}^{\gamma}$ is formal power series in $\hbar$, hiding indices;

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}^{(0)} + \sum_{n=1}^{\infty} \hbar^n \boldsymbol{\mathcal{A}}^{(n)} \tag{3.50}$$

where $\boldsymbol{\mathcal{A}}^{(n)}$ for all $n$ are function of $\{t^\alpha\}$ independent to $\hbar$. Let us denote for simplicity in notation

$$\boldsymbol{\mathcal{A}}^{(0)} = \mathcal{A}, \qquad \boldsymbol{\mathcal{A}}^{(1)} = \mathcal{B}, \qquad \boldsymbol{\mathcal{A}}^{(2)} = \mathcal{C}, \quad \ldots, \tag{3.51}$$

etc. From (3.34) we find the following infinite sequence of equations starting the lowest order in $\hbar$

$$\begin{aligned}
0 =&\mathcal{A}_{\alpha\beta}^{\rho}\mathcal{A}_{\rho\gamma}^{\sigma} - \mathcal{A}_{\beta\gamma}^{\rho}\mathcal{A}_{\rho\alpha}^{\sigma},\\
0 =&\partial_\alpha \mathcal{A}_{\gamma\beta}^{\sigma} - \partial_\gamma \mathcal{A}_{\alpha\beta}^{\sigma}\\
&+ \left(\mathcal{B}_{\alpha\beta}^{\rho}\mathcal{A}_{\rho\gamma}^{\sigma} + \mathcal{A}_{\alpha\beta}^{\rho}\mathcal{B}_{\rho\gamma}^{\sigma} - \left(\mathcal{B}_{\beta\gamma}^{\rho}\mathcal{A}_{\rho\alpha}^{\sigma} + \mathcal{A}_{\beta\gamma}^{\rho}\mathcal{B}_{\rho\alpha}^{\sigma}\right)\right),\\
0 =&\partial_\alpha \mathcal{B}_{\gamma\beta}^{\sigma} - \partial_\gamma \mathcal{B}_{\alpha\beta}^{\sigma} + (-1)^{|\beta||\gamma|}\left(\mathcal{B}_{\alpha\beta}^{\rho}\mathcal{B}_{\rho\gamma}^{\sigma} - \mathcal{B}_{\beta\gamma}^{\rho}\mathcal{B}_{\rho\alpha}^{\sigma}\right)\\
&+ \left(\mathcal{A}_{\alpha\beta}^{\rho}\mathcal{C}_{\rho\gamma}^{\sigma} + \mathcal{C}_{\alpha\beta}^{\rho}\mathcal{A}_{\rho\gamma}^{\sigma} - \left(\mathcal{A}_{\beta\gamma}^{\rho}\mathcal{C}_{\rho\alpha}^{\sigma} + \mathcal{C}_{\beta\gamma}^{\rho}\mathcal{A}_{\rho\alpha}^{\sigma}\right)\right),\\
\vdots =&\vdots
\end{aligned} \tag{3.52}$$

etc. etc. From (3.38) we find the following sequence of equations starting the lowest order in $\hbar$

$$\begin{aligned}
\mathcal{A}_{\alpha\beta}^{\sigma} &= \mathcal{A}_{0\alpha}^{\rho}\mathcal{A}_{\rho\beta}^{\sigma},\\
\mathcal{B}_{\alpha\beta}^{\sigma} &= \mathcal{A}_{0\alpha}^{\rho}\mathcal{B}_{\rho\beta}^{\sigma} + \mathcal{B}_{0\alpha}^{\rho}\mathcal{A}_{\rho\beta}^{\sigma} - \frac{\partial \mathcal{A}_{0\alpha}^{\sigma}}{\partial t^\beta},\\
\mathcal{C}_{\alpha\beta}^{\sigma} &= \mathcal{B}_{0\alpha}^{\rho}\mathcal{B}_{\rho\beta}^{\sigma} + \mathcal{A}_{0\alpha}^{\rho}\mathcal{C}_{\rho\beta}^{\sigma} + \mathcal{C}_{0\alpha}^{\rho}\mathcal{A}_{\rho\beta}^{\sigma} - 2\frac{\partial \mathcal{B}_{0\alpha}^{\sigma}}{\partial t^\beta},
\end{aligned} \tag{3.53}$$

etc. etc.

Here are some remarks on the semiclassical $\hbar \to 0$ limit of the master equation (3.8);

$$\boldsymbol{\Delta}\boldsymbol{\mathcal{S}} = 0, \qquad (\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}) = 0. \tag{3.54}$$

Then we have

$$\boldsymbol{\mathfrak{Q}}_{\boldsymbol{\mathcal{S}}}^2 = 0, \qquad \boldsymbol{\Delta}\boldsymbol{\mathfrak{Q}}_{\boldsymbol{\mathcal{S}}} + \boldsymbol{\mathfrak{Q}}_{\boldsymbol{\mathcal{S}}}\boldsymbol{\Delta} = 0, \tag{3.55}$$

and we can set $\mathbf{S} = \mathbf{S}^{(0)}$ such that both $\mathbf{S}$ and $\mathbf{Q_S}$ are independent to $\hbar$. It also follows that $\mathbf{\Delta S}_\alpha = \mathbf{Q_S S}_\alpha = 0$ and the relation (3.28) implies that

$$\mathbf{S}_\alpha \mathbf{S}_\beta = \boldsymbol{\mathcal{A}}^{(0)\gamma}_{\alpha\beta} \mathbf{S}_\gamma + \mathbf{Q_S} \mathbf{\Lambda}^{(0)}_{\alpha\beta} \tag{3.56}$$

and

$$\left(\boldsymbol{\mathcal{A}}^{(1)}\right)^\gamma_{\alpha\beta} \mathbf{S}_\gamma + \mathbf{Q_S} \mathbf{\Lambda}^{(1)}_{\alpha\beta} = -\frac{\partial \mathbf{S}_\beta}{\partial t^\alpha} + \mathbf{\Delta \Lambda}^{(0)}_{\alpha\beta}, \tag{3.57}$$

while, for $n \geq 2$

$$\left(\boldsymbol{\mathcal{A}}^{(n)}\right)^\gamma_{\alpha\beta} \mathbf{S}_\gamma + \mathbf{Q_S} \mathbf{\Lambda}^{(n)}_{\alpha\beta} = \mathbf{\Delta \Lambda}^{(n-1)}_{\alpha\beta}. \tag{3.58}$$

Note that the relation (3.56) is just classical algebra of $\mathbf{Q}_{\mathbf{S}^{(0)}}$ cohomology classes in the basis $\{\mathbf{S}^{(\mathbf{0})}{}_\alpha\}$ under the product.

Assuming the semi-classical master equation (3.54) holds, we shall call the coordinates $\{t^\alpha\}$ or the basis $\{\mathbf{S}_\gamma\}$ *classical* if the following equations are satisfied;

$$\mathbf{\Delta \Lambda}^{(0)}_{\alpha\beta} = \frac{\partial \mathbf{S}_\beta}{\partial t^\alpha}. \tag{3.59}$$

Combining above with (3.56), we have

$$\begin{aligned} -\hbar \frac{\partial \mathbf{S}_\beta}{\partial t^\alpha} + \mathbf{S}_\alpha \mathbf{S}_\beta &= \boldsymbol{\mathcal{A}}^{(0)\gamma}_{\alpha\beta} \mathbf{S}_\gamma - \hbar \mathbf{\Delta \Lambda}^{(0)}_{\alpha\beta} + \mathbf{Q_S} \mathbf{\Lambda}^{(0)}_{\alpha\beta} \\ &= \boldsymbol{\mathcal{A}}^{(0)\gamma}_{\alpha\beta} \mathbf{S}_\gamma + \mathbf{K_S} \mathbf{\Lambda}^{(0)}_{\alpha\beta} \end{aligned} \tag{3.60}$$

If follows, from (3.57) and (3.58), that $\boldsymbol{\mathcal{A}}^{(n)} = 0$ for $\forall n \geq 1$ as $\{\mathbf{S}_\gamma\}$ is a basis of $\mathbf{Q_S}$-cohomology group. Assuming the system admit a classical coordinates, the structure constants $\boldsymbol{\mathcal{A}}^\gamma_{\alpha\beta}$ of algebra of $\mathbf{K_S}$-cohomology, in such a coordinates, are the same with the structure constants $\mathcal{A}^\gamma_{\alpha\beta} := \boldsymbol{\mathcal{A}}^{(0)\gamma}_{\alpha\beta}$ of algebra of $\mathbf{Q_S}$-cohomology.

