New exact treatment of the perturbed Coulomb
interactions

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Abstract

A novel method for the exact solvability of quantum systems is discussed and used to obtain closed analytical expressions in arbitrary dimensions for the exact solutions of the hydrogenic atom in the external potential \( \Delta V(r) = br + cr^2 \), which is based on the recently introduced supersymmetric perturbation theory.

One of the challenging problems in nonrelativistic quantum mechanics is to find exact solutions to the Schrödinger equation for potentials that are used in different fields of physics. In particular, the perturbed Coulomb potentials represent simplified models of many situations found in atomic, molecular, condensed matter and particle physics. There has been much interest in obtaining analytical solutions of such potentials in arbitrary dimensions. These problems have been studied for years and a general solution has not yet been found.

Such class of potentials,

\[ V(r) = -\frac{a}{r} + br + cr^2, \]  

are possible candidates for the quarkonium potential as has been indicated by the quarkonium spectroscopy. In the special case of \( c = 0 \) and \( b > 0 \) such potentials reduce to the well known charmonium potential. Apart from its relevance in heavy quarkonium spectroscopy, this class of potentials with \( c = 0 \) has important applications in atomic physics. The Stark effect in a hydrogen atom in one dimension is given exactly by the charmonium like potential (\( b \) being the electric field parameter). The more general class of these potentials with \( c > 0 \) is also relevant in atomic physics. This could be interpreted as the potential seen by an electron of an atom exposed to a suitable admixture of electric and magnetic fields. In addition, nuclei in the presence of an electron background form a system which is important for condensed matter physics and for laboratory and stellar plasmas. The potential between two nuclei embedded in such a plasma is approximately Coulomb plus harmonic oscillator, which corresponds to \( b = 0 \) in (1).

As the exact form of such interactions are being unknown to a great extent, it is thus desirable to study the general analytical properties of a large class of potentials in (1). In connection with this, the analyticity of the energy levels for these kind of potentials was investigated rigorously by many authors using different theories [1]-[14] in relation to their potential applications in spectroscopic problems.

In this letter, we introduce an alternative, simple formalism for an algebraic solution of the Schrödinger equation with the perturbed Coulomb potential and find exact solutions in \( N \)-dimensional space. The new formalism is based on the supersymmetric quantum mechanics and
uses the spirit of perturbation theory, in which there is no room for the drawbacks encountered in the calculations with the perturbation theories available in the literature. The work introduced in this letter also clarifies, within the powerful frame of the present formalism, that the potential in (1) is in fact an exactly solvable shape invariant potential unlike the claim in a recent work [11] where the authors stated that the supersymmetric quantum mechanics yields exact solutions for a single state only for the perturbed Coulomb potential in (1) which was treated as a quasi-exactly solvable potential.

Let us now introduce the formalism. The reduced radial wave Schrödinger equation for a spherically symmetric potential in $N$-dimensional space reads

$$\frac{\hbar^2}{2m} \frac{\Psi''_n(r)}{\Psi_n(r)} = V(r) - E_n \ , \ V(r) = V_{ES}(r) + \Delta V(r) \ , \ V_{ES}(r) = V_0(r) + \frac{\Lambda (\Lambda + 1) \hbar^2}{2mr^2} ,$$

where $V_0$ is one of the exactly solvable potential, such as the Coulomb and harmonic oscillator potentials used through the present work, $n = 0, 1, 2, \ldots$ being the radial quantum number, and $\Lambda = (M - 3)/2$ with $M = N + 2\ell$. We see that the radial Schrödinger equation in $N$ dimensions has the same form as the three-dimensional one. Consequently, given that the potential has the same form in any dimension, the solution in three dimensions can be used to obtain the solution in arbitrary dimensions simply by the use of the substitution $\ell \rightarrow \Lambda$. In the above equation, $\Delta V$ stands for the small perturbation ($=br + cr^2$) assuming the potential parameter $a$ is large when compared to other coupling parameters $b$ and $c$ in (1) with the consideration of perturbed Coulomb interactions while for the case of large $c$ and relatively small $a$ and $b$ the perturbing potential involves $\Delta V = -a/r + br$ leading to a perturbed harmonic oscillator potential. This significant feature will be used later in this work to test our results.

