Conformal Invariance
and Degrees of Freedom
in the QCD String

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Abstract

We propose a method for determining the degrees of freedom of the effective QCD string which makes use of the observed numbers of meson states. In particular, we find the $D_\perp = 2$ scalar string to be in excellent agreement with data. Our results are consistent with those obtained via a complementary method involving the static quark potential, suggesting that conformal invariance and modular invariance are reflected in the hadronic spectrum.

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It is by now a well-established fact that many aspects of QCD can be successfully modelled by strings. Indeed, an effective QCD string theory would simultaneously explain the existence of Regge trajectories, linear confinement, the exponential rise in hadron-state densities, and $s$- and $t$-channel duality. In fact, it has recently been proposed that the string symmetry known as modular invariance might even explain relative meson/baryon abundances [1].

Over the years many different string models have been proposed for describing the QCD color flux tube deemed responsible for quark confinement in mesons: early examples include the scalar (Nambu) string, the Ramond string, and the Neveu-Schwarz (NS) string, and more recent examples include Polyakov’s “rigid string” [2], Green’s “Dirichlet string” [3], and the Polchinski-Strominger effective string [4]. While all of these models are endowed with a certain number $D_\perp$ of bosonic degrees of freedom on the two-dimensional string worldsheet (corresponding to the vibrational and rotational string degrees of freedom in the external spacetime), the early models differ from the scalar string by placing additional, purely internal degrees of freedom on the string worldsheet. The later string models instead introduce effective interactions for the spacetime degrees of freedom which alter their short-distance behavior.

Conformal invariance nevertheless plays an important role in each case. In the early models, conformal invariance is exact (though classical), and indeed we now understand that in fact an infinite array of models of this type can be constructed by placing on the string worldsheet virtually any two-dimensional conformal field theory (CFT). The quantum excitations of the corresponding new worldsheet fields should then also produce new hadronic states. Conformal invariance in some of the more recent models, by contrast, is an effective symmetry, valid in the long-string limit. The effective CFT’s which emerge in this limit are nevertheless equally responsible for reproducing the salient features of hadronic spectroscopy.

A comparison with data is therefore necessary in order to constrain the class of strings appropriate for modelling QCD, or, more generally, to test the validity of the string approach by determining the extent to which conformal invariance is actually reflected in the observed hadron spectrum. In particular, we seek to use data to constrain the worldsheet central charge of the effective QCD string, since this parameter is a unique measure of the degrees of freedom in a general two-dimensional the-
ory. Most previous efforts in this direction have employed the static quark potential method. We shall here propose a different method which yields information concerning not only the total central charge, but also its distribution between spacetime and internal degrees of freedom. As we shall see, the fact that these two independent methods favor similar values for the central charge strongly suggests that the QCD string possesses conformal — and indeed modular — invariance. As a by-product, we will also find that the \( D_\perp = 2 \) scalar string is in excellent agreement with data.

Let us first review the basic ideas behind the static quark potential \( V(R) \). If the inter-quark radius \( R \) is sufficiently large that confinement can be modelled by a color flux tube, this potential can be shown [5] to take the exact form

\[
V(R) = \left[ (\sigma R)^2 + M_0^2 \right]^{1/2} \\
\sim \sigma R + (M_0^2 / 2\sigma) R^{-1} + O(R^{-2}) ,
\]

(1)

where \( \sigma \) is the string tension of the flux tube and \( M_0^2 \leq 0 \) is a constant independent of \( R \). While the first (linear) term in the large-\( R \) expansion of \( V(R) \) represents the classical energy in the effective string, the second term (the so-called “pseudo-Coulomb” term) is an attractive universal quantum correction (or Casimir energy) which arises due to transverse zero-point vibrations of the string. As such, this term is to be distinguished from the true attractive Coulomb term which has the same form and which arises at short distances from gluon exchange. The form of the exact string result (1) indicates that while \( \sigma R \) plays the role of a string “momentum,” the quantity \( M_0 \) appears as the string “rest mass” (or ground-state energy). The fact that \( M_0^2 \leq 0 \) (or equivalently that the long-distance pseudo-Coulomb term is attractive) implies that the ground state of the effective QCD string is tachyonic, yet this is argued to cause no inconsistency in the large-\( R \) limit [6].

