Real-Time Perturbation Theory in de Sitter Space

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Abstract

We consider scalar field theory in de Sitter space with a general vacuum invariant under the continuously connected symmetries of the de Sitter group. We begin by reviewing approaches to define this as a perturbative quantum field theory. One approach leads to Feynman diagrams with pinch singularities in the general case, which renders the theory perturbatively ill-defined. Another approach leads to well-defined perturbative correlation functions on the imaginary time continuation of de Sitter space. When continued to real-time, a path integral with a non-local action generates the time-ordered correlators. Curiously, observables built out of local products of the fields show no sign of this non-locality. However once one couples to gravity, we show acausal effects are unavoidable and presumably make the theory ill-defined. The Bunch-Davies vacuum state is the unique de Sitter invariant state that avoids these problems.

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I. INTRODUCTION

There has been much recent debate about whether quantum field theory in de Sitter space has a unique vacuum invariant under all the continuously connected symmetries of the space. The resolution of this question is crucial to the understanding of possible trans-Planckian effects on the predictions of inflation [1], and observable effects today such as ultra high energy cosmic ray production [2, 3]. These questions are all the more pressing given recent experimental results confirming general predictions of inflation for the cosmic microwave background [4], and of supernova observations consistent with a positive cosmological constant today [5].

At the level of free field theory, de Sitter space has a one-complex parameter family of vacua, dubbed the $\alpha$-vacua [6, 7, 8, 9]. It was been argued cutoff versions of these can be relevant during inflation, where $\alpha$ parameterizes the effects of trans-Planckian physics [10, 11, 12, 13, 14, 15, 16]. Others have argued the $\alpha$-vacua suffer from inconsistencies [17, 18, 19, 20] once interactions are included, and that the Bunch-Davies/Euclidean vacuum state is the unique consistent state.

In this paper we review existing approaches to this issue, and elaborate on the connections between them. The most straightforward approach, where one treats the vacuum state as a squeezed state fails due to the appearance of pinch singularities, which renders the perturbation theory ill-defined [18]. We emphasize this is not a problem with the ultra-violet structure of the theory, but rather Feynman integrals become ill-defined when propagators on internal lines are null separated.

A potentially more promising approach based on an imaginary time formulation [21] leads to a sensible perturbation theory, and propagators that agree with the imaginary time continuations of the free propagators of [8, 9]. This perturbative expansion can be continued to real-time and written in terms of a path integral with a non-local kinetic term, but local potential and source terms. For the pure scalar field theory, the algebra of observables built out of local products of the fields remains local. However once the theory is coupled to gravity the acausality becomes unavoidable and presumably renders the theory ill-defined, in keeping with the chronology protection conjecture [22].
II. FREE PROPAGATOR

To establish notation, we begin by reviewing the results of [6,7,8,9] for the free vacua in de Sitter space, invariant under the elements of the de Sitter group continuously connected to the identity. Fields may be decomposed as mode sums

\[
\phi(x) = \sum_n \phi_n^E(x) a_n + \phi_n^{E*}(x) a_n^\dagger
\]

\[
= \sum_n \phi_n^\alpha(x) b_n + \phi_n^{\alpha*}(x) b_n^\dagger.
\]

One then defines the Bunch-Davies/Euclidean vacuum as

\[ a_n|E\rangle = 0 \]

and the \( \alpha \)-vacua as

\[ b_n|\alpha\rangle = 0. \tag{1} \]

The respective mode functions are related as

\[
\phi_n^\alpha = N_\alpha (\phi_n^E + e^{\alpha} \phi_n^{E*}),
\]

\[
\phi_n^E = N_\alpha (\phi_n^\alpha - e^{\alpha} \phi_n^{\alpha*}), \tag{2}
\]

with \( N_\alpha = 1/\sqrt{1 - \exp(\alpha + \alpha^*)} \). The creation and annihilation operators are then related by a mode number independent Bogoliubov transformation

\[ b_n = N_\alpha (a_n - e^{\alpha^*} a_n^\dagger). \tag{3} \]