Using the spirit of the standard perturbation theory, we put forward here a more general formalism to treat quantum mechanical perturbation problems efficiently in a simple framework, which has been already discussed in detail through the recent works [15, 16] involving different applications of the model. Start with

$$\Psi_n(r) = \chi_n(r) \phi_n(r) ,$$

in which $\chi_n$ is the known normalized eigenfunction for the exactly solvable potential in (1) including the barrier term and $\phi_n$ is a moderating function due to the perturbing potential. Substituting (3) into (2) yields

$$\frac{\hbar^2}{2m} \left( \frac{\chi''_n}{\chi_n} + \frac{\phi''_n}{\phi_n} + 2 \frac{\chi'_n \phi'_n}{\chi_n \phi_n} \right) = V - E_n .$$

With the new definitions,

$$W_n = -\frac{\hbar}{\sqrt{2m}} \chi'_n , \quad \Delta W_n = -\frac{\hbar}{\sqrt{2m}} \phi'_n ,$$

one arrives at

$$\frac{\hbar^2}{2m} \chi''_n = W_n^2 - \frac{\hbar}{\sqrt{2m}} W_n' = V_{ES} - \epsilon_n ,$$

where $\epsilon_n$ is the eigenvalue for the exactly solvable potential of interest, and

$$\frac{\hbar^2}{2m} \left( \frac{\phi''_n}{\phi_n} + 2 \frac{\chi'_n \phi'_n}{\chi_n \phi_n} \right) = \Delta W_n^2 - \frac{\hbar}{\sqrt{2m}} \Delta W'_n + 2W_n \Delta W_n = \Delta V - \Delta \epsilon_n ,$$
in which $\Delta\epsilon_n$ is the correction term to the energy due to $\Delta V$, and $E_n = \epsilon_n + \Delta\epsilon_n$. Subsequently, Eq. (4) reduces to

$$(W_n + \Delta W_n)^2 - \frac{\hbar}{\sqrt{2m}} (W_n + \Delta W_n)' = V - E_n .$$

As the complete spectrum and wave functions are known in the analytical form for the exactly solvable potentials appeared in (6), one needs here to solve (7) to obtain in a closed form the corrections to both, energy and wave function.

Proceeding with the perturbed Coulomb potential in arbitrary dimensions,

$$V(r) = \left[-\frac{a}{r} + \frac{\Lambda(\Lambda+1)}{2mr^2}\right] + br + cr^2,$$

and set the superpotential, for the ground solutions,

$$W_{n=0}(r) = \sqrt{\frac{ma}{(\Lambda + 1)\hbar^2}} - \frac{(\Lambda + 1)\hbar}{\sqrt{2mr}},$$

leading to the first part of the potential in the bracket, and from the literature, corresponding normalized wave function and energy are

$$\chi_{n=0}(r) = N_0 r^{\Lambda+1} \exp\left[-\frac{ma}{(\Lambda + 1)\hbar^2}r\right], \quad \epsilon_{n=0} = -\frac{ma^2}{2\hbar^2(\Lambda + 1)^2}.$$

For the perturbing potential, $\Delta V = br + cr^2$, the physically acceptable unique choice is

$$\Delta W_{n=0}(r) = \sqrt{cr},$$

which satisfies Eq. (7) from where one readily sees that

$$\Delta\epsilon_{n=0} = \frac{M(M-1)b\hbar^2}{4ma}, \quad b = \frac{2\sqrt{2mc}}{(M-1)\hbar}.$$

It is stressed that such solutions in general have constraints on the potential parameters as appeared in (13). These constraints differ for each eigenvalue, and hence various solutions do not correspond to the same potential and are not orthogonal.

From (12), the moderating function

$$\phi_{n=0}(r) = \exp\left[-\frac{\sqrt{2m}}{\hbar} \int^r \Delta W_{n=0}(r)\right] = \exp\left[-\frac{b(M-1)}{4a}r^2\right].$$

Therefore, the full wave function $\Psi_{n=0}$ for the potential in (9) takes the form

$$\Psi_{n=0}(r) = \chi_n(r)\phi_{n=0}(r) = N_0 r^{\frac{M-1}{2}} \exp\left[-\frac{2ma}{(M-1)\hbar^2}r - \frac{b(M-1)}{4a}r^2\right].$$

Finally, the exact ground state energy is

$$E_{n=0} = \epsilon_{n=0} + \Delta\epsilon_{n=0} = -\frac{2ma^2}{\hbar^2(M-1)^2} + \frac{M(M-1)b\hbar^2}{4ma}.$$