**Question**: Assuming the semi-classical master equation, does a classical system of coordinates always exist?

Assume a classical coordinates system exist and Let $\{t^\alpha\}$ be such a system of classical coordinates. Then the equations (3.30), (3.34), and (3.38) of quantum-flat structure (in the semi-classical limit) reduce to the following relations (remember that the structure constants does not depends on $\hbar$ under the present circumstance);

1. Commutativity

$$\mathcal{A}^\gamma_{\alpha\beta} = \mathcal{A}^\gamma_{\beta\alpha}. \tag{3.61}$$

2. Associativity

$$\mathcal{A}^{\rho}_{\alpha\beta}\mathcal{A}^{\sigma}_{\rho\gamma} = \mathcal{A}^{\rho}_{\beta\gamma}\mathcal{A}^{\sigma}_{\rho\alpha}. \tag{3.62}$$

3. "Potentiality"

$$\partial_{\alpha}\mathcal{A}^{\sigma}_{\gamma\beta} = \partial_{\gamma}\mathcal{A}^{\sigma}_{\alpha\beta}. \tag{3.63}$$

4. Identity

$$\mathcal{A}^{\sigma}_{\alpha\beta} = \mathcal{A}^{\rho}_{0\alpha}\mathcal{A}^{\sigma}_{\rho\beta}, \qquad \frac{\partial \mathcal{A}^{\rho}_{0\alpha}}{\partial t^{\beta}} = 0. \tag{3.64}$$

We also note that $\boldsymbol{\mathcal{S}} = \boldsymbol{S} + \sum_{n=1}^{\infty} \frac{1}{n!} t^{\alpha_1} \dots t^{\alpha_n} \left( \left. \frac{\partial^n \boldsymbol{\mathcal{S}}}{\partial t^{\alpha_1} \dots \partial t^{\alpha_n}} \right|_{\vec{v}=\vec{0}} \right)$ while in the classical coordinates we have (3.59) , i.e., $\frac{\partial^2 \boldsymbol{\mathcal{S}}}{\partial t^{\alpha} \partial t^{\beta}} = \boldsymbol{\Delta\Lambda}^{(0)}_{\alpha\beta}$. Thus we obtain

$$\boldsymbol{\mathcal{S}} = \boldsymbol{S} + t^{\alpha}\boldsymbol{O}_{\alpha} + \sum_{n=2}^{\infty} \frac{1}{n!} t^{\alpha_1} \dots t^{\alpha_n} \boldsymbol{\Delta\Gamma}_{\alpha_1 \dots \alpha_n}, \tag{3.65}$$

where

$$\begin{aligned} \boldsymbol{O}_{\alpha} &= \left. \boldsymbol{\mathcal{S}}_{\alpha} \right|_{\vec{t}=\vec{0}}, \\ \boldsymbol{\Gamma}_{\alpha_1 \dots \alpha_n} &= \left. \frac{\partial^{n-2} \boldsymbol{\Lambda}^{(0)}_{\alpha_{n-1}\alpha_n}}{\partial t^{\alpha_1} \dots \partial t^{\alpha_{n-2}}} \right|_{\vec{t}=\vec{0}} \end{aligned} \tag{3.66}$$

Now we note that the semi-classical flat structure together with existence of classical system coordinates is closely related with the well-known flat or Frobenius structure. The missing piece is an invariant metric $g_{\alpha\beta}$ satisfying

$$\mathcal{A}^{\rho}_{\alpha\beta} g_{\rho\gamma} = \mathcal{A}^{\rho}_{\beta\gamma} g_{\rho\alpha}, \qquad \frac{\partial g_{\alpha\beta}}{\partial t^{\gamma}} = 0. \tag{3.67}$$

On the other hand we already started from generating functional $\boldsymbol{\mathcal{Z}}$ for the family of QFTs.

Historically the flat structure was first discovered by K. Saito in his study of the period mapping for a universal unfolding of a function with an isolated critical point (in the context of singularity theory) [33]. His main motivation was to have a new constructions of modular functions by a certain generalization of the well-known theory of elliptic integrals and modular functions (see a recent exposition [34]). Here the concept of primitive form, which essentially plays the role of $\boldsymbol{\mathcal{Z}}$ is crucial. We may interpret his work as a kind of the rigorous and complete description of, perhaps, the simplest example of flat family of QFTs, which also give us a hint on the global issues of flat family of QFTs . In the physics literature flat structure first appear,

independently, in the works of Witten-Dijkraaf-Verlinde-Verlinde (WDDV) on topological conformal field theory in 2-dimensions [42, 8]. Later Dubrovin formalized those structures in the name of Frobenius manifolds [10]. Another construction of flat or Frobenius structure is due to Barannikov-Kontsevich on the extended moduli space of complex structures on Calabi-Yau manifold in the context of B-model [2]. We should remark that the construction of Frobenius structure by Barannikov-Kontsevich is more general than their original context and closely related with our semi-classical case. All of the above constructions of flat or Frobenius structures can be viewed as special limits of the quantum flat structures of various family of QFTs

# 4 Down to Earth

One may start from a ghost number $U = 0$ function $\boldsymbol{S}^{(0)}$ on an infinite dimensional graded space $\boldsymbol{\mathcal{T}}$ admitting odd symplectic structure $\boldsymbol{\omega}$ carrying the ghost number $U = -1$ satisfying the classical master equation

$$\left(\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(0)}\right) = 0, \tag{4.1}$$

where $(\bullet, \bullet)$ is the graded Poisson bracket, carrying the ghost number $U = 1$, defined by $\boldsymbol{\omega}$. An usual classical action functional $\boldsymbol{s}$, then, corresponds to the restriction of $\boldsymbol{S}^{(0)}$ to a Lagrangian subspace $\boldsymbol{\mathcal{L}}$ of $\boldsymbol{\mathcal{T}}$. More precisely classical action functional $\boldsymbol{s}$ is typically supported only on certain subspace, which is called the space of classical fields, of $\boldsymbol{\mathcal{L}}$. The complimentary of space of classical fields in $\boldsymbol{\mathcal{L}}$ consists of space of ghosts and their anti-ghosts multiplets, ghosts of ghosts and their anti-ghost multiplets, etc., etc., depending on the nature of symmetry and constraints of the classical action functional.

In this section we describe a systematic ways of constructing the classical BV action functional for a large class of $d$-dimensional QFTs related with symplectic $d$-algebras. The upshot is that for any symplectic $d$-algebra one can associate QFT on $d$-dimensions. For QFT on $d$-dimensional manfold with boundary we shall see that there exist a structure of strongly homology $(d-1)$-algebra on boundary and the structure of symplectic $d$-algebra in the bulk, which corresponds to the structure on the cohomology of Hochschild complex of the boundary algebra. This setup may be used to develop an universal quantization machine of $(d-1)$-algebras. Also the construction in this section can be used to "define" differential-topological invariants of low $(d = 3, 4)$-dimensions for any symplectic $d = 3, 4$ algebras. We remark that this section is an obvious generalization of the authors previous work [27],

which was motivated by the seminal paper [23] of Kontsevich [23] as well as the papers [1, 22, 5].

### 4.1 Closed $s$-Braneoids

Let $M_{s+1}$ be an $(s+1)$-dimensional oriented and smooth manifold. Being an $(s+1)$-dimensional QFT, a classical BV action functional $\boldsymbol{S}^{(0)}$ must be defined by integration of certain top-form over $M_{s+1}$. The differential forms on $M_{s+1}$ may most suitably be described as smooth functions on the total space $T[1]M_{s+1}$ of twisted by $U=1$ tangent space to $M_{s+1}$. Without any loss of generality we may pick a local coordinates system $\{\sigma^\mu\}$, $\mu = 1,\ldots,s+1$, on $M_{s+1}$ and assign $U=0$. Then the fiber coordinates are set to $\{\theta^\mu\}$ assigned to $U=1$, such that $U(\sigma^\mu)+U(\theta^\mu)=1$ and $\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu$. Let $\mathfrak{s} = \bigoplus_{k=0}^{s+1}\mathfrak{s}_k$ be the $\mathbb{Z}$-graded space of functions on $T[1]M_{s+1}$. The space $\mathfrak{s}$ is isomorphic to the space of differential forms on $M_{s+1}$, where the wedge product is replaced with ordinary product. The exterior derivative $d$ on $M_{s+1}$ induces an odd nilpotent vector field $\widehat{d} = \vartheta^\mu\frac{\partial}{\partial\sigma^\mu}$ carrying $U=1$ on $T[1]M_{s+1}$. Thus we have the following complex

$$\left(\widehat{d};\mathfrak{s} = \bigoplus_{k=0}^{s+1}\mathfrak{s}_k\right), \tag{4.2}$$

which is isomorphic to de Rham complex on $M_{s+1}$. Now the integration of certain $(s+1)$-from over $M_{s+1}$ is equivalent to

$$\oint_{T[1]M_{s+1}}\alpha := \int_{M_{s+1}} d\sigma^1\ldots d\sigma^{s+1}\int d\theta^1\ldots d\theta^{s+1}\alpha \tag{4.3}$$

where $\alpha\in\mathfrak{s}_{s+1}$, which may take certain values of extra structures. We note that the integration over odd variable (Berezin integral) is defined as $\int d\theta^\mu\theta^\nu = \delta^{\mu\nu}$. Thus the integral $\int d\theta^1\ldots d\theta^{s+1}$ shifts $U$ by $-(s+1)$. We should also note that the total integration measure is coordinates independent as the Jacobians of even and odd variable cancel each others.