As shown in [9] we can choose mode functions so that

\[ \phi_n^E(x)^* = \phi_n^E(\bar{x}) \tag{4} \]

with \( \bar{x} \) the anti-podal point to \( x \). These mode functions are normalized with respect to the norm

\[
(\phi_1,\phi_2) = i \int_{\Sigma} (\phi_1^* \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1^*) d\Sigma^\mu
\]

with \( \Sigma \) any Cauchy surface.
The Wightman function is
\[
G^\alpha(x, y) = \langle \alpha | \phi(x) \phi(y) | \alpha \rangle = \sum_n \phi_n^\alpha(x) \phi_n^{\alpha*}(y) = N_\alpha^2 \left( G^E(x, y) + e^\alpha G^E(\bar{x}, y) + e^{\alpha*} G^E(x, \bar{y}) + |e^\alpha|^2 G^E(\bar{x}, \bar{y}) \right).
\]
(5)

The state |\alpha\rangle can be thought of as a squeezed state with respect to the Euclidean vacuum

\[
|\alpha\rangle = U |E\rangle
\]

with the unitary operator \( U \) defined as
\[
U = \exp \left( \sum_n \beta \left( a_n^E \right)^2 - \beta^* \left( a_n^E \right)^2 \right), \quad \beta = \frac{1}{4} \left( \log \tanh \frac{-\text{Re} \alpha}{2} \right) e^{-i \text{Im} \alpha}.
\]

It is then natural to construct \[21\]
\[
\tilde{\phi}(x) = U^\dagger \phi(x) U = N_\alpha \sum_n \left( \phi_n^E(x) + e^\alpha \phi_n^E(\bar{x}) \right) a_n + \left( \phi_n^E(x) + e^{\alpha*} \phi_n^E(\bar{x}) \right) a_n^\dagger
\]
which suggests that from the Euclidean vacuum viewpoint, creation of a particle in the \( \alpha \)-vacuum can be thought of as creating a particle with respect to the Euclidean vacuum at \( x \) together with a particle at the anti-podal point \( \bar{x} \).

## A. Real-time ordering

Having discussed the Wightman functions, we now need to discuss more carefully time-ordering prescriptions. First let us represent de Sitter space as a hyperboloid in flat \( \mathbb{R}^5 \) with metric \( \eta_{ab} = \text{diag}(-1, 1, 1, 1, 1) \) and coordinates \( X^a \) with \( a = 1 \cdots 5 \)
\[
X^a X^b \eta_{ab} = H^{-2}.
\]

Following \[9\] we define the signed geodesic distance between points as
\[
\bar{d}(x, y) = H^{-1} \arccos \bar{Z}(x, y)
\]
where
\[
\bar{Z}(x, y) = \begin{cases} H^2 \eta_{ab} X^a(x) X^b(y) + i \epsilon, & \text{if } x \text{ to the future of } y \\ H^2 \eta_{ab} X^a(x) X^b(y) - i \epsilon, & \text{if } x \text{ to the past of } y. \end{cases}
\]
With this definition \( \tilde{d}(x, y) = -\tilde{d}(\bar{x}, \bar{y}) \). Not that although only points with \( Z \geq -1 \) are connected by geodesics, \( \tilde{d}(x, y) \) can be defined by analytic continuation for \( Z < -1 \).

The Euclidean vacuum Wightman function is given by

\[
G^E(x, y) = c \, _2F_1(h_+, h_-; 2; \frac{1 + \tilde{Z}}{2})
\]

where

\[
h_{\pm} \equiv \frac{3}{2} \pm i\mu
\]

\[
\mu \equiv \sqrt{m^2 - \left( \frac{3H}{2} \right)^2}
\]

\[
c \equiv \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^2}.
\]

Unless otherwise stated, we consider the case \( m > 3H/2 \) in this paper. Some of the properties of this function are as follows:

- a pole when points coincide (\( \tilde{Z} = 1 \))
- a branch cut running along \( \tilde{Z} = (1, \infty) \), where the imaginary part changes sign
- and asymptotically as \( |\tilde{Z}| \to \infty \)

\[
G^E(x, y) \propto \left( -\tilde{Z} \right)^{-h_+} \frac{\Gamma(h_+ - h_-)}{\Gamma(h_-)} + \left( -\tilde{Z} \right)^{-h_-} \frac{\Gamma(h_- - h_+)}{\Gamma(h_+)} \frac{\Gamma(h_-)}{\Gamma(h_- - 1)}.
\]

Using (5) and (7), the general \( \alpha \)-vacuum Wightman function is then explicitly constructed. Note the general \( \alpha \)-vacuum Wightman function has poles both at \( \tilde{Z} = 1 \) and \( \tilde{Z} = -1 \).