The results in Eqs. (15) and (16) are exact and agree with those in [11, 12].
As noted earlier, the potential in (9) behaves also like a harmonic oscillator in case for large \(c\) and relatively small \(a\) and \(b\). This enables one to check out explicitly the results obtained above and the reliability of the formalism introcuded. For this consideration, we set the superpotentials

\[
W_{n=0}(r) = \sqrt{c}r - \frac{(\Lambda + 1)\hbar}{\sqrt{2mr}}, \quad \Delta W_{n=0}(r) = \frac{b/2}{\sqrt{c}},
\]

which yield, through the use of either Eqs. (6) and (7) together, or (8) alone,

\[
\epsilon_{n=0} = \frac{\hbar\sqrt{c}}{\sqrt{2m}} (2\Lambda + 3), \quad \Delta \epsilon_{n=0} = -\frac{b^2}{4c},
\]

which are exactly equivalent to those found for the perturbed Coulomb case in (16). Similarly, using (17) one readily arrives at (15).

For the generalization, we further make clear that the present technique is also applicable to excited states, for which one needs to use the shape invariance property and the relation between supersymmetric partners [17]

\[
V^+(r, \alpha_0) = V^-(r, \alpha_1) + R(\alpha_1),
\]

where \(\alpha_0 = \Lambda\) and \(\alpha_1 = f(\alpha_0) = \Lambda + 1\) are the position independent parameters while \(V^\pm\) are the supersymmetric partners

\[
(W_{n=0} + \Delta W_{n=0})^2 \mp \frac{\hbar}{\sqrt{2m}} (W_{n=0} + \Delta W_{n=0}) = V^\pm - E_{n=0}^-,
\]

which depend upon superpotentials in (10) and (12) for the case \(a > b, c\). Bearing in mind that the replacement of \(\Lambda\) with \(\Lambda + 1\) in \(V^-(r, \Lambda)\) leads to \(V^-(r, \Lambda + 1)\) and keeping the potential parameters \(b\) and \(c\) fixed as in [11, 12] while allowing the change in \(a\), see Eq. 13, we obtain by the straightforward calculations

\[
E_n^- = E_n^--\frac{b^2}{4c} + \frac{\hbar\sqrt{c}}{\sqrt{2m}} (2\Lambda + 3),
\]

\[
E_n^- = \sum_{k=1}^{n} R(\alpha_k) = -\frac{\hbar\sqrt{c}}{\sqrt{2m}} [(2\Lambda + 3) - 2(n + \Lambda) + 3],
\]

\[
E_n = -\frac{b^2}{4c} + \frac{\hbar\sqrt{c}}{\sqrt{2m}} [2(n + \Lambda) + 3], \quad n = 0, 1, 2, \ldots,
\]

which agrees with [12] in \(N\)-dimensional space and with [9] in three dimensions. Furthermore, one can easily construct the bound \(n^{th}\) state wave functions from the ground state wave function using the supersymmetric definition [17]

\[
\Psi_{n+1}^-(r, \alpha_0) \propto A^+(r, \alpha_0)\Psi_n^-(r, \alpha_1), \quad A^+(r, \alpha_0) = -\frac{\hbar}{\sqrt{2m}} \left[ \frac{d^2}{dr^2} + \frac{\Psi_{n=0}^+(r, \alpha_0)}{\Psi_{n=0}^-(r, \alpha_0)} \right].
\]

We conclude with two remarks. First, the present formalism can be generalized to all the polynomial forces

\[
V_m(r) = A_r^{2m} + B_r^{2m-1} + \ldots + F/r + G/r^2
\]

(23)
as an alternative but simple treatment to the other works [18], and the references therein. With the consideration of the perturbed Coulomb problem we have clarified that the explicit solution of the related Schrödinger equation with the potential family in (23) remains feasible in an almost complete parallel with their $m = 1$ predecessors. Along this line, the works are in progress.

The second remark we wish to make is that apart from the inherent interest one has in the existence of exact solutions, the results reported here are likely to be useful in perturbation calculations for the excited state energies and wave functions, particularly if the method of other perturbation theories appeared in the literature for evaluating second- and higher-order corrections can be extended to cover such cases. In this context, we believe that the simple form of our new approach to such problems opens a new direction of development towards many practical applications yet to be constructed and appreciated.

References