Now the infinite dimensional graded space $\boldsymbol{\mathcal{T}}$ may be viewed as space of certain functions on $T[1]M_{s+1}$, more precisely, the space of all sections of certain graded bundle over $T[1]M_{s+1}$. On such a space $\boldsymbol{\mathcal{T}}$ the odd symplectic form $\boldsymbol{\omega}$ must be induced from something. A natural choice is to identify $\boldsymbol{\mathcal{T}}$ with the space of all ghost number and parity preserving maps

$$\Phi : T[1]M_{s+1} \to \mathbb{T}_{s+1}, \tag{4.4}$$

where $\mathbb{T}_{s+1}$ is a finite dimensional smooth graded space admitting symplectic form $\omega_s$ of ghost number $U = s$ and of the same parity, even or odd, with $s$.

We denote $\mathfrak{t}$ the space of functions on $\mathbb{T}_{s+1}$, graded by the ghost number $U$,

$$\mathfrak{t} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{t}_k. \tag{4.5}$$

Associated with $\omega_s$ we have the graded Poisson bracket $[\star, \star]_{-s}$ of degree $U = -s$

$$[\star, \star]_s : \mathfrak{t}_{k_1} \otimes \mathfrak{t}_{k_2} \longrightarrow \mathfrak{t}_{k_1+k_2-s}. \tag{4.6}$$

We call the algebra functions on $\mathbb{T}_s$ $(\mathfrak{t}, [\star, \star]_s, \cdot)$ endowed with $[\star, \star]_{-s}$ and ordinary (super) commutative and (super) associative product $\cdot$ a *symplectic* $(s+1)$*-algebra*. We remark that the notion of $d = (s+1)$-algebra has been first introduced by Getzler-Jones [18] and refined by Tarmarkin and Kontsevich (see ref. [23] for some history and operadic viewpoint, which we shall not use directly here). We put symplectic as the bracket $[\star, \star]_{-s}$ is non-degenerated.

Now we consider an element $H_o \in \mathfrak{t}_{s+1}$ satisfying

$$[H_o, H_o]_{-s} = 0. \tag{4.7}$$

Define $Q_{H_o}$ as the Hamiltonian vector field on $\mathbb{T}_{s+1}$

$$Q_{H_o} = [H_o, \ldots]_{-s}. \tag{4.8}$$

Note that $Q_{H_o}^2 = 0$ and $Q_{H_o}$ is odd carrying $U = 1$. Thus we have a complex

$$\left( Q_{H_o}, \mathfrak{t} = \bigoplus_k \mathfrak{t}_k \right), \tag{4.9}$$

to be compared with (4.2).

Remark that $[\star, \star]_{-s} : \mathfrak{t}_s \otimes \mathfrak{t}_s \to \mathfrak{t}_s$, which means that $[\star, \star]_{-s}$ induces a structure of Lie algebra on $\mathfrak{t}_s$. Thus we may define adjoint adjoint action by an element $b \in \mathfrak{t}_s$ an any $\gamma \in \mathfrak{t}$;

$$e^{ad_b} \circ (\gamma) := \gamma + [b, \gamma]_{-s} + \frac{1}{2!}[b, [b, \gamma]_{-s}]_{-s} + \ldots, \tag{4.10}$$

which is equivalent to a degree preserving canonical transformation connected to the identity. We call the two solutions $H, H' \in \mathfrak{t}_{s+1}$ of (4.7) equivalent if they are related by the above adjoint action. Thus we can define a moduli space $\mathcal{N}$ as the set of equivalence classes of solutions of (4.7);

$$\mathcal{N} = \{H \in \mathfrak{t}_{s+1} | [H, H]_{-s} = 0\} / \sim, \tag{4.11}$$

such that the solution $H_o$ corresponds to a point $o \in \mathcal{N}$. Here we consider the case that the ghost number $U = s$ part of $Q_{H_o}$ cohomology of the complex (4.9) is trivial. Otherwise we assume small resolution of the pair $(\mathbb{T}_{s+1}, H_o)$ to $(\widetilde{\mathbb{T}}_{s+1}, \widetilde{H}_o)$ following the similar procedure described in Sect. 2.3. We shall also assume the "anomaly free" condition that the ghost number $U = s+2$ part of $Q_{H_o}$ cohomology of the complex (4.9) is trivial. Then the moduli space $\mathcal{N}$ is unobstructed (around $o$).

We describe a map (4.4) locally by a "local" coordinates on $\mathbb{T}_{s+1}$, which are regarded as functions on $T[1]M_{s+1}$. Let $\{x^I\}$ be a "local" coordinates system on $\mathbb{T}_{s+1}$. We denotes the ghost number of $x^I$ by $U(x^I) \in \mathbb{Z}$. We parametrize a map by $\{\widehat{x}^I\}$, where

$$\widehat{x}^I := x(\sigma,\theta)^I = x(\sigma)^I + \frac{1}{n!}\sum_{i=1}^{s+1} x(\sigma)^I_{\mu_1\ldots\mu_{s+1}}\theta^{\mu_1}\ldots\theta^{\mu_{s+1}}. \tag{4.12}$$

By the ghost number preserving maps we meant

$$U(x^I) = U(\widehat{x}^I), \qquad |x^I| = |\widehat{x}^I|. \tag{4.13}$$

We define associated $n$-form components on $M_{s+1}$ by

$$x^I_{[n]} := \frac{1}{n!}x(\sigma)^I_{\mu_1\ldots\mu_n}d\sigma^{\mu_1}\wedge\ldots\wedge d\sigma^{\mu_n}. \tag{4.14}$$

Note that the ghost number $U$ of $x^I_{(n)}$ is $U(x^I) - n$. From the symplectic form $\omega_s(dx,dx)$ of $U = s$ we have the following induced odd symplectic form $\boldsymbol{\omega}$ carrying $U = -1$ on $\boldsymbol{\mathcal{T}}$

$$\boldsymbol{\omega} = \oint_{T[1]M_{s+1}} \omega_s(\delta\widehat{x}, \delta\widehat{x}), \tag{4.15}$$

since $\oint_{T[1]M_{s+1}}$ shifts $U$ by $-(s+1)$ and has the parity $(s+1)$ mod 2.

Now we can describe classical BV action functional associated to $H_o$ as follows

$$\begin{aligned}\boldsymbol{S}^{(0)}_{H_o} &= \oint_{T[1]M_{s+1}} \Big(\omega(\widehat{x}, \widehat{dx})_s + \Phi^*(H_o)\Big)\\ &\equiv \oint_{T[1]M_{s+1}} \Big(\omega(\widehat{x}, \widehat{dx})_s + \widehat{H}_o\Big)\end{aligned} \tag{4.16}$$

Note that $\boldsymbol{S}^{(0)}_{H_o}$ carries the ghost number $U = 0$ and is an even function(al). One can check that $\boldsymbol{S}^{(0)}_{H_o}$ satisfies the classical master equation $\Big(\boldsymbol{S}^{(0)}_{H_o}, \boldsymbol{S}^{(0)}_{H_o}\Big) =$

0 provided that the boundary of $M_{s+1}$ is empty, as the result of $[H_o, H_o]_{-s} = 0$, after integration by parts and using the Stokes theorem.

Define odd Hamtionian vector of $\boldsymbol{S}^{(0)}_{H_o}$ on $\boldsymbol{\mathcal{T}}$ as follows;

$$\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}} := \left(\boldsymbol{S}^{(0)}_{H_o}, \ldots\right), \qquad \boldsymbol{Q}^2_{\boldsymbol{S}^{(0)}_{H_o}} = 0, \tag{4.17}$$

and by definition $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}} \boldsymbol{S}^{(0)}_{H_o} = 0$. We note that

$$\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}} \widehat{x}^I = \widehat{d}\widehat{x}^I + \widehat{Q_{H_o} x^I}. \tag{4.18}$$

Since $\boldsymbol{Q}^2_{\boldsymbol{S}^{(0)}_{H_o}} = 0$, we have the following complex

$$\left(\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}, \mathfrak{T} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{T}_k\right). \tag{4.19}$$

An classical observable of the theory is an element of cohomology of the above complex. A class of classica observables can be constructed as follows.