There are a number of options for defining real-time ordering of the two-point functions described above. Conventional definitions \( [9] \) correspond to

\[
iG^E_F(x, y) = \theta(x^0 - y^0)G^E(x, y) + \theta(y^0 - x^0)G^E(y, x)
\]

and

\[
iG^\alpha_F(x, y) = \theta(x^0 - y^0)G^\alpha(x, y) + \theta(y^0 - x^0)G^\alpha(y, x).
\]

with \( x^0 \) a global real-time coordinate. These Green functions satisfy the inhomogeneous Klein-Gordon equation

\[
(\Box - m^2)G_F(x, y) = -\frac{\delta^4(x - y)}{\sqrt{-g(x)}}, \quad g(x) = \det g_{\mu\nu}(x)
\]
with $g_{\mu\nu}$ the spacetime metric. Note $\delta(x - \bar{y})$ does not appear. The propagator $\mathcal{G}$ can be written as

$$iG_E^\alpha(x, y) = c_2 F_1(h_+, h_-; 2; \frac{1 + Z'}{2})$$

where $Z'$ is defined with the new $i\epsilon$ prescription

$$Z'(x, y) = H^2 \eta_{ab} X^a(x) X^b(y) + i\epsilon.$$ 

However, as discussed in $[23]$, another natural time-ordering in the $\alpha$-vacuum is obtained by ordering the respective terms of $\mathcal{G}$ according to the arguments of the mode functions, $x$ and $\bar{x}$ (when $\alpha$ is real)

$$i\mathcal{G}_\alpha^{EL}(x, y) = N_\alpha^2 \left( \theta(x, y) \left( 1 + |\alpha|^2 \right) G^E(x, y) + \theta(y, x) \left( 1 + |\alpha|^2 \right) G^E(y, x) + 2\theta(\bar{y}, x) \epsilon^\alpha G^E(\bar{y}, x) + 2\theta(x, \bar{y}) \epsilon^\alpha G^E(x, \bar{y}) \right).$$

Note these Green functions do not satisfy the asymptotic boundary condition $([1])$ in the infinite past or future. So contrary to $[23]$, do not satisfy the correct boundary conditions for the description of a squeezed state. These propagators satisfy the non-local inhomogeneous Klein-Gordon equation $[23]$

$$(\Box - m^2)\tilde{G}_\alpha^{EL}(x, y) = -N_\alpha^2 \left( 1 + |\alpha|^2 \right) \frac{\delta^4(x - y)}{\sqrt{-g(x)}} + 2\epsilon^\alpha \frac{\delta^4(x - \bar{y})}{\sqrt{-g(x)}}$$

so are to be interpreted as corresponding to source boundary conditions on a linear combination of nodal $(y)$ and anti-nodal points $(\bar{y})$.

Another natural time ordering is

$$i\mathcal{G}_E^\alpha(x, y) = N_\alpha^2 \left( \theta(x, y) \left( 1 + |\alpha|^2 \right) G^E(x, y) + \theta(y, x) \left( 1 + |\alpha|^2 \right) G^E(y, x) \right)$$

$$\theta(\bar{y}, x) \left( e^\alpha + e^{\alpha^*} \right) G^E(\bar{y}, x) + \theta(x, \bar{y}) \left( e^\alpha + e^{\alpha^*} \right) G^E(x, \bar{y})$$

$$= iN_\alpha^2 \left( \left( 1 + |\alpha|^2 \right) G^E_F(x, y) + \left( e^\alpha + e^{\alpha^*} \right) G^E_F(x, \bar{y}) \right)$$

which is obtained by time ordering the arguments of the propagators appearing in $[23]$. This agrees with the propagator of $[23]$ when $\alpha$ is real, and generalizes it when $\alpha$ is complex.