If we assume the dynamics of the color flux tube to be modelled by a two-dimensional conformal field theory, then this ground state energy \( M_0 \) and the central charge \( c \) of the corresponding worldsheet theory are related by

\[
\alpha' M_0^2 = h - c/24 .
\]

(2)

Here \( \alpha' \equiv (2\pi\sigma)^{-1} \) is the Regge slope, and \( h \) is the conformal dimension of that primary field in the worldsheet theory which is responsible for producing the ground
state. In most cases this primary field is merely the identity field with $h = 0$, whereupon we see that the coefficient of the pseudo-Coulomb term directly yields the corresponding central charge. Note that in all other cases, however, the coefficient of this term yields information concerning only the difference $h - c/24 \equiv \tilde{c}/24$. This is dramatically illustrated in the Ramond string: in this case $h = c/24 = (D - 2)/16$, whereupon $\tilde{c} = 0$, the ground state is massless, and the long-range pseudo-Coulomb term is absent.

By fitting the parameters of the quark potential in Eq. (1) to heavy-quark spectroscopic data and/or results from lattice QCD, many authors [7] have attempted to determine the ground-state energy $M_0^2$ of the QCD string and thereby determine its central charge. While the string tension $\sigma$ is generally found to be in good agreement with the value $\alpha' \approx 0.85$ (GeV)$^{-2}$ determined by Regge-trajectory analyses, values of $\tilde{c}$ have been obtained throughout the range $0 < \tilde{c} \leq 4$, clustering near $\tilde{c} \approx 2$. Perhaps the largest source of error in these methods is the fact that they rely upon a full separation of the effects of the long-range Coulomb term from those of the true Coulomb interaction: while the latter are a priori unrelated to $\tilde{c}$, fits to spectroscopic data and/or lattice QCD results undoubtedly contain their contributions. Furthermore, as discussed, $\tilde{c}$ is not always an accurate measure of the degrees of freedom in the string worldsheet theory.

We shall now propose a different approach towards determining the central charge $c$, one which avoids all of the above difficulties and is, in a sense to be clarified later, complementary to that involving the static quark potential. In string theory (or more generally in any CFT), the number or degeneracy of states $g_n$ at any excitation level $n$ is given by the coefficients in a certain polynomial $\chi_h(x)$ called the character of the sector $[h]$ of the worldsheet CFT:

$$\chi_h(x) = x^{-\tilde{c}/24} \sum_{n=0}^{\infty} g_n x^n.$$  \hspace{1cm} (3)

Since the spacetime mass $M_n$ of a given string excitation level $n$ is given by $\alpha' M_n^2 = n - \tilde{c}/24$, we see that the ground state energy $M_0^2$ in each sector $[h]$ is indeed given by Eq. (2). Thus, the static quark potential method, by fitting $M_0$, is essentially a test of the string $n \to 0$ limit. However, as has been well-known from the earliest days of the dual-resonance models, information can also be extracted from the high-energy
limit \((n \rightarrow \infty)\), for in this limit string theories predict an exponential rise in the degeneracy of states \(g_n\) with excitation number \(n\):

\[
g_n \sim A [C^2(n - \hat{c}/24)]^{-B} e^{C\sqrt{n - \hat{c}/24}} = A \left(\frac{M}{T_H}\right)^{-2B} e^{M/T_H}. \tag{4}
\]

Here \(A, B,\) and \(C\) are constants, with \(T_H \equiv (C\sqrt{\alpha'})^{-1}\). The form of these expressions demonstrates that \(T_H\) is the famous Hagedorn temperature \([8]\), a critical temperature beyond which the partition functions of such theories (and indeed all of their thermodynamic quantities) cannot be defined. The exponent \(B\) also has profound physical consequences. Since the internal energy of such a hadronic system near the Hagedorn temperature \(T_H\) grows as \(U(T) \sim (T_H - T)^{2B-3}\) \([9]\), we see that \(T_H\) is a true maximum temperature if \(B \leq 3/2\), and merely the site of a second-order QCD phase transition otherwise.

What makes these observations useful for our purposes, however, is the fact that the Hagedorn temperature \(T_H\) defined via Eq. (4) is directly related to the total central charge of the underlying worldsheet CFT:

\[
c = \frac{3}{2\pi^2} \left(\alpha' T_H^2\right)^{-1}. \tag{5}
\]

Indeed, this result holds independently of the sector \([h]\) in which our string states are presumed to reside, yielding a value for the true central charge \(c\) rather than \(\hat{c}\). Thus, a determination of the Hagedorn temperature can be used to obtain a value for the total central charge \(c\). Furthermore, the exponent \(B\) can also be interpreted in terms of an underlying string theory, for \(B\) is universally related to \(D_\perp\), the effective number of spacetime dimensions in which transverse string oscillations can take place (or equivalently the number of uncompactified bosonic degrees of freedom in the worldsheet CFT \([10]\)):

\[
B = \frac{1}{4} (3 + D_\perp). \tag{6}
\]

Thus, a fit of Eq. (4) to hadronic data yields information concerning not only the total number of degrees of freedom, but also their effective distribution between spacetime and internal excitations.