Consider a function $\gamma \in \mathfrak{t}$ on $\mathbb{T}_s$ with certain ghost number $U(g)$. The pullback $\widehat{\gamma} := \boldsymbol{\Phi}^*(\gamma)$ of $\gamma$ can be viewed as a functional $\gamma(\sigma^\mu, \theta^\nu)$ on $T[1]M_{s+1}$. The relation (4.18) implies that $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}} \widehat{\gamma} = \widehat{d}\widehat{\gamma} + \widehat{Q_{H_o}\gamma}$. Let $\gamma$ be an elment of the $Q_{H_o}$-cohomology of the complex (4.9). Then we have

$$\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}} \widehat{\gamma} = \widehat{d}\widehat{\gamma}, \tag{4.20}$$

for $\gamma \in H(\mathfrak{t}, Q_{H_o})$. We call the above relation *descent equations*. We can expand $\widehat{\gamma}$ as

$$\widehat{\gamma} = \gamma(\sigma) + \sum_{n=1}^{s+1} \gamma(\sigma)_{\mu_1\ldots\mu_n} \theta^{\mu_1} \ldots \theta^{\mu_n}, \tag{4.21}$$

where $\gamma(\sigma)_{\mu_1\ldots\mu_n}$ is a functional on $M_{s+1}$ transforming as totally antisymmetric $n$-tensor. Then we have associated $n$-form $\gamma_{[n]}$ on $M_{s+1}$ defined by

$$\gamma_{[n]} := \frac{1}{n!}\gamma(\sigma)_{\mu_1\ldots\mu_n} d\sigma^{\mu_1} \wedge \ldots \wedge d\sigma^{\mu_n}. \tag{4.22}$$

Note that the ghost number $U$ of $\gamma_{[n]}$ is $U(\gamma) - n$. Then the equation (4.20)

becomes

$$\begin{aligned} \boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}\gamma_{[0]} &= 0,\\ d\gamma_{[0]} - \boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}\gamma_{[1]} &= 0,\\ d\gamma_{[1]} - \boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}\gamma_{[2]} &= 0,\\ &\vdots,\\ d\gamma_{[s]} - \boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}\gamma_{[s+1]} &= 0,\\ d\gamma_{[s+1]} &= 0, \end{aligned} \tag{4.23}$$

where $d$ denotes the exterior derivative on $M_{s+1}$. Now consider a homology $n$-cycle $C_n \in H_*(M_{s+1})$ on $M_{s+1}$ and define $\int_{C_n}\gamma_{[n]}$. Then (4.6) implies that $\int_{C_n}\gamma^{[n]}$ is an observable and the BRST cohomology of $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}$ depends only on the homology class of $C_n \in H_*(M_{s+1})$.

Now we *assume* that one can define $\boldsymbol{\Delta}$ after suitable regularization such that $\boldsymbol{\Delta}^2 = 0$. We shall consider the situation that there exist $\boldsymbol{S}_{H_o}$ satisfying

$$-\hbar\boldsymbol{\Delta}\boldsymbol{S}_{H_o} + \frac{1}{2}\left(\boldsymbol{S}_{H_o}, \boldsymbol{S}_{H_o}\right) = 0, \tag{4.24}$$

where

$$\boldsymbol{S}_{H_o} = \boldsymbol{S}^{(0)}_{H_o} + \hbar\boldsymbol{S}^{(1)}_{H_o} + \ldots. \tag{4.25}$$

Now we back to the target space $\mathbb{T}_{s+1}$. Let $\{\gamma_\alpha\}$ be a basis of the cohomology of the complex (4.9) and let $\{t^\alpha\}$ be the dual basis such that

$$U(\gamma_\alpha) + U(t^\alpha) = s+1. \tag{4.26}$$

Now we assume that there is solution of

$$[\mathcal{H},\mathcal{H}]_{-s} = 0, \tag{4.27}$$

such that

$$\mathcal{H} = H + t^\alpha\gamma_\alpha + \sum_{n=2}^{\infty}\frac{1}{n!}t^{\alpha_1}\ldots t^{\alpha_n}\gamma_{\alpha_1\ldots\alpha_n}, \qquad U(\mathcal{H}) = s+1. \tag{4.28}$$

Let's define $\mathcal{Q}_{\mathcal{H}} = [\mathcal{H},\ldots]_{-s}$ which is odd nilpotent vector of degree $U=1$.

In general we consider certain graded Artin ring with maximal ideal

$$\mathfrak{a} = \bigoplus_{-\infty<k\leq s+1}\mathfrak{a}_k. \tag{4.29}$$

The bracket $[\star,\star]_{-s}$ on $\mathfrak{t}$ can be naturally extended to $\mathfrak{t}\otimes\mathfrak{a}$. Then we may define extended moduli space $\mathcal{M}$;

$$\mathcal{M}=\{\mathcal{H}\in(\mathfrak{t}\otimes\mathfrak{a})_{s+1}|[\mathcal{H},\mathcal{H}]_{-s}=0\}/\sim \tag{4.30}$$

where the equivalence is defined by adjoint action of element $\beta\in(\mathfrak{t}\otimes\mathfrak{a})_s$. Then we may regard the fixed $H\in\mathcal{N}\subset\mathcal{M}$ as a basepoint $o$ in $\mathcal{M}$ and interpret $Q_H$-cohomology as the tangent space $T_o\mathcal{M}$.

Now we have corresponding families of classical BV action functional

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(0)}_{\mathcal{H}}&=\int_{T[1]M_{s+1}}\Big(\omega(\widehat{x},\widehat{dx})_s+\widehat{\mathcal{H}}\Big)\\&=\boldsymbol{S}^{(0)}_H+\int_{T[1]M_{s+1}}\left(t^\alpha\widehat{\gamma}_\alpha+\sum_{n=2}^{\infty}\frac{1}{n!}t^{\alpha_1}\dots t^{\alpha_n}\widehat{\gamma}_{\alpha_1\dots\alpha_n},\right)\end{aligned} \tag{4.31}$$

From (4.27), and as the boundary of $M_{s+1}$ is empty, we have

$$\Big(\boldsymbol{\mathcal{S}}^{(0)}_{\mathcal{H}},\boldsymbol{\mathcal{S}}^{(0)}_{\mathcal{H}}\Big)=0, \tag{4.32}$$

Thus

$$\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{S}}^{(0)}_{\mathcal{H}}}=\Big(\boldsymbol{\mathcal{S}}^{(0)}_{\mathcal{H}},\dots\Big),\qquad \boldsymbol{\mathcal{Q}}^2_{\mathcal{S}^{(0)}_{\mathcal{H}}}=0,\qquad \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{S}}^{(0)}_{\mathcal{H}}}\widehat{x}^I=\widehat{d\widehat{x}}^I+\widehat{\mathcal{Q}_{\mathcal{H}}x^I}. \tag{4.33}$$

Thus we have

> *Associated with any symplectic $(s+1)$-algebra with non-empty moduli space $\mathcal{M}$ we have family of pre QFTs which action functional satisfies the classical BV master equations.*

Now we *assume* that one can define $\boldsymbol{\Delta}$ after suitable regularization such that $\boldsymbol{\Delta}^2=0$. We shall consider the situation that for any $\boldsymbol{\mathcal{S}}^{(0)}_{\mathcal{H}}$, $\mathcal{H}\in\mathfrak{M}$ there exist $\boldsymbol{\mathcal{S}}_{\mathcal{H}}$ satisfying

$$-\hbar\boldsymbol{\Delta}\boldsymbol{\mathcal{S}}_{\mathcal{H}}+\frac{1}{2}\left(\boldsymbol{\mathcal{S}}_{\mathcal{H}},\boldsymbol{\mathcal{S}}_{\mathcal{H}}\right)=0, \tag{4.34}$$

where

$$\boldsymbol{\mathcal{S}}_{\mathcal{H}}=\boldsymbol{\mathcal{S}}^{(0)}_{\mathcal{H}}+\hbar\boldsymbol{\mathcal{S}}^{(1)}_{\mathcal{H}}+\dots. \tag{4.35}$$

Then we have

> *Family of QFTs parametrized by the moduli space $\mathfrak{M}$.*

Now our earlier discussion endows quantum flat structure on $\mathfrak{M}$ via the family of $(s+1)$-dimensional QFTs, which define the function $\mathcal{Z}$ on $\mathfrak{M}$;

$$\mathcal{Z} = \int_{\boldsymbol{\mathcal{L}}} d\mu \, e^{-\mathcal{S}_{\mathcal{H}}/\hbar} \tag{4.36}$$

Recall that $\mathcal{Z}$ depends on homology classes of the Lagrangian subspace $\mathcal{L}$ in $\boldsymbol{\mathcal{T}}$. So we have any many inequivalent flat structures on $\mathfrak{N}$ as homology classes of Lagrangian subspaces.

We also note that, by construction,

> *The quantum flat structures on $\mathfrak{N}$ depend on the smooth structures on $M_{s+1}$.*

It can be argued that the following is true.