As we will see, these propagators appear on internal lines, when one analytically continues from the imaginary time formulation of $[21]$ to real-time. This propagator satisfies the inhomogeneous Klein-Gordon equation

$$(\Box - m^2)\tilde{G}_E^\alpha_F(x, y) = -N_\alpha^2 \left( 1 + |\alpha|^2 \right) \frac{\delta^4(x - y)}{\sqrt{-g(x)}} + \left( e^\alpha + e^{\alpha^*} \right) \frac{\delta^4(x - \bar{y})}{\sqrt{-g(x)}}.$$ 

This Feynman propagator can be written in terms of hypergeometric functions using $[10]$. 

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III. INTERACTING THEORY

A. Squeezed state approach

The most direct approach to setting up the real-time perturbation theory is to define Green functions using the interaction picture representation

$$G(x_1, \cdots, x_n) = \langle E| U^\dagger T \left( \phi(x_1) \cdots \phi(x_n) e^{i S_{int}^{(2)}} \right) U|E \rangle$$

(14)

where $S_{int}$ is the interacting part of the action, and $T$ denotes time-ordering with respect to global time. One can view $U|E\rangle$ as a squeezed state defined in the infinite asymptotic past/future of de Sitter and these Green functions may then be used to extract an S-matrix. This expression may be expanded in powers of the interaction by conventional means, and the end result involves Feynman propagators ordered with respect to the global time coordinate $\tau$.

As emphasized in [18], pinch singularities arise in these expressions which render the integrals ill-defined. This may be seen in the following example of a diagram that occurs with a $\lambda \phi^4$ interaction. The lower loop gives rise to the factor

$$\int_{dS_4} d^4x \sqrt{-g} G_{\mu\nu}^\alpha(x, y)^2$$

(15)
which contains terms like

\[ \int_{dS^4} d^4 x \sqrt{-g} G^E_F(x, y) G^E_F(\bar{x}, \bar{y}) . \]

Writing this in terms of hypergeometric functions \[[1]\], we see the \(i\epsilon\) prescriptions differ in the two propagators. The contribution coming from the region of integration where \(x\) is null related to \(y\) gives a term proportional to

\[ \int dZ \frac{1}{Z - 1 + i\epsilon} \frac{1}{Z - 1 - i\epsilon} = \frac{\pi}{\epsilon} \]

(16)

where \(Z\) is integrated along the real axis. As \(\epsilon \to 0\) the poles pinch the integration contour and the integral diverges. We emphasize this divergence has nothing to do with the ultraviolet structure of the theory, so cannot be regulated with local (or even bi-local) counter-terms.

This divergence implies that perturbation theory does not make sense as it stands. It is conceivable that some resummation of perturbation theory does make sense, but we lack methods to address this kind of approach in a completely systematic way. We note this type of resummation is attempted in non-equilibrium statistical field theory where one similarly encounters pinch singularities \[[24, 25]\]. This resummation can lead to a shifting in the poles of the propagator, so that the \(i\epsilon\) in (16) is replaced by an \(i\Gamma\) where \(\Gamma\) is a decay rate, proportional to some power of the interactions. It is then clear from (16) that the resummed theory will be difficult to handle due to the appear of inverse powers of the coupling.

**B. Imaginary-time approach**

Since the direct approach to treating the \(\alpha\)-vacua as squeezed states in de Sitter space is a nonstarter, one can try to appeal to an imaginary time formulation \[[21]\]. This provides us with a straightforward way to deal with spacetimes with event horizons, since for imaginary time the event horizon shrinks to a point. In the black hole case, field theory on the imaginary time continuation (also called the Euclidean section) of a black hole background leads to a density matrix description from the real time point of view. One might have hoped a similar novel interpretation of the \(\alpha\)-vacua emerges in the real-time point of view, due to the cosmological horizon of de Sitter space.