In practice, however, a number of subtleties arise which must be addressed before an adequate fit can be performed. First, by tabulating the experimentally measured
masses $M_i$ and widths $\Gamma_i$ of observed mesons [11], we have calculated the density
\( \rho_{\text{exp}}(M) \) of meson states:

\[
\rho_{\text{exp}}(M) \equiv \sum_i W_i \mathcal{S}(M_i; M, \Gamma_i) .
\]  

(7)

Here $W_i \equiv \gamma_i(2I_i + 1)(2J_i + 1)$ is the number of states per resonance (with $\gamma_i = 1$
for charge self-conjugate states and $\gamma_i = 2$ otherwise), and $\mathcal{S}$ represents a statistical
distribution function. The result is plotted in Fig. 1, with error bars determined by
varying $\mathcal{S}$ between Breit-Wigner, Gaussian, and fixed-width $(\Gamma_i = 200$ MeV) distributions. These error bars also include an estimate of the uncertainties resulting from
the possible subtraction of quark masses. While it is clear that this density experiences
the predicted exponential growth over much of the plotted mass range, we see that the rate of growth sharply diminishes beyond $1.7$ GeV. This is attributable
to experimental difficulties, for at higher energies it becomes harder to distinguish
mesons from background. Indeed, a comparison with explicit quark-model calculations [12]
shows $1.7$ GeV to be the first energy where fewer mesons are observed than
predicted. Similarly, at the other extreme, the data below $0.3$ GeV depends on pion
contributions whose small masses result from approximate chiral symmetries which
string theory is not expected to model. Thus, we shall limit our attention to the
experimental meson data for $0.3 \leq M \leq 1.7$ GeV.

This in turn requires a more sophisticated treatment of the theoretical string
predictions than was sketched above. In particular, taking $\alpha' \approx 0.85$ (GeV)$^{-2}$, we see that a mass near $1.7$ GeV corresponds only to a string excitation of $n \approx 3$. While
the $n \to \infty$ asymptotic function quoted in Eq. (4) is remarkably accurate even for
relatively small values of $n$, it differs from the true string degeneracies $g_n$ for $n \leq 3$
by as much as $95\%$ for the scalar string. We thus require from string theory a more
precise functional form, one which includes a sufficient number of subleading terms
so that the true values of $g_n$ for the known strings are accurately reproduced. It is
a straightforward matter [13] to determine the first set of these subleading terms,
however, and together these yield the following improved “asymptotic” form for the
state degeneracies $g_M$ and corresponding density $\rho(M)$:

\[
g_M \sim \sqrt{2\pi A} \xi^\nu I_{I\nu}(\xi) \quad \text{where } \xi \equiv M/T_H
\]

\[
\equiv (2\alpha'M)^{-1} \rho_{\text{string}}(M) .
\]  

(8)

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Here $I_{\nu}$ is the modified Bessel function of order $|\nu|$, with $\nu \equiv 1/2 - 2B$; note that this result reproduces Eq. (4) in the limit $M \to \infty$ [14]. By substituting the proper values of $A$, $B$, and $T_H$ for known strings, we have verified that Eq. (8) is indeed accurate over the required range to within 2%. This is fortunate, since the forms of any additional subleading terms are dependent on model-specific parameters other than $B$ and $T_H$.

We then performed a fit [15] comparing the experimental and theoretical values of $\int_{m_i}^{m_i} \rho(M) dM$ as a function of $m$, with $m_i = 0.3$ GeV. Our results are as follows. Taking $B = 5/4$ (i.e., $D_\perp = 2$) resulted in a best fit with

$$T_H = 300 \text{ MeV} \implies c = 1.97,$$

where we have taken $\alpha' = 0.85$ (GeV)$^{-2}$. This is remarkably close to the central charge $c = 2$ of the $D_\perp = 2$ scalar string, and demonstrates that a string picture is indeed consistent with the data obtained from counting the numbers of hadronic states.

Constraining $B$ to different values, on the other hand, yields a different set of Hagedorn temperatures. In Fig. 2, the singly- and doubly-shaded regions indicate the range of values $(B, T_H)$ for which fits to the data with a $\chi^2$/d.o.f. $\leq 1$ can be obtained: the dashed line indicates the central value of the fits (i.e., the “best fits”), and the heavy solid lines indicate the border of the region allowed by string theory (corresponding to the constraints $B \geq 3/4$, $c \geq D_\perp$ [16]). We see that there indeed exists a region of overlap between the data-allowed and string-allowed regions; furthermore, the “best fit” line passes directly through this overlap region in the range $3/4 \leq B \leq 5/4$. Indeed, since the scalar strings for general $D_\perp$ lie along the upper (curved) boundary of this string region, we see that the “best fit” line intersects this scalar string line almost exactly at the expected value $B = 5/4$, or $D_\perp = 2$.