> *There exist a suitable choice of homology class $[\boldsymbol{\mathcal{L}}]$ of Lagrangian subspaces and its representative such that $\mathcal{Z}$ define family of differential-topological invariants on $M_{s+1}$.*

The most interesting case for the above perspective would be $s = 3$, that is, quantum field theoretic definition of smooth invariants of 4-manifold.[13] Note that the Hodge star operator $*$ on smooth oriented 4-manifold satisfies $*^2 = 1$ and maps 2-form to 2-form. This implies that we can always choose a contiuos family of Lagrangian subspaces $\boldsymbol{\mathcal{L}}$ depending on continuous family of metric, which property implies that $\mathcal{Z}$ define family of smooth invariants of 4-manifold. The philosophy here is to use all symplectic 4-algebras to prove smooth structures of 4-manifold via associated QFTs.

We remark that the whole construction described above can be generalized by replacing $T[1]M_{s+1}$ with any smooth graded manifold adimiting a non-degenerated volume form of the parity of $(s+1)$ with $U = -(s+1)$ and an odd nilpotent vector field with $U = 1$.

[13] Allow me to give a simple example for this case. Let $V$ is a finite dimesional vector space over $R$ and let $V[1]$ be the suspension of $V$ by $U = 1$. Then consider $\mathbb{T}_4 = T^*[3]V[1] \simeq V[1] \oplus V^*[2]$, which has a structure of symplectic 4-algebra. The resulting 3-braneoid leads to the celabrated Donaldson-Witten theory [10, 41] after suitable gauge fixing for $dimV < 4$. In general we have certain deformation of Donaldson-Witten theory, where semi-simple Lie algebra is replaced with semi-simple weakly homotopy Lie algebra, where the Jacobi identity is violated. Nonetheless the path integral gives smooth invariants of 4-manifolds. The similar deformation is also possible for physical Yang-Mills theory.

### 4.1.1 Hamitonian picture and dimensional reduction

Now we turn to Hamiltonian picture and dimensional reduction.

Consider the classical BV action functional $\boldsymbol{S}^{(0)}_{H_o}$ in (4.16). It is not difficult to see that $\boldsymbol{S}^{(0)}_{H_o}$ is invariant under odd symmetry generated by $\boldsymbol{V}_\mu$ carrying $U = -1$;

$$\boldsymbol{V}_\mu \widehat{x}^I := \frac{\partial}{\partial \theta^\mu}\widehat{x}^I, \tag{4.37}$$

as $\boldsymbol{S}^{(0)}_{H_o}$ is defined by an integral over $T[1]M_{s+1}$. We have the following commutation relation

$$\{\boldsymbol{V}_\mu, \boldsymbol{V}_\nu\} = 0, \qquad \{\boldsymbol{V}_\mu, \boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}\} = \frac{\partial}{\partial \sigma^\mu}, \tag{4.38}$$

which is a form of world-volume supersymmetry. Now we assume that $M_{s+1} = M_s \times \mathbb{R}$, where $M_s$ is an oriented smooth $s$-dimensional manifold. We can decompose $(\sigma^\mu)$ as $(\sigma^i, \sigma^0)$, $i = 1, 2, \ldots, s$, where $s^0$ is the time ($\mathbb{R}$) coordinates. Then the component $\boldsymbol{V}_0$ is defined globally as $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}$. Note that both $\boldsymbol{V}_0$ and $\boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}$ are odd nilpotent vector fields on $\boldsymbol{\mathcal{T}}$. Let $\boldsymbol{Q}^*$ and $\boldsymbol{Q}$ denote corresponding charges. Then the commutation relation $\{\boldsymbol{V}_0, \boldsymbol{Q}_{\boldsymbol{S}^{(0)}_{H_o}}\} = \frac{\partial}{\partial \sigma^0}$ implies that

$$\boldsymbol{Q}^*\boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{Q}^* = Ham \tag{4.39}$$

where $Ham$ means "Hamiltonian" of the theory (the usual Hamiltonian can be obtained from $Ham$ after "gauge fixing") . We may interested in the ground state $Ham|0>= 0$.[14]

The dimensional reduction means dropping the dependence of the theory on the "time-direction" $\mathbb{R}$. Note that the superfields $\widehat{x}^I = x(\sigma^m, \theta^\mu)$ are decomposed as

$$\widehat{x}^I = x^I(\sigma^i, \theta^i, \theta^0) = y^I(\sigma^i, \theta^i) + \theta^0 z^I(\sigma^i, \theta^i) := \widehat{y}^I + \theta^0 \widehat{z}^I \tag{4.40}$$

such that $U(\widehat{y}^I) = U(\widehat{x}^I)$ and $U(\widehat{z}^I) = U(\widehat{x}^I) - 1$. It follows that the dimensional reduction of the theory of the maps $T[1]M_{s+1} \to \mathbb{T}_{s+1}$ becomes a theory of maps $T[1]M_s \to T^*[s-1]\mathbb{T}_{s+1}$.[15]

[14]Applying this construction to the case in the foonote[12] leads to the Floer homology of 3-manifolds [15, 41] for $dimV < 4$ and its deformation in general. This implies Floer-like homology of 3-manifolds has a full featured generalization associated with symplecitc 4-algebras.

[15]Applying this construction to the case in the foonote[12] leads to the Casson invariant of 3-manifold W.

## 4.2 Open $s$-Braneoids

So far we assumed that the $(s+1)$-dimensional manifold $M_{s+1}$ has no boundary. Now we consider the cases that the boundaries of $M_{s+1}$ are non-empty. For simplicity we shall begin with the case that there is only one boundary component.

### 4.2.1 Boundary Condition

Assume that we have the same data as the empty boundary case and consider $\boldsymbol{S}^{(0)}$ given by (4.16). Recall that the BV bracket $\Big(\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(0)}\Big)$ involves a total derivative term from the bracket between the first term $\oint_{T[1]M_{s+1}} \omega_s(\widehat{x}, \widehat{dx})$ in (4.16). It is not difficult to check the total derivative term vanish if we impose the following boundary condition

$$\begin{aligned} &\Phi : T[1]M_{s+1} \to \mathbb{T}_{s+1}, \\ &\Phi(T[1](\partial M_{s+1}) \subset \mathbb{L}, \end{aligned} \tag{4.41}$$

where $\mathbb{L}$ is a any Lagrangian subspace of $\mathbb{T}_{s+1}$ with respect to the symplectic form $\omega_s$. We remark that for $s = \textit{even}$ $\mathbb{T}_{s+1}$ may not admits any Lagrangian subspace. From now on we always consider $(\mathbb{T}_{s+1}, \omega_s)$ admitting Lagrangian subspace. The BV bracket $\Big(\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(0)}\Big)$ involves another total derivative term from the bracket between the first term $\oint_{T[1]M_{s+1}} \omega_s(\widehat{x}, \widehat{dx})$ and the second term $\oint_{T[1]M_{s+1}} \widehat{H}_o$ in (4.16). With the above boundary condition such total derivative term vanishes if

$$H|_{\mathbb{L}} = 0. \tag{4.42}$$

Finally the bracket between the second term in (4.16) vanishes iff

$$[H, H]_{-s} = 0. \tag{4.43}$$

### 4.2.2 Algebraic Digression

We may identify $\mathbb{T}_{s+1}$ in the neighborhood of $\mathbb{L}$ with the total space $T^*[s]\mathbb{L}$ of cotangent bundle over $\mathbb{L}$ with the fiber twisted by $U = s$; We denote a system of Darboux coordinates of $T^*[s]\mathbb{L}$ by $(q^a, p_a)$, $(base|fiber)$ such that

$$\omega_s = dp_a dq^a, \qquad U(q^\alpha) + U(p_\alpha) = s, \tag{4.44}$$

and $\mathbb{L}$ is defined by $p_a = 0$ for all $a$. Then one may Taylor expand $H \in \mathfrak{t}_{s+1}$ around $\mathbb{L}$

$$
\begin{aligned}
H &= \sum_{n=0}^{\infty} M_n, \\
M_n &= \frac{1}{n!} m(q)^{a_1 \ldots a_n} p_{a_1} \ldots p_{a_n}.
\end{aligned}
\tag{4.45}
$$