To discuss the continuation from real time to imaginary time we use global coordinates

\[ ds^2 = -dt^2 + (\cosh t)^2 d\Omega_3^2 \]
so $t \to i\tau$ takes us to imaginary time. Here we have chosen units where $H = 1$. The imaginary time continuation of de Sitter is the four-sphere. The imaginary time approach of \cite{21} proceeds by using the transformation \cite{11} to express $\alpha$-vacuum Green functions as linear combinations of Euclidean vacuum Green functions

$$G(x_1, \ldots, x_n) = \langle E | U^\dagger \phi(x_1) \cdots \phi(x_n) e^{i\Sigma_{int}(\phi)} U | E \rangle = \langle E | \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) e^{i\Sigma_{int}(\phi)} | E \rangle . \quad (17)$$

This is to be understood as an interaction picture expression. An important point is that on the Euclidean section, the free propagators $G^\alpha(x, y)$ are symmetric functions of their arguments because points are spacelike separated. The ordering of operators in this expression is therefore irrelevant. As described in \cite{21}, the ultraviolet divergences that appear in this approach can be canceled by de Sitter invariant local counter-terms. This approach yields free two-point functions that match those of Mottola and Allen on the Euclidean section, and it is in this sense the approach is a generalization of the $\alpha$-vacuum to the interacting case. However the real-time physics of this approach is so far mysterious, and we wish to explore this question in the following.

C. Continuation to real time

Let us examine what happens when we continue integrals of products of $G^\alpha(x, y)$ on the sphere to integrals over de Sitter space. We are free to expand \cite{17} as in \cite{15} and order the arguments in any way convenient for analytically continuing to real-time. Let us first focus on the case of the Euclidean vacuum $e^\alpha = 0$. We begin with the imaginary time contour as shown in figure \cite{22} running from $-i\pi$ to $i\pi$. This may be continued to the contour shown in figure \cite{23}. The horizontal component (with a small positive slope) running from $-\infty$ to $\infty$ through points $x_1$ to $x_4$ gives rise to the expected real-time contour. The corresponding $i\epsilon$ prescription is $t \to t + i\epsilon \text{ sgn } t$ which gives a Feynman propagator connecting internal lines \cite{10}. The vertical components of the contour are closely analogous to those that appear in the real-time formulation of finite temperature field theory in flat space \cite{26, 27}. There the vertical components of the contour typically factorize for Green functions evaluated at finite values of the time. However this factorization is quite subtle \cite{28} even for flat space, so we will not assume it here in general.

The other horizontal components to the time-contour correspond to the fact that one is
not computing a transition amplitude, but rather the expectation value of some time-ordered string of field operators with respect to a density matrix specified at some specific time \( t_0 \)
(where \( t_0 = 0 \) in figure 3)

\[
G(x_1, \cdots, x_n) = \text{Tr}_0 \rho(t_0) T (\phi(x_1) \cdots \phi(x_n)) .
\] (18)

The additional time contour represents the time-evolution back to the initial time, as is easy to see when \([15]\) is written in Schröedinger picture. The density matrix formalism is discussed in the general context of Friedman-Robertson-Walker backgrounds in \([29]\). In this work a deformation of the contour shown in figure 3 is used that only contains two horizontal real-time components (and a single vertical component).

In \([20]\) the same two-component time-contour is used (neglecting the vertical components of the contour), which can be re-expressed in terms of a two-component field formalism. However they use a time-ordering prescription analogous to \([21]\), which as we will see is not obtained via analytic continuation of the imaginary time approach. Their main claim was that there exist ultra-violet divergences that cannot be canceled with de Sitter invariant counter-terms. They approximated the \( \alpha \)-vacuum by starting with the free field theory state, and then turned on interactions at a finite time. Given that their boundary conditions did not respect de Sitter invariance, the appearance of de Sitter non-invariant counter-terms should not be too surprising. Such features cannot arise in a manifestly de Sitter invariant formulation, so are more properly regarded as renormalization effects associated with the symmetry breaking initial state.