The overlap region can be further narrowed if we constrain the normalization constant $A$ in Eq. (8). Within the allowed region in Fig. 2, we find that $A_{\text{fit}}$ ranges from $\mathcal{O}(1)$ to $\geq \mathcal{O}(10^3)$. String theory, however, places a precise limit on the value of $A$:

$$A_{\text{string}} \leq \sqrt{2\pi} \left(4\pi \alpha'T_H^{-2}\right)^\nu,$$

with equality occurring for the scalar string. Thus, if we assume that there are 36
independent strings which contribute to the \((u, d, s)\)-quark meson spectrum (corresponding to 36 quark degrees of freedom: 9 possible quark/anti-quark flavor combinations, 4 spin states, and one color singlet state), we can require \(r \equiv A_{\text{fit}}/A^{\text{max}}_{\text{string}} \leq 36\). This then restricts us to the lower (doubly-shaded) portion of the data-allowed region in Fig. 2. In Fig. 3 we have plotted versus \((c, D_{\perp})\) that region of allowed parameter space which satisfies all three of these string constraints. The best fit line is superimposed.

We have already seen in Eq. (9) that the \(c = D_{\perp} = 2\) scalar string lies directly on the best fit line; we now see from Fig. 2 that this point is also exactly on the \(r = 36\) border. This implies that the scalar string should accurately model the absolute numbers of states in the meson spectrum, and not merely their rate of growth. In Fig. 1, for example, we have superimposed on the meson data the actual numbers of states predicted by 36 copies of the scalar string (solid line). The agreement is excellent.

In conclusion, we have seen that the density of meson states in the QCD spectrum is consistent with string-theoretic predictions; moreover, estimations of the central charge of the QCD string obtained via measurements of the Hagedorn temperature are consistent with those independently obtained via static quark potential methods. This latter agreement is especially significant, for these two methods depend separately on quantities which are \emph{a priori} independent, namely the rate of growth of the numbers of mesons, and the energy of the ground state of the corresponding flux tube. Indeed, only an underlying effective two-dimensional conformal invariance — in particular, modular invariance and the associated symmetry under \(x \to -1/x\) in Eq. (3) — serve to relate them. Our results thus constitute strong additional evidence that the confinement phase of QCD is consistent with a string theory in which conformal symmetry and modular invariance play a significant role. This certainly warrants further study.

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References


[10] The general relation is $B = 3/4 - k/2$ where $k$ is the modular weight of the characters $\chi$ associated with the worldsheet CFT [13]. Each transverse spacetime dimension (and the Liouville mode, if present) contributes $k = -1/2$. There are, however, no contributions from compactified bosonic worldsheet fields (or equivalently from the worldsheet fermions of the R/NS strings).


[13] G.H. Hardy and S. Ramanujan, *Proc. Lond. Math. Soc.* 17, 75 (1918); I. Kani and C. Vafa, *Commun. Math. Phys.* 130, 529 (1990). Since $I_{-\nu}(x) = I_{\nu}(x)$ if $\nu \in \mathbb{Z}$ or $x \gg 1$, we restrict ourselves to Bessel functions with positive order in Eq. (8); these are better-behaved in the $x \sim \mathcal{O}(1)$ and unphysical ($\nu \not\in \mathbb{Z}$) regions.

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[14] Since the states in a non-interacting string theory are infinitely narrow and populate discrete energy levels \( M_n \), their density is \( \rho(M) = g_M \sum_n \delta(M - M_n) = g_M(2\alpha'M) \sum_n \delta(\alpha'M^2 + \hat{\alpha}/24 - n). \) We replace the latter sum of \( \delta \)-functions with unity in order to provide a more realistic estimate of the generic broadening effects that interactions would have on the string spectrum. Furthermore, in the asymptotic limit \( n \to \infty \), this yields \( \rho(M) = g_M/\Delta M \) where \( \Delta M \) is the mass difference between adjacent string levels.

[15] Note that since Hagedorn’s original 1967 fits [8] were made to a bootstrap-motivated functional form with no connection to string theory, his results cannot be used to constrain the central charge.

[16] It is possible, however, to violate the constraint \( c \geq D_\perp \) through renormalization group effects: see, e.g., A.B. Zamolodchikov, JETP Lett. 43, 730 (1986).
Fig. 1: Number of meson states with masses $\leq M$ as function of $M$, compared with scalar-string result.
Fig. 2: Data- and string-allowed values of \((B,T_H)\), as discussed in text. Best data fits (dashed line) and \(D_\perp = 2\) scalar string point (dot) also shown.
Fig. 3: Values of \((D_1, c)\) satisfying both data and string constraints, with best fit line superimposed and scalar string shown (dot).