Now the condition $[H, H]_{-s} = 0$ becomes

$$
\sum_{p+q=n} [M_p, M_q]_{-s} = 0, \qquad \forall n \geq 0. \tag{4.46}
$$

For each $M_n$ one may assign $n$-poly differential operator $m_n$ acting on the $n$-th tensor product $\mathfrak{l}^{\otimes m}$ of the space $\mathfrak{l}$ of functions on $\mathbb{L}$ such as;

$$
\begin{aligned}
m_0 &: \mathfrak{l} \to \mathbb{k}, \\
m_n &: \mathfrak{l}^{\otimes n} \to \mathfrak{l}, \qquad \text{for } n \geq 1
\end{aligned}
\tag{4.47}
$$

by canonically "quantization", i.e., replacing the BV bracket $(p_a, q^b) = \delta_a{}^b$ to commutators of operators, naively, $\widehat{q}^b = q^b$ and $\widehat{p}_a = \frac{\partial}{\partial q^a}$. It is not difficult to check $m_n$ carry ghost number

$$
U(m_n) = -ns + s + 1, \tag{4.48}
$$

i.e., $U(m_0) = s + 1$, $U(m_1) = 1$, $U(m_2) = -s + 1$, etc. Then we may take the condition (4.46) as a definition of a structure $(m_0, m_1, m_2, \ldots)$ of *weakly homotopy Lie $s$-algebroid* on $\mathbb{L}$. We note that $H|_{\mathbb{L}} = m_0$. Thus the condition $H|_{\mathbb{L}} = 0$ means that $m_0 = 0$ and together with the condition $[H, H]_{-s} = 0$ we have a a structure $(m_1, m_2, \ldots)$ of *strongly homotopy Lie $s$-algebroid* on $\mathbb{L}$.[16]

[16] A structure of strongly homotopy Lie $s$-algebroid is certain homotopy generalization of structure of Lie algebroid on $\mathbb{L}$. Note that the equation (4.46) with the condition $M_0 = 0$ reads

$$
\begin{aligned}
[M_1, M_1]_{-s} &= 0, \\
[M_1, M_2]_{-s} &= 0, \\
\frac{1}{2}[M_2, M_2]_{-s} + [M_1, M_3]_{-s} &= 0, \\
[M_2, M_3]_{-s} + [M_1, M_4]_{-s} &= 0, \\
&\vdots
\end{aligned}
\tag{4.49}
$$

etc. Note that $m_1 = m(q)^a \frac{\partial}{\partial q^a}$ is an odd vector with $U = 1$ on $\mathbb{L}$, which satisfies $m_1^2 = 0$, due to the first equation above. We may take the existence of such odd vector field as a

We call two structures of strongly homotopy Lie $s$-algebroids on $\mathbb{L}$ are equivalent if they are related by change of the Lagrangian compliment of $\mathbb{L}$ in $\mathbb{T}_{s+1} \simeq T^*[s]\mathbb{L}$, called skrooching. It is obvious a skrooching always leads to another structure of strongly homotopy Lie $s$-algebroids on $\mathbb{L}$ as a skrooching always preserve all the conditions in (4.42).

Now we consider the following infinitesimal canonical transformations

$$\begin{aligned} q^a &\to q^a, \\ p_a &\to p_a + \frac{\partial \Gamma}{\partial q^a}, \end{aligned} \tag{4.50}$$

generated by $\Gamma(q) \in \mathfrak{l}_s$. Let $H_\Gamma$ be the result of the above transformation, which automatically satisfies $[H_\Gamma, H_\Gamma]_{-s} = 0$, while $H_\Gamma|_{\mathbb{L}} \neq 0$ in general; we have

$$\begin{aligned} H_\Gamma &= \sum_0^\infty M_{\Gamma n}, \\ M_{\Gamma 0} &= \sum_{n=1}^\infty \frac{1}{n!} m_n(\Gamma, \ldots, \Gamma), \\ M_{\Gamma 1} &= \sum_{n=1}^\infty \frac{1}{(n-1)!} m^{a_1 \ldots a_n} \left(\frac{\partial \Gamma}{\partial q^{a_1}}\right) \cdots \left(\frac{\partial \Gamma}{\partial q^{a_{n-1}}}\right) p_{a_n}, \\ &\vdots \end{aligned} \tag{4.51}$$

Consider the generating functional $\Gamma \in \mathfrak{l}_s$ satisfying $H_\Gamma|_{\mathbb{L}} \equiv M_{\Gamma 0} \equiv m_{\Gamma 0} = 0$;

$$m_1(\Gamma) + \frac{1}{2} m_2(\Gamma, \Gamma) + \frac{1}{3!} m_3(\Gamma, \Gamma, \Gamma) + \ldots = 0. \tag{4.52}$$

Then $H_\Gamma$ induces another structure $(m_{\Gamma 1}, m_{\Gamma 2}, m_{\Gamma 3}, \ldots)$ of strongly homotopy Lie $s$-algebroid on $\mathbb{L}$. We remark that the two structures $(m_1, m_2, m_3, \ldots)$ and $(m_{\Gamma 1}, m_{\Gamma 2}, m_{\Gamma 3}, \ldots)$ of sh Lie $s$-algebroids on $\mathbb{L}$ are *not* equivalent. We note, as an example, that

$$m_{\Gamma 1} = m_1 + \sum_{n=2}^\infty \frac{1}{(n-1)!} m_n(\Gamma, \ldots, \Gamma, \quad ) \tag{4.53}$$

structure of of Lie algebroid on the graded space $\mathbb{L}$. The second equation above may be viewed as the condition that $M_2$ defines a cocycle. Assume that $M_n = 0$ for $\forall n \geq 3$, we may call above as a structure of $s$-Lie bi-algebroid. For $M_n = 0$ for $\forall n \geq 4$, we may call above as a structure of quasi $s$-Lie bi-algebroid etc. Some important examples of such structure are in refs. [11, 32, 27, 20, 25].

and, by construction, $m^2_{\Gamma 1} = 0$, which is equivalent to the equation (4.51). We call a solution $\Gamma \in \mathfrak{l}_s$ of the equation (4.51) strongly homotopy Dirac $s$-structure. As $\partial_\Gamma : \mathfrak{l}_k \to \mathfrak{l}_{k+1}$ and $\partial_\Gamma^2 = 0$ we have complex $(\partial_\Gamma, \mathfrak{l})$ and associated non-linear (co)homology by "Ker $\partial_\Gamma / \operatorname{Im} \partial_\Gamma$".

Now we specialize to the case that $s$ is odd. Then the structure of symplectic $(s+1)$-algebra on $T^*[s]\mathbb{L}$ enhances to the structure of BV $(s+1)$-algebra. Namely there exist $\Delta : \mathfrak{t}_k \to \mathfrak{t}_{k-s}$ satsifying $\Delta^2 = 0$;

$$\Delta = (-1)^{|q^a|+1} \frac{\partial_r^2}{\partial q^a \partial p_a}, \tag{4.54}$$

and generate the bracket $[.,.]_{-s}$. For $s = -1$ this is (finite dimensional version) of the original BV structure in Sect. 2. We remark that the above $\Delta$ should *not* be confused with the BV operator $\boldsymbol{\Delta}$ in the space of all fields $\boldsymbol{T}$.

This is the end of the digression and let's justify why the above considerations are relevant.

#### 4.2.3 Boundary Deformations

Consider the classical BV action functional

$$\mathbf{S}_H^{(0)} = \oint_{T[1]M_{s+1}} \Big( \widehat{p}^a \widehat{d q}^a + H(\widehat{q}, \widehat{p}) \Big), \tag{4.55}$$

where we assume boundary condition (4.41) and $H \in \mathfrak{t}_{s+1}$ satisfies

$$[H, H]_{-s} = 0, \qquad H|_{\mathbb{L}} = 0. \tag{4.56}$$

Then we may rewrite (4.55) as follows

$$\mathbf{S}_H^{(0)} = \oint_{T[1]M_{s+1}} \left( \widehat{p}^a \widehat{d q}^a + \sum_{n=1}^{\infty} \frac{1}{n!} m(\widehat{q})^{a_1 \ldots a_n} \widehat{p}_{a_1} \cdots \widehat{p}_{a_n} \right). \tag{4.57}$$

Let's now consider canonical transformation generated by

$$\boldsymbol{\Psi} = \oint_{M_{s+1}} \Psi(\widehat{q}, \widehat{p}), \tag{4.58}$$

where $\Psi(q,p) \in \mathfrak{t}_s$ such that $\boldsymbol{\Psi} \in \mathfrak{T}_{-1}$. Let $\Gamma(q) = \Psi(q,p)|_{\mathbb{L}} \in \mathfrak{l}_{-s}$. The action functional $\mathbf{S}_{H_o}^{\boldsymbol{\Psi}}$ after the resulting canonical transformation is given

by

$$\begin{aligned}\mathcal{S}_H^{(0)\Psi} =& \oint_{T[1]M_{s+1}} \left(\widehat{p}_a d\widehat{q}^a + H_\Psi\left(\widehat{q},\widehat{p}\right)\right) + \oint_{T[1]M_{s+1}} \widehat{d}\Psi(\widehat{q},\widehat{p}) \\ =&\mathcal{S}_{H_\Psi} + \oint_{T[1](\partial M_{s+1})} \Gamma(\widehat{q}),\end{aligned} \tag{4.59}$$

where we used the boundary condition after using the Stokes theorem. The above action functional also satisfy the master equation if and only if $H_\Psi|_{\mathbb{L}} = 0$. On the other hand the value $H_\Psi|_{\mathbb{L}}$ equals to $H_\Gamma|_{\mathbb{L}}$. Thus we have $H_\Gamma|_{\mathbb{L}} = 0$ for the master equation. We also note that the canonical transformation generated a boundary interaction term depending only $\Gamma = \Psi|_{\mathbb{L}}$. Thus it is obvious that $\Psi \in \mathfrak{t}_s$ satisfying $\Psi|_{\mathbb{L}} = 0$ leads to the same physical theory. We called canonical transformation generated by such a $\Psi$ skrooching. On the other hand $\Gamma \in \mathfrak{l}_s$ leads to a non-zero boundary interaction terms. According to our definition an element $\Gamma \in \mathfrak{l}_s$ satisfying $H_\Gamma|_{\mathbb{L}} = 0$ is a strongly homotopy Dirac $s$-structure on $\mathbb{L}$, which is defined as a solution of Maurer-Cartan equation (4.52) of the structure of strongly homotopy Lie $s$-algebroid, defined by $H$, on $\mathbb{L}$. Thus the set of equivalence classes of strongly homology Dirac $s$-structure on $\mathbb{L}$ is isomorphic to the moduli space of boundary deformations for the fixed bulk background $H \in \mathfrak{t}_{s+1}$.