It is natural to conjecture that in-out transition amplitudes relating the vacuum in the infinite past to vacuum in the infinite future, may be computed by including only the horizontal component of the contour running along the real time axis (with small positive slope) as shown in figure 3. This allows us to express the real-time physics using a single component field. This fits nicely with the expectation that pure states do not evolve into mixed states in a fixed de Sitter background, since the spacetime is globally hyperbolic \([37]\).

Now let us examine what happens when \( e^\alpha \neq 0 \). Again we start with the imaginary time contour, with a product of imaginary time propagators \([5]\). These may be decomposed into Euclidean vacuum correlators as in the second line of \([5]\). To continue to real-time, we continue to the contour shown in figure 3. The real-time continuation of the imaginary time formalism \([21]\) therefore yields a set of amplitudes of the form \([18]\), with \( \phi \)'s replaced by \( \bar{\phi} \)'s.
FIG. 2: Imaginary time contour.

If we wish to compute only in-out matrix elements, then we retain only the horizontal contour running along the real axis, and we find nodal and anti-nodal points are to be ordered according to their global time \( t \) \[23\]. Thus the propagators \[13\] will appear on internal lines in the general \( \alpha \)-vacuum expression for real-time ordered Green functions. Because all singularities appearing in the propagator have the same \( i\epsilon \) prescription, the resulting integrals are integrals of analytic functions, and no pinch singularities arise. Likewise, no pinch singularities will arise if we use the full contour to compute the continuation of the amplitudes of \[21\], though one must then use a multi-component field formalism analogous to \[29\] to directly perform the real-time computations.

One might wonder whether the single horizontal component plus the vertical components of the contour might lead one back to the in-out amplitudes of the squeezed state
FIG. 3: Real-time contour. The horizontal components of contours are to be understood to run off to \( t = \pm \infty \).

approach. This cannot be the case, because the vertical components will not generate pinch singularities, nor will the vertical components change the boundary conditions on the free propagators from \([\mathbf{1}]\) to \([\mathbf{3}]\).

D. Path Integral Formulation

The real-time continuation of the imaginary time formalism just described yields a set of finite renormalized in-out amplitudes of the form

\[
G(x_1, \cdots, x_n) = \langle E | T \left( U^\dagger \phi(x_1) \cdots \phi(x_n) e^{iS_{\text{int}}(\phi)} U \right) | E \rangle = \langle E | T \left( \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) e^{iS_{\text{int}}(\tilde{\phi})} \right) | E \rangle
\]

(19)
where the time-ordering prescription is as in [13]. Note that by definition local sources couple to the \( \tilde{\phi} \) [21], which is a linear combination of the field at nodal and anti-nodal points as in (4). We will work under the hypothesis that the set of amplitudes [19] define a consistent set of probability amplitudes. For example, these could be used to approximate transition amplitudes corresponding to observations of a comoving observer. The general set of local observables should correspond to Wightman functions of the \( \tilde{\phi} \).

The time-ordered correlators [19] can be generated from the following path integral

\[
Z = \int \mathcal{D}\phi e^{iS[\phi]} \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n)
\]

where the action \( S \) is

\[
S = \frac{1}{2} \int d^4x \sqrt{-g(x)} \, d^4y \sqrt{-g(y)} \tilde{\phi}(x) K(x, y) \tilde{\phi}(y) - \int d^4x \sqrt{-g(x)} \left( V(\tilde{\phi}) + j(x) \tilde{\phi}(x) \right)
\]

with the non-local kinetic term determined by

\[
K(x, y) = \left( a \frac{\delta^4(x-y)}{\sqrt{-g(x)}} + b \frac{\delta^4(x-\bar{y})}{\sqrt{-g(x)}} \right) \left( \square_x - m^2 \right)
\]

where

\[
a = \frac{1 - |\alpha|^4}{(1 - e^{2\alpha})(1 - e^{2\alpha^*})}, \quad b = \frac{- (e^{\alpha} + e^{\alpha^*})(1 - |\alpha|^2)}{(1 - e^{2\alpha})(1 - e^{2\alpha^*})}.
\]

This kernel is the inverse of the Feynman propagator [13]

\[
\int d^4z \sqrt{-g(z)} K(x, z) \tilde{G}^\alpha_F(z, y) = - \frac{\delta^4(x-y)}{\sqrt{-g(x)}}.
\]