### 4.2.4 Extended Bulk/Boundary Deformations

Now we consider bulk deformations compatible with boundary condition. Consider a deformation $\mathcal{H}$ (4.28) of $H$ satisfying the following equation

$$\begin{cases} [\mathcal{H},\mathcal{H}] = 0, \\ \quad \mathcal{H}|_{\mathbb{L}} = 0. \end{cases}, \tag{4.60}$$

where $\mathcal{H} \in (\mathfrak{t}\otimes\mathfrak{a})_{s+1}$. Thus $\mathcal{H}$ satisfying above induce a structure of *extended* strongly homotopy Lie $s$-algebroid $(\mathfrak{l}, (\mu_1, \mu_2, \mu_3, \ldots))$ on $\mathbb{L}$. Note that $\mu_1^2 = 0$. As $\mu_1 : (\mathfrak{l}\otimes\mathfrak{a})_k \to (\mathfrak{l}\otimes\mathfrak{a})_{k+1}$ and $\mu_1^2 = 0$ we have complex $(\mu_1, \mathfrak{l})$ and associated (co)homology by "Ker $\mu_1/\operatorname{Im}\mu_1$". We define the extended bulk moduli space $\mathfrak{N}(\mathbb{L})_{s+1}$ by the set of solutions of (4.60) modulo equivalence, defined by the adjoint action of an element in $(\mathfrak{t}\otimes\mathfrak{a})_1$ vanishing on the Lagrangian subspace $\mathbb{L}$ in $\mathbb{T}_{s+1} \simeq T^*[s]\mathbb{L}$ - this may be called extended skrooching. We call two structures of extended strongly homotopy Lie $s$-algebroids on $\mathbb{L}$ are equivalent if they are related by extended skrooching.

Thus $\mathfrak{N}(\mathbb{L})_{s+1}$ parametrize the set of equivalence classes of structures of extended sh Lie $s$-algebroid on $\mathbb{L}$.

It is now natural to consider extended boundary interactions via extend sh Dirac 1-structure on $\mathbb{L}$ defined by elements $\Upsilon \in (\mathfrak{l} \otimes \mathfrak{a})_s$ satisfying

$$\mathcal{H}_\Upsilon|_\mathbb{L} = 0. \tag{4.61}$$

Equivalently

$$\mu_1(\Upsilon) + \frac{1}{2}\mu_2(\Upsilon, \Upsilon) + \frac{1}{3!}\mu_3(\Upsilon, \Upsilon, \Upsilon) + \ldots = 0. \tag{4.62}$$

Then $\mathfrak{H}_\Upsilon$ induces another quantizable structure $(\mu_{\Upsilon 1}, \mu_{\Upsilon 2}, \mu_{\Upsilon 3}, \ldots)$ of strongly homotopy Lie 1-algebroid on $\mathbb{L}$, where

$$\mathfrak{d}_\Upsilon = \mathfrak{d} + \sum_{n=2}^{\infty} \frac{1}{(n-1)!}\mu_n(\Upsilon, \ldots, \Upsilon, \quad) \tag{4.63}$$

and, by construction, $\mathfrak{d}_\Upsilon^2 = 0$, which is equivalent to the equation (4.62). The two structures $(\mathfrak{d}, \mu_2, \mu_3, \ldots)$ and $(\mathfrak{d}_\Upsilon, \mu_2^\Upsilon, \mu_2^\Upsilon, \ldots)$ of extended sh Lie 1-algebroids on $\mathbb{L}$ are not equivalent. As $\mathfrak{d}_\Upsilon : (\mathfrak{l} \otimes \mathfrak{a})_k \to (\mathfrak{l} \otimes \mathfrak{a})_{k+1}$ and $\mathfrak{d}_\Upsilon^2 = 0$ we have complex $(\mathfrak{d}_\Upsilon, (\mathfrak{l} \otimes \mathfrak{a}))$ and associated non-linear (co)homology by "Ker $\partial_\Gamma / \operatorname{Im} \partial_\Gamma$".

It is obvious that the boundary interaction depend on the bulk background or the bulk moduli space $\mathcal{M}_\mathbb{L}$. Thus the total moduli space $\mathfrak{F}_\mathbb{L}$ of both the bulk and boundary deformations has a structure of fibered space $\mathfrak{F}_\mathbb{L} \to \mathfrak{M}_\mathbb{L}$ such that

$$\begin{array}{ccc} \mathfrak{B}_t & \subset & \mathfrak{F}_\mathbb{L} \\ \uparrow & & \downarrow \\ \{t\} & \in & \mathfrak{M}_\mathbb{L} \end{array} \tag{4.64}$$

### 4.2.5 Deformation Quantization of $d$-Algebra

Our program is closely related with the deformation theory of $d$- algebra [23]. Recently there has been many spectacular developments in deformation theory of associative algebras [16] (or 1-algebras from now on), following the first solution of deformation quantization by Kontsevich [22]. An amusing result is that deformation theory of 1-algebras is closely related with the geometry of configuration space of points on 2-dimensions. Another beautiful result is that the deformation complex, the Hochschild complex $\mathtt{Hoch}(A_1)$, of 1-algebra ($d$-algebra in general) $A_1$ has a structure of 2-algebra ($(d+1)$-algebra) [23, 38].

Let $\mathbb{L}$ be any $\mathbb{Z}$-graded smooth algebraic variety. Let $A_s(\mathbb{L})$ be the algebra of functions on $\mathbb{L}$ regarded as an $s$-algebra. Let $\mathtt{Hoch}(A_s(\mathbb{L}))$ be the Hochschild complex of $A_s(\mathbb{L})$ and let $H^\bullet(\mathtt{Hoch}(A_s(\mathbb{L})))$ be the cohomology. An important lemma of Kontsevich is that the space $\oplus H^\bullet(\mathtt{Hoch}(A_s(\mathbb{L}))$ is isomorphic to the space $\mathfrak{t}$ of functions on the total space $\mathbb{T}_{s+1} = T^*[s]\mathbb{L}$ of twisted by $U = [s]$ cotangent bundle to $\mathbb{L}$.

It is natural to identify a degree $(s+1)$-function $H^{s+1}$ on $T^*[s]\mathbb{L}$ with $H|_{\mathbb{L}_s} = 0$ as an element of $H^1(\mathtt{Hoch}(A_s(\mathbb{L})))$. The first cohomology of any Hochschild complex of an algebra is naturally corresponds to the infinitesimal determining the first order deformation of the algebra. In the present case our $H^{s+1}$ satisfying the "master" equations (2.7) can be interpreted as the infinitesimal for deformations of the $s$-algebra $A_s(\mathbb{L})$ as an $s$-algebra. This is the first mathematical clue for what kind of quantum algebras we are dealing with. Our approach also gives a natural "explanation" why deformation theory of $s$-algebra is related with differential-geometry of $(s+1)$-dimensions.

We should note that the open $s$-braneoid theory at the level of action functional see the structure of $(s+1)$-algebra of the cohomology $H^*(\mathtt{Hoch}(A(\mathbb{L})_s))$ rather than that of the Hochschild complex $\mathtt{Hoch}(A(\mathbb{L})_s)$. There is a fundamental theorem that there is a structure of $(s+1)$-algebra on the Hochschild complex of any $s$-algebra. Such is an $(s+1)$-algebra is called *formal* if it is quasi-isomorphic to its cohomology. The formality means that the set of equivalence class of solutions of Maurer-Cartan equation (2.7) of the cohomological $(s+1)$-algebra is isomorphic to the set of equivalence class of solutions of Maurer-Cartan equation of a $(s+1)$-algebra structure on $\mathtt{Hoch}(A_s(\mathbb{L}))$.