It is possible to make a field redefinition to write the kinetic term in local form, but then the potential term becomes non-local. It is also worth noting that the amplitudes [19] cannot be interpreted simply as amplitudes in a squeezed state background, contrary to [23]. This would require the \( U \) operators to be commuted past the time-ordering symbol, so that one could define an asymptotic state \( U|E\rangle \) in the infinite past. However this step cannot be made due to the non-locality of the theory, as one can easily check using the explicit mode expansions of the amplitudes. Summing up, this approach differs from the squeezed state approach described in section [14A] due to the different time-ordering prescription, which in turn leads to a different non-local kinetic term.
E. Algebra of observables

Local sources couple directly to $\tilde{\phi}$ and likewise interactions are local \cite{20,21}. If we are interested in the scalar field theory with possible local scalar couplings of other fields to $\tilde{\phi}$, then the set of observables will be built out of local products of $\tilde{\phi}$ and derivatives. As shown in \cite{9}, the commutator algebra of the $\tilde{\phi}$ is actually independent of $\alpha$, and so vanishes at spacelike separations. The same will therefore be true of local products of the $\tilde{\phi}$. Apparently then the pure scalar field theory can give rise to a self-consistent set of probability amplitudes in this approach, which does not allow faster than light signaling. This provides us with a posteriori justification for taking the single-component real-time contour leading to \cite{19}.

However gravity couples locally to the stress-energy tensor

\[
T_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}} \frac{\delta S[\phi]}{\delta g^{\mu\nu}(x)}
\]

\[
= a \left( \tilde{\phi}_\mu(x) \tilde{\phi}_\nu(x) - \frac{1}{2} g_{\mu\nu}(x) \tilde{\phi}_\rho(x) \tilde{\phi}_\sigma(x) + \frac{1}{2} m^2 \left( \tilde{\phi}(x) \right)^2 g_{\mu\nu}(x) \right) + \left( V(\phi(x)) + j(x) \tilde{\phi}(x) \right) g_{\mu\nu}(x) +
\]

\[
b \left( \frac{1}{2} \tilde{\phi}_{\mu}(x) \tilde{\phi}_{\nu}(x) + \frac{1}{2} \tilde{\phi}_\mu(\bar{x}) \tilde{\phi}_\nu(\bar{x}) - \frac{1}{2} g_{\mu\nu}(x) g^{\rho\sigma}(x) \tilde{\phi}_\rho(x) \tilde{\phi}_\sigma(\bar{x}) + \frac{1}{2} g_{\mu\nu}(x) m^2 \tilde{\phi}(x) \tilde{\phi}(\bar{x}) \right)
\]

which is non-local in the $\tilde{\phi}$’s due to the non-local kinetic term \cite{21}. The commutator of $T^{\mu\nu}$ with a local product of $\tilde{\phi}$ can therefore be non-vanishing at spacelike separations. Therefore once the scalar field is coupled to gravity, the locality of the observables is spoiled and faster than light signaling becomes possible. This should be taken as a sign that the theory is non-perturbatively ill-defined once coupled to gravity. Once faster than light signaling is possible, it should be possible to consider processes analogous to closed timelike curves, which typically lead to uncontrollable quantum backreaction \cite{22}. We emphasize this is not acausality at Planck scale separations, but macroscopic acausality induced by propagation of the massless graviton. Even if these terms appear with tiny coefficients (as they would based on the arguments of \cite{3,13}), there is no known theoretical framework for handling such processes.

One could try to fix this problem by placing the graviton itself in an $\alpha$-state, by instead demanding a linear combination of $g_{\mu\nu}(x)$ and $g_{\mu\nu}(\bar{x})$ couple locally to $T^{\mu\nu}$ . As with the $\tilde{\phi}$ field, the new graviton will then have a non-local kinetic term. This may well work at the linearized level around a fixed de Sitter background, but once we include gravitational
interactions and proceed to write down a diffeomorphism invariant action, one will run into problems. For the pure scalar field theory to work it was important that interactions were local, despite the non-local kinetic term. However, if we start with the non-local gravitational kinetic term and add interactions order by order in Newton’s constant to achieve diffeomorphism invariance, we will induce non-local gravitational interaction terms. Again it seems impossible to avoid problems with faster than light signaling.