Assume that $N_s$ bounds an oriented compact $(s+1)$-dimensional manifold $M_{s+1}$. We may regard a topological open $s$-braneoid theory on $M_{s+1}$ as closed $(s-1)$-brane theory on $\partial M_{s+1} = N_s$ with 'bulk" deformations specified by $\mathfrak{H}^{s+1}$. The boundary sector of the theory may be viewed as the theory maps $\boldsymbol{\varphi} : \partial M_{s+1} \longrightarrow \mathbb{L}$. Recall a solution of the "master" equation (2.7) induces a structure of strongly homotopy $(s-1)$-Poisson structure on $\mathbb{L}$ or, equivalently the structure $(\mathfrak{f}_{\mathbb{L}}; (\mu_1, \mu_2, \mu_3, \ldots)_{\mathbb{L}})$ of strongly homotopy Lie $(s-1)$-algebroid or, simply, of $s$-algebra. A special case of such structure is a degree $U = s-1$ symplectic structure on $\mathbb{L}$ or, equivalently, the structure of strongly homotopy Lie $(s-1)$-algebroid with $\mu_n = 0$ for all $n$ except for $n = 2$ and $\mu_2$ is non-degenerated. Then we just have the standard topological closed $(s-1)$-brane theory associated with $\mathbb{T}_s = \mathbb{L}$ such that $\{.,.\}_{s-1} \equiv \mu_2$. In general we may use the above topological open $s$-brane on $M_{s+1}$ to define differential-topological invariants of the boundary $N_s = M_{s+1}$

by the correlation functions of the boundary BV observables. In other words topological open $s$-brane associates any $s$-algebra -strongly homotopy $(s-1)$-algebroid, with differential-topological invariants of $s$-dimensional manifold, which bounds $(s+1)$-dimensional space.

It seems to be reasonable to believe that the perturbation expansions of the open $s$-brane theory above generates elements of Hochschild complex $\mathtt{Hoch}(A(\mathbb{L})_s)$ and the BV Ward identity of the theory[17] is equivalent to the Maurer-Cartan equation for $\mathtt{Hoch}(A(\mathbb{L})_s)$. The above "principle" is beautifully demonstrated by Cattaneo-Felder for $s=1$ and $\mathbb{L}=X$ is a Poisson manifold in their path integral approach to Kontsevich's formality theorem [5]. We emphasis that the Maurer-Cartan equation for the cohomological $(s+1)$-algebra $H^*(\mathtt{Hoch}(A(\mathbb{L})_s))$ is equivalent to the BV master equation of our $s$-brane theory and the BV Ward identity is a direct consequence, at lease formally, of the BV master equation. A crucial point is that the BV Ward identity depends differential-topology of $M_{s+1}$ (including configurations space of points on $M_{s+1}$).

The above discussion seem to indicate that they may be fundamental relations between differential-topology of $d$-dimensions and the world of $d$-algebras. It could be more precise to state that (quantum) $d$-algebra should be defined in terms of differential-topology of $d$-dimensions as detected by path integrals. Kontsevich conjectured that there are structures of $d$-algebra in conformal field theory on $\mathbb{R}^d$ with motivic Galois group action on the moduli space [23]. Our construction naturally suggest that the conjecture can be naturally generalized to $d$-braneoids on any orient smooth $(d+1)$-dimensional manifold. All those direct us to certain universal properties of Feynman path integrals of the theory related with differential-topology, arithmetic geometry as well as number theory.

#### 4.2.6 Multiple Boundaries

So far we assume that $M_{s+1}$ has only a single boundary component. Now we relax the condition by allowing $\partial M_{s+1}$ has multiple components. For each component $N_i$ of $\partial M_{s+1}$, we pick a Lagrangian subspace $\mathbb{L}_i$ in $(\mathbb{T}_{s+1},\omega_s)$ and assign boundary condition prescribed before. Now the classical BV master equations requires that

$$H|_{\mathbb{L}_i}=0, \quad \text{for } \forall i \tag{4.65}$$

[17] The BV Ward identity is an identity of path integral as the result of BV master equations

in addition to the condition $[H,H]_{-s}=0$. Then under the canonical transformation generated by $\boldsymbol{\Psi}$ in (4.58) we have

$$\boldsymbol{S}_H^{(0)\boldsymbol{\Psi}} = \boldsymbol{S}^{(0)}_{H_\Psi} + \sum_i \oint_{T[1]N_i} \Gamma_i, \tag{4.66}$$

where $\Gamma_i = \Psi|_{\mathbb{L}_i}$. Above action functional satisfies the master equation if and only if $H_\Psi|_{\mathbb{L}_i} = 0$ for $\forall i$. Thus it is obvious that $\Psi \in \mathfrak{t}_s$ satisfying $\Psi|_{\cup_i \mathbb{L}_i} = 0$ leads to the same physical theory.

Let's consider, as a simplest example, the case that $\partial M_{s+1} = N_1 \cup N_2$ and the associated Lagrangian subspaces $\mathbb{L}_1$ and $\mathbb{L}_2$ in $(\mathbb{T}_{s+1}, \omega_s)$ are complementary with each others defined by $p_a = 0$ and $q^a = 0$,respectively, for $\forall a$. We may expand $H$ as $H = \sum_{k,\ell=1}^{\infty} \frac{1}{k!\ell!} m^{a_1\ldots a_k}_{b_1\ldots b_\ell} q^{b_1}\cdots q^{b_\ell} p_{a_1}\cdots p_{a_n}$, and for each $\mathbb{L}_1$ and $\mathbb{L}_2$ we have structures of sh Lie $s$-algebroid, which, together, may be called sh Lie $s$-bialgebroids. Now we have the associated classical BV action functional

$$\boldsymbol{S}_H^{(0)} = \oint_{T[1]M_{s+1}} \left( \widehat{p}_a \widehat{d}\widehat{q}^a + \sum_{k,\ell=1}^{\infty} \frac{1}{k!\ell!} m^{a_1\ldots a_k}_{b_1\ldots b_\ell} \widehat{q}^{b_1}\cdots \widehat{q}^{b_\ell} \widehat{p}_{a_1}\cdots \widehat{p}_{a_n} \right). \tag{4.67}$$

Note that the (super) propagator exist between $\widehat{p}_a$ and $\widehat{q}^a$ only such that at each boundary there are no propagation and interaction takes place only at the bulk. The tree-level interactions correspond to the strucuture of *classical* sh Lie $s$-bialgebroids, while the higher order (in $\hbar$) corrections would lead to *quantum* sh Lie $s$-bialgebroids We may say the QFT defines some kind of quantum cobordism.

### 4.3 Toward Quantum Clouds

This paper has a serious limitation to unveil quantum world. The general results in section 3 is based on assumption that $\boldsymbol{\Delta}$ exists without proper definition of it, while using $\boldsymbol{\Delta}$ crucially. In section 5, where we present realistic case of $d$-dimensional QFTs and their family, we simply ignored $\boldsymbol{\Delta}$, while written in the quantum perspective relying on $\boldsymbol{\Delta}$. Let $M_n$ be a $n$-manifold with or without boundary. Let $\{\phi(\sigma)^a, \phi(\sigma)^{\bullet}_a\}$ be `fields` and `anti-fields` of certain model discussed in section 5 with suitable boundary conditions, where $\{\sigma^\mu\}$ is a local coordinates system in $M_n$ and $\{a\}$ denotes all the discrete indices in the model. Then $\boldsymbol{\Delta}$ is naively "defined" as follows

$$\boldsymbol{\Delta}\text{"}=\text{"} \lim_{\tau^\mu\to\sigma^\mu} \sum_a \int \sqrt{g} d^n\tau \int \sqrt{g} d^n\sigma \left( (-1)^{|\phi^a|+1} \frac{\delta^2}{\delta\phi(\tau)^{\bullet}_a \delta\phi(\sigma)^a} \right) \tag{4.68}$$

involving diagonal of $M_n \times M_n$ which requires suitable regularization. Related to above Feynman propagators are certain differential forms on suitably compactified configuration space of points on $M_n$. Also, due to nonlocal observables, supported on various cycles, we need to worry about space of cycles in $M_n$. The pressing problem is to define $\mathbf{\Delta}$ correctly and universally (before doing any gauge fixing) in each dimensions for general smooth manifold with or without boundary (this will take care of the crucial renormalization of QFT). We remark that a closely related problem, though in a limited situation (loop space), has been dealt in string topology of Chase and Sullivan [6]. Once this is achieve QFTs are largely characterized by certain $\mathcal{K}_{\mathbb{S}}$-cohomology ring, for $\mathbf{\Delta} e^{-\mathbb{S}/\hbar} = 0$, as we demonstrated in section 3. Then we may concentrate on structures associated with (correct) moduli space of QFTs for both generic part and singular points, which give rise to various perturbative QFTs. As for mathematical side, it will give us an universal quantization machine for $d$-algebras decorated by algebro-differential-topology of $d/(d+1)$-manifolds.

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