Furthermore the anti-podal symmetry is a special feature of de Sitter space that will not generalize in a background independent way. One could try to define the theory on the Lorentzian continuation of $\mathbb{RP}^4$, making the identification $x \sim \bar{x}$. Here the gravitational action takes the conventional Einstein-Hilbert form, but it is not clear how to make sense of physics on such a spacetime. One could take as a fundamental region the inflationary patch

$$ds^2 = 1/\eta^2 \left( -d\eta^2 + d\bar{x}^2 \right),$$

and treat $\eta$ as the global time coordinate. However as you extend a time-like geodesic to the past, you eventually hit the line $\eta = -\infty$ in finite proper time, and then begin moving forward in time at a different spatial position. Therefore the spacetime is not time orientable. As described in [30], this implies global quantization of a free scalar field on this spacetime is not possible. In [30] it is argued scalar field quantization within a single static patch can be done self-consistently. However without a global quantization method, one must go well beyond the conventional framework of semi-classical quantum gravity to make sense of the coupling of such a system to gravity.

For the Bunch-Davies vacuum all these problems are avoided, because the kinetic term is local for $e^\alpha = 0$. We conclude that the Bunch-Davies vacuum is the only de Sitter invariant vacuum state that yields a consistent conventional perturbative quantum field theory when coupled to gravity.

IV. CONCLUSIONS

We have seen that from the current theoretical standpoint, the $\alpha$-vacua are in general inconsistent. The most straightforward treatment as a squeezed state leads to pinch singularities which render the perturbative expansion ill-defined. Another approach derived from imaginary time methods leads to a well-defined perturbative expansion, however the theory
becomes non-local once coupled to gravity. We stress this non-locality is over macroscopic scales due to the fact it is induced by the massless graviton, so does not have a local effective description even at arbitrarily low energies. Conventional wisdom then suggests the vacua cannot be consistently coupled to gravity at the non-perturbative level. Hopefully these results serve to pin-point the problems with the so-called $\alpha$-vacua, and establish the Bunch-Davies vacuum as the unique de Sitter invariant vacuum state that survives coupling to gravity.

These results have a number of important implications for trans-Planckian effects during inflation. Certain classes of trans-Plankian effects can be modelled as an $\alpha$-vacua with an explicit ultra-violet cutoff as advocated in $^{13,15}$. In these models it is presumed unknown ultra-violet physics place modes in an $\alpha$-vacuum below some proper cutoff wavenumber. The present results indicate it is unlikely this unknown ultraviolet physics can be described by a local perturbative effective field theory. Within the context of local effective field theory, one can still fine-tune the initial state so that it gives rise to unusual effects at the end of inflation. However, we now can convincingly argue that generic perturbations will inflate away and the unique de Sitter invariant Bunch-Davies state will be left behind $^{38}$. Up to fine tuning issues, the influence of high energy physics on inflation can then be captured by a local low-energy effective action analysis around the Bunch-Davies state, which leads to the conclusion that high energy physics corrections to the cosmic microwave background will typically be beyond the cosmic variance limits $^{17,31}$ (notwithstanding some loopholes $^{32,33}$). One can also view these results as highlighting the type of modification of conventional gravity needed to make sense of a variety of proposed trans-Planckian effects. Finally, it should also be noted our results do not apply to trans-Planckian corrections to inflation that do not asymptote to a de Sitter $\alpha$-vacuum (see for example $^{34}$).

Acknowledgments

We thank R. Brandenberger for helpful comments. D.L. thanks the Departamento de Física, CINVESTAV for hospitality. This research is supported in part by DOE grant DE-FE0291ER40688-Task A and US Israel Bi-national Science Foundation grant #2000359.
[37] In general, when continuing from imaginary-time Green functions to real-time Green functions, pure states evolve to mixed states.
[38] One might hope that inflation was sufficiently short for certain perturbations to survive - see